Chapter 8
Optimization Under Equality Constraints

Mathematics for Economist, Alpha Chiang

EE320: 1/2016

Contents

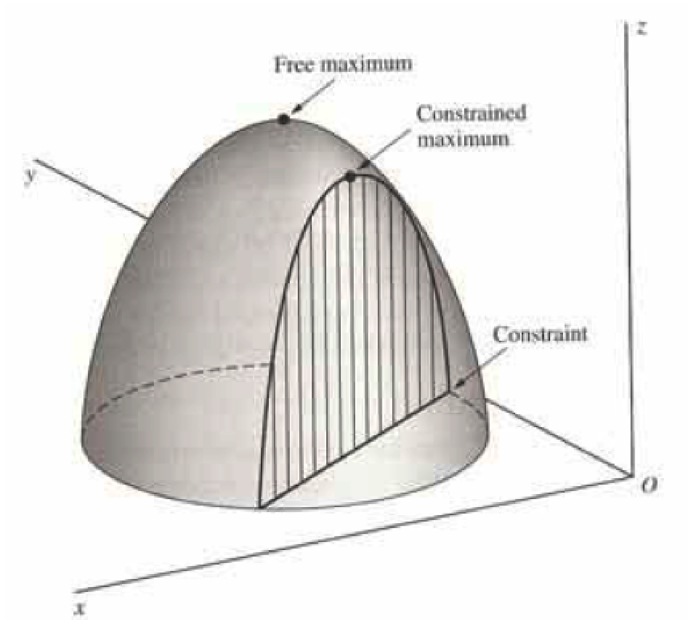
1 Effects of a constraint

- In the previous topic, we studied optimization problems, where the choice variables can be chosen freely.
- Example: A profit-maximizing firm's objective is:

$$\pi = PQ – [C(Q_1) + C(Q_2)]$$

where $Q = Q_1 + Q_2$.

- Q_1 and Q_2 can be chosen freely in order to maximize profit.
  $\Rightarrow$ Here, Q_1^* and Q_2^* are *free optimal values*.
- **Question**: If there is a *production constraint*, e.g.$Q_1 + Q_2 = 900$, then how would the firm change its behavior?
  $\Rightarrow$ ${Q_1}^{**}$ and ${Q_2}^{**}$ will now become the *constrained optimal values*
- Figure: "Free extremum" vs. "Constrained extremum"



  - Lower the optimum value of the objective function
  - Our goal is to come up with the solution method that help finding the constrainted optimum point

- Statement of the problem : Comparison , Constrain-free optimization and Constraint Optimization

Statement of the problem: Comparison

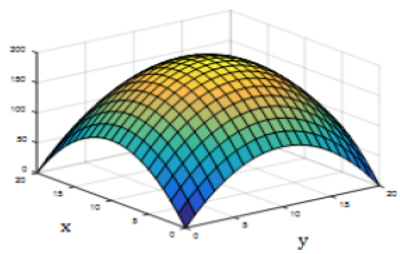
Constrain-free optimization	Constraint optimization
$\max\ (\min)_{x,y}\ f(x,y)$	$\max\ (\min)_{x,y}\ f(x,y)$
	such that
	$g(x,y) = c$

3

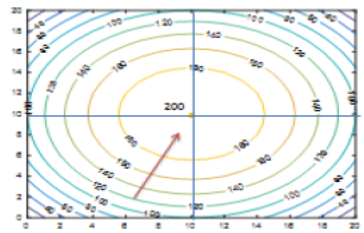
Statement of the problem: Comparison

Constrain-free optimization	Constraint optimization
$\max\ 200-(x-10)^2-(y-10)^2$	$\max\ 200-(x-10)^2-(y-10)^2$
	such that
	$x+y=10$

3-D curve of the objective function



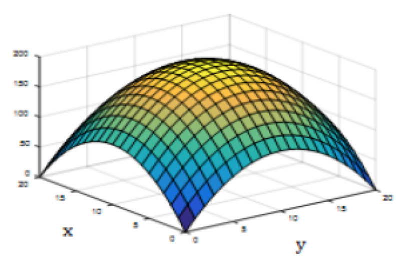
2D representation of the objective function



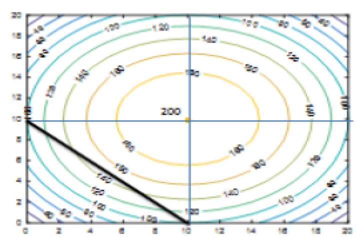
X= 10 and y = 10 is the optimum point under constraint-free optimization problem.

Introducing a constraint set

3-D curve of the objective function



2D representation of the objective function



Can we still choose x = 10 and y = 10 when the constraint is imposed?

2 Solution method to find the optimal value(s) : Elimination method and Lagrange multiplier

2.1 Elimination method (Substitution Method)

2.1.1 Examples

- **Example** : $\underset{x,\,y}{MAX}\; Z = xy$, subject to $x + y = 7$.

- **Example**: $\underset{x_1,x_2}{MAX}\; U = x_1x_2 + 2x_1$, subject to $4x_1 + 2x_2 = 60$.

2.1.2 Limitation

- Functional form of the constraint function is too complicate.
  - for example : $ln\left(\frac{x}{y}\right) + x^2 - \frac{y^2}{x^2+1} = \frac{3}{xy} + 2$
- More than single constraint set.
  - In general case, there can be more than one constraint.
  - Doable, but difficult/impractical to reduce the problem into unconstrained optimization problem.

2.2 Lagrange multiplier

- Construct a new function called the LaGrange function.
- The function takes the following form:

$$L(x, y, \lambda; c) = f(x, y) + \lambda \left[c - g(x, y)\right]$$

where λ is LaGrange multiplier.
(an auxiliary variable introduced for computational purpose, to be explained later.)

- Under some regularity conditions, the solution to the constrained optimization is the stationary point for the LaGrange function.
  - A stationary point of a function is the point where "first-order derivatives" are equal to zero.
- Consider the following maximization problem.

  $\underset{x,y}{Max}\ f(x,y)$

  subject to $\quad g(x,y)=c$

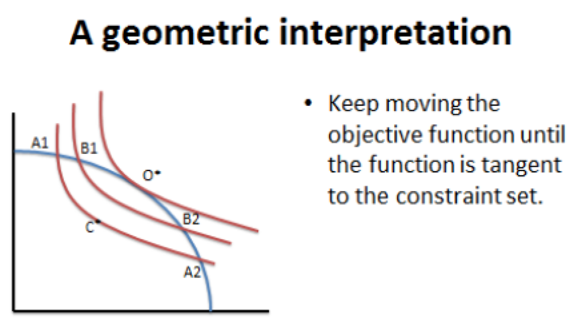
  - For LaGrange function, they are (x^*, y^* and λ^*) such that

$$\frac{\partial L}{\partial x}=\frac{\partial L}{\partial y}=\frac{\partial L}{\partial \lambda}=0$$

$$L(x,y,\lambda) \quad = \quad f(x,y)+\lambda[c-g(x,y)]$$

F.O.N.C.

$$\begin{aligned} L_x(x,y) &= f_x(x,y)-\lambda g_x(x,y)=0 \\ L_y(x,y) &= f_y(x,y)-\lambda g_y(x,y)=0 \\ L_\lambda(x,y) &= c-g_x(x,y)=0 \end{aligned}$$



2.2.1 Examples.

- **Example** : $\underset{x,\,y}{MAX}\ Z=xy$, subject to $x+y=7$.

- **Example**: $\underset{x_1,x_2}{MAX}\ U = x_1x_2 + 2x_1$, subject to $4x_1 + 2x_2 = 60$.

- Example : $\underset{x_1,x_2}{MAX}\ x_1^2 + x_2^2$, subject to $x_1 + 4x_2 = 2$

2.2.2 Intepretation of Lagrange Multiplier

- Consider the problem $\underset{x,\,y}{max}(min)\ f(x,y)$ subkect to $g(x,y) = c$.
  - solutions to this problem: $x^*(c)$, $y^*(c)$, and $f^*(c)$.
  - $f^*(c)$ is called optimal value function.
  - First order condition : $f_x - \lambda g_x = 0$
- We can show that $\dfrac{df^*(c)}{dc} = \lambda(c)$.

  Note that the Lagrange multiplier λ is the rate at which the optimal value of the objective function changes with respect to changes in the constraint constant, c.

  - Proof : $\dfrac{df^*(c)}{dc} = \lambda(c)$

$$\begin{aligned}
\mathcal{L}(c) &= f\left[x(c), y(c)\right] + \lambda(c)\left[c - g(x(c), y(c))\right] \\
\frac{d\mathcal{L}(c)}{dc} &= \left(f_x\frac{dx}{dc} + f_y\frac{dy}{dc}\right) + \lambda(c)\left[1 - g_x\frac{dx}{dc} - g_y\frac{dy}{dc}\right] + \left[c - g(x(c), y(c))\right]\frac{d\lambda}{dc} \\
&= \left(f_x - \lambda(c)g_x\right)\frac{dx}{dc} + \left(f_y - \lambda(c)g_y\right)\frac{dy}{dc} + \lambda(c) + \left[c - g(x(c), y(c))\right]\frac{d\lambda}{dc}
\end{aligned}$$

  At the optimal point, $f_x = \lambda^* g(x^*) = f_y - \lambda^* g(y^*) = c - g(x^*, y^*)$.

  Hence, $\dfrac{d\mathcal{L}^*(c)}{dc} = \lambda^*(c)$

  Moreover, since at the optimal point $c - g(x,y) = 0$. $\dfrac{d\mathcal{L}^*(c)}{dc} = \dfrac{df^*(c)}{dc}$.

- In economic applications, c denotes the available stock of some resources, $f(x,y)$ denotes utility or profit.
- $\lambda(c)dc$ is the change in utility or profit that can be obtained from dc units more of the resource.
- Thus, λ is called a "**shadow price**" of the resource.
- Examples:
  - For cost minimization problem, $\dfrac{dC}{dQ_0} = MC$.
  - For utility maximization problem, $\dfrac{dU}{dY_0} = MU$.
  - For profit maximization problem, $\dfrac{d\pi}{dQ_0}$
  - For output maximization problem, $\dfrac{dQ}{dC_0}$

3 Second-order conditions for constrained optimization

- Similar to the unconstrained optimization problem, we need to confirm our result.
- If the objective function is concave at the stationary point (along the constraint set), solution to the first-order condition is then confirmed to be a constrained maximizer.
- To do this, we check for the property of the second-order derivative matrix of the LaGrange function.
- By checking the property of the so called "Bordered" Hessian matrix.

3.1 Bordered Hessian Matrix

- **Proof**:
- The second-order sufficient conditions are:
  - For maximum of z: d^2z is negative definite, subject to dg = 0
  - For minimum of z: d^2z is positive definite, subject to dg = 0
  - where $dg = g_x dx + g_y dy = 0$.
- Define the determinant of the Bordered Hessian matrix as:

$$\bar{H} = \begin{bmatrix} \mathcal{L}_{\lambda\lambda} & \mathcal{L}_{\lambda,x} & \mathcal{L}_{\lambda,y} \\ \mathcal{L}_{x,\lambda} & \mathcal{L}_{x,x} & \mathcal{L}_{x,y} \\ \mathcal{L}_{y,\lambda} & \mathcal{L}_{y,x} & \mathcal{L}_{y,y} \end{bmatrix}$$

- Example . $\underset{x,y}{Max}\ f(x,y)$ subject to : $g(x,y) = c$

$$\bar{H} = \begin{bmatrix} 0 & g_x & g_y \\ g_x & \mathcal{L}_{x,x} & \mathcal{L}_{x,y} \\ g_y & \mathcal{L}_{y,x} & \mathcal{L}_{y,y} \end{bmatrix}$$

3.2 Second Order Condition

- For maximum, determinant of the bordered Hessian must be POSITIVE.
- For minimum, determinant of the bordered Hessian must be NEGATIVE.
- The sign definiteness of d^2z can be determined from the following criterion:*
  1. d^2z is **negative definite** subject to $g_x dx + g_y dy = 0$: iff $\left|\bar{H}\right| < 0$.
  2. d^2z is **positive definite** subject to gxdx + gydy = 0 iff $\left|\bar{H}\right| > 0$.

Summary:

- **Step 1:** Forming the LaGrange function : for a stationary value of $f(x,y)$ subject to : $g(x,y) = c$
- **Step 2:** Derive stationary points of the LaGrange function (Solving the FOCs of LaGrange function)
- **Step 3:** Checking concavity/convexity of function along the constraint set. –
  The second-order sufficient conditions are:
  1. For **maximum** of z, the **bordered Hessian** is **positive** $\left|\bar{H}\right| > 0$.
     – Concave: Determinant of Bordered Hessian is POSITIVE. (Maximum)
  2. For **manimum** of z, the **bordered Hessian** is **negative** $\left|\bar{H}\right| < 0$.
     – Convex: Determinant of Bordered Hessian is NEGATIVE. (Minimum)

- **Example**: Determine whether the extremum of the objective function ,

$$U(x_1, x_2) = x_1 x_2 + 2x_1$$

$$\text{subject to} \qquad 4x_1 + 2x_2 = 60$$

gives a minimum or a maximum.

- **Example**: Determine whether the extremum of the objective function ,

$$U(x_1, x_2) = x_1 x_2 + 2x_1$$

$$\text{subject to} \qquad 4x_1 + 2x_2 = 60$$

gives a minimum or a maximum.

4 Economic applications

4.1 Utility Maximization Problem

- Objectiuve function $\underset{Q_1,Q_2}{MaxU} = U(Q_1, Q_2)$
- Subject to $Y_0 = p_1Q_1 + p_2Q_2$
- Lagrangian funciton : $L(Q_1, Q_2, \lambda) = U(Q_1, Q_2) + \lambda [Y_0 - p_1Q_1 - p_2Q_2]$
- F.O.N.C.
  $L_1 = U_1(Q_1, Q_2) - \lambda p_1 = 0$
  $L_2 = U_2(Q_1, Q_2) - \lambda p_2 = 0$
  $Y_0 - p_1Q_1 - p_2Q_2 = 0$
  $\Rightarrow (Q_1^*, Q_2^*)$ is such that $\dfrac{U_1(Q_1,Q_2)}{U_2(Q_1,Q_2)} = \dfrac{p_2}{p_1}$ and $Y_0 = p_1Q_1 + p_2Q_2$.
- Second order sufficient condition.

$$
\begin{aligned}
|\bar{H}| &= \begin{vmatrix} 0 & g_x & g_y \\ g_x & \mathcal{L}_{x,x} & \mathcal{L}_{x,y} \\ g_y & \mathcal{L}_{y,x} & \mathcal{L}_{y,y} \end{vmatrix} \\
&= \begin{vmatrix} 0 & p_1 & p_2 \\ p_1 & U_{11} & U_{12} \\ p_2 & U_{21} & U_{22} \end{vmatrix} \\
&= -p_1^2U_{22} + 2p_1p_2U_{12} - p_2^2U_{11} > 0
\end{aligned}
$$

- **Example** : Suppose that $U = 50A + 200B - 0.5A^2 - 2.5B^2$. Find A and B that maximize utility given that $P_A = 10$, $P_B = 5$ and $Y = 490$.

4.2 Expenditure Minimization

- Objectiuve function $\underset{Q_1,Q_2}{Min}\, E = p_1 Q_1 + p_2 Q_2$
- Subject to $\bar{U} = U(Q_1, Q_2)$
- Lagrangian funciton : $L(Q_1, Q_2, \lambda) = p_1 Q_1 + p_2 Q_2 + \lambda\left[\bar{U} - U(Q_1, Q_2)\right]$
- F.O.N.C.

$$\left.\begin{array}{lcl} L_1 & = & p_1 - \lambda U_1(Q_1, Q_2) = 0 \\ L_2 & = & p_2 - \lambda U_2(Q_1, Q_2) = 0 \\ L_\lambda & = & \bar{U} - U(Q_1, Q_2) = 0 \end{array}\right\} \quad \begin{array}{c} (Q_1^*, Q_2^*) \\ \dfrac{U_1(Q_1, Q_2)}{U_2(Q_1, Q_2)} = \dfrac{p_1}{p_2} \end{array} \quad \text{and} \quad \begin{array}{c} \text{where} \\ \bar{U} = U(Q_1, Q_2) \end{array}$$

- Second order sufficient condition.

$$\begin{aligned} \left|\bar{H}\right| &= \begin{vmatrix} 0 & g_x & g_y \\ g_x & \mathcal{L}_{x,x} & \mathcal{L}_{x,y} \\ g_y & \mathcal{L}_{y,x} & \mathcal{L}_{y,y} \end{vmatrix} \\ &= \begin{vmatrix} 0 & -U_1 & -U_2 \\ -U_1 & -\lambda U_{11} & -\lambda U_{12} \\ -U_2 & -\lambda U_{21} & -\lambda U_{22} \end{vmatrix} \\ &= \\ &> 0 \end{aligned}$$

- Example. Suppose that $U = 50A + 200B - 0.5A^2 - 2.5B^2$. Find A and B that maximize utility given that $P_A = 10$, $P_B = 5$ and $U = U_0$.

4.3 Profit Maximization

- Objectiuve function $\underset{Q_1,Q_2}{Max}\pi = p_1Q_1 + p_2Q_2 - c(Q_1,Q_2)$
- Subject to $\bar{Q} = Q_1 + Q_2$
- Lagrangian funciton : $L(Q_1,Q_2,\lambda) = p_1Q_1 + p_2Q_2 - c(Q_1,Q_2) + \lambda\left[\bar{Q} - Q_1 - Q_2\right]$
- F.O.N.C.

$$\left.\begin{array}{rcl} L_1 &=& p_1 - c_1(Q_1,Q_2) - \lambda = 0 \\ L_2 &=& p_2 - c_2(Q_1,Q_2) - \lambda = 0 \\ L_\lambda &=& \bar{Q} - Q_1 - Q_2 = 0 \end{array}\right\} \quad \begin{array}{c} (Q_1^*,Q_2^*) \\ p_1 - c_1 = p_2 - c_2 \end{array} \quad \begin{array}{c} \\ \text{and} \end{array} \quad \begin{array}{c} \text{where} \\ \bar{Q} = Q_1 + Q_2 \end{array}$$

- Second order sufficient condition.

$$\begin{aligned} \left|\bar{H}\right| &= \begin{vmatrix} 0 & g_x & g_y \\ g_x & \mathcal{L}_{x,x} & \mathcal{L}_{x,y} \\ g_y & \mathcal{L}_{y,x} & \mathcal{L}_{y,y} \end{vmatrix} \\ &= \begin{vmatrix} 0 & & \\ & & \\ & & \end{vmatrix} \\ &= \\ &> \quad 0 \end{aligned}$$

- **Example.** Suppose $P_1 = 70$, $P_2 = 100$, $TC = 100 + 0.1Q_1^2 + 0.2Q_2^2$ and $Q_1 + Q_2 = 325$. Find Q_1 and Q_2 that maximize the profit.

4.4 Output Maximization

- Objectiuve function $\underset{Q_1,Q_2}{Max} Q = Q(K,L)$
- Subject to $\bar{C} = wL + rK$
- Lagrangian funciton : $L(K,L,\lambda) = Q(K,L) + \lambda\left[\bar{C} - wL - rK\right]$
- F.O.N.C.

$$\left.\begin{array}{rcl} L_1 &=& Q_K(K,L) - \lambda r = 0 \\ L_2 &=& Q_L(K,L) - \lambda w = 0 \\ L_\lambda &=& \bar{C} - wL - rK = 0 \end{array}\right\} \quad \begin{array}{c} (K^*, L^*) \\ \dfrac{Q_K(K,L)}{Q_L(K,L)} \end{array} \quad \text{and} \quad \begin{array}{c} \text{where} \\ \bar{C} = wL + rK \end{array}$$

- Second order sufficient condition.

$$\begin{aligned} \left|\bar{H}\right| &= \begin{vmatrix} 0 & g_x & g_y \\ g_x & \mathcal{L}_{x,x} & \mathcal{L}_{x,y} \\ g_y & \mathcal{L}_{y,x} & \mathcal{L}_{y,y} \end{vmatrix} \\ &= \begin{vmatrix} 0 & & \\ & & \\ & & \end{vmatrix} \\ &= \\ &> \quad 0 \end{aligned}$$

- **Example.** Suppose $Q = KL$, $w = 6$, $r = 10$, $C_0 = 60$. Find K and L that maximizes output.

4.5 Cost Minimization

- Objectiuve function $\underset{Q_1,Q_2}{Min}\, C = wL + rK$
- Subject to $\bar{Q} = F(K, L)$
- Lagrangian funciton : $L(K, L, \lambda) = wL + rK + \lambda \left[\bar{Q} - F(K, L)\right]$
- F.O.N.C.

$$\left.\begin{array}{rcl} L_1 & = & r - \lambda F_K(K,L) = 0 \\ L_2 & = & w - \lambda F_L(K,L) = 0 \\ L_\lambda & = & \bar{Q} - F(K,L) = 0 \end{array}\right\} \quad \begin{array}{c} (K^*, L^*) \\ \dfrac{F_K(K,L)}{F_L(K,L)} = \dfrac{r}{w} \end{array} \quad \text{and} \quad \begin{array}{c} \text{where} \\ \bar{Q} = F(K,L) \end{array}$$

- Second order sufficient condition.

$$\begin{array}{rcl} \left|\bar{H}\right| & = & \begin{vmatrix} 0 & g_x & g_y \\ g_x & \mathcal{L}_{x,x} & \mathcal{L}_{x,y} \\ g_y & \mathcal{L}_{y,x} & \mathcal{L}_{y,y} \end{vmatrix} \\ & = & \begin{vmatrix} 0 & & \\ & & \\ & & \end{vmatrix} \\ & = & \\ & < & 0 \end{array}$$

- Graph Expansion Path:
  - Expansion path describes the least cost combination of K^* and L^* required to produce varying levels of quantities.

- **Example.** Suppose $Q = KL$, $w = 6$, $r = 10$, $C_0 = 60$. Find K and L that maximizes output.

5 Extensions: n-variable and multiconstraint cases

5.1 Optimization with Equality Constraints: n-Variable and One Equality Constraint

- The objective function

$$z = f(x_1, x_2, x_{3,}..., x_n)$$

subject to the contraint $g(x_1, x_2, x_3, ..., x_n) = c$.
The Lagrangian function :
$\mathcal{L}(x_1, x_2, x_3, ...x_n, \lambda) = f(x_1, x_{2,}x_3, ..., x_n) + \lambda[c - g(x_1, x_2, x_3, ..., x_n)]$

- FONC.

$$\begin{array}{lcll}
\mathcal{L}_1 & = & f_1(x_1, x_2, x_3, ..., x_n) - \lambda g_1(x_1, x_2, x_3, ..., x_n) & = 0 \\
\mathcal{L}_2 & = & f_2(x_1, x_2, x_3, ..., x_n) - \lambda g_2(x_1, x_2, x_3, ..., x_n) & = 0 \\
\vdots & & \vdots & \vdots \\
\mathcal{L}_n & = & f_n(x_1, x_2, x_3, ..., x_n) - \lambda g_n(x_1, x_2, x_3, ..., x_n) & = 0 \\
\mathcal{L}_\lambda & = & c - g(x_1, x_2, x_3, ..., x_n) & = 0
\end{array}$$

- Bordered Hessian

$$\bar{H} = \begin{bmatrix} 0 & g_1 & g_2 & \cdots & g_{...} \\ g_1 & \mathcal{L}_{11} & \mathcal{L}_{12} & \cdots & \mathcal{L}_{....} \\ g_2 & \mathcal{L}_{21} & \mathcal{L}_{y22} & \cdots & \mathcal{L}_{....} \\ \vdots & & & \ddots & \vdots \\ g_{....} & \mathcal{L}_{....} & \mathcal{L}_{....} & \cdots & \mathcal{L}_{....} \end{bmatrix}$$

- Bordered leading principal minors can be defined as :

$$|\bar{H}_2| = \begin{vmatrix} 0 & g_1 & g_2 \\ g_1 & \mathcal{L}_{11} & \mathcal{L}_{12} \\ g_2 & \mathcal{L}_{21} & \mathcal{L}_{y22} \end{vmatrix}$$

$$|\bar{H}_3| = \begin{vmatrix} 0 & g_1 & g_2 & g_{...} \\ g_1 & \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{....} \\ g_2 & \mathcal{L}_{21} & \mathcal{L}_{y22} & \mathcal{L}_{....} \\ g_{....} & \mathcal{L}_{....} & \mathcal{L}_{....} & \mathcal{L}_{....} \end{vmatrix}$$

...

$$|\bar{H}_n| = \begin{vmatrix} 0 & g_1 & g_2 & \cdots & g_{...} \\ g_1 & \mathcal{L}_{11} & \mathcal{L}_{12} & \cdots & \mathcal{L}_{....} \\ g_2 & \mathcal{L}_{21} & \mathcal{L}_{y22} & \cdots & \mathcal{L}_{....} \\ \vdots & & & \ddots & \vdots \\ g_{....} & \mathcal{L}_{....} & \mathcal{L}_{....} & \cdots & \mathcal{L}_{....} \end{vmatrix}$$

- The conditions for positive and negative definiteness of d^2z are :

$$d^2z \text{ is } \begin{Bmatrix} \text{postive definite} \\ \text{negative definite} \end{Bmatrix} \text{ subject to } \quad dg = 0 \quad \text{iff } \begin{cases} |\bar{H}_2|, |\bar{H}_3|, ..., |\bar{H}_n| < 0 \\ |\bar{H}_2| > 0, |\bar{H}_3| < 0, |\bar{H}_4| > 0... \end{cases}$$

Condition	Maximum	Minimum
First-order necessary condition	$L_\lambda = L_1 = L_2 = ... = L_n = 0$	$L_\lambda = L_1 = L_2 = ... = L_n = 0$
Second-order necessary condition*	$\lvert\overline{H}_2\rvert > 0; \lvert\overline{H}_3\rvert < 0;$ $\lvert\overline{H}_4\rvert > 0; ...; (-1)^n \lvert\overline{H}_n\rvert > 0$	$\lvert\overline{H}_2\rvert, \lvert\overline{H}_3\rvert, ..., \lvert\overline{H}_n\rvert < 0$

Example : Cost minimization

- Objectiuve function $\underset{Q_1,Q_2}{Min}\, C = wL + iK + rT$
- Subject to $\bar{Q} = F(K, L, T)$
- Lagrangian funciton : $L(K, L, \lambda) = wL + rK + \text{............} + \lambda\left[\bar{Q} - F(K, L, T)\right]$
- F.O.N.C.

$$\left.\begin{array}{lcll} L_1 & = & i - \lambda F_K(K,L,T) & = 0 \\ L_2 & = & w - \lambda F_L(K,L,T) & = 0 \\ L_3 & = & & = 0 \\ L_\lambda & = & \bar{Q} - F(K,L) = 0 & = 0 \end{array}\right\} \quad \begin{array}{ccc} (K^*, L^*, T^*) & & \text{where} \\ \dfrac{F_K(K,L,T)}{F_L(K,L,T)} = \text{........} & \text{and} & \bar{Q} = F(Q,L,T) \\ \dfrac{F_K(K,L,T)}{F_T(K,L.T)} = \text{.........} & , & \dfrac{F_L(K,L,T)}{F_T(K,L,T)} = \text{.........} \end{array}$$

- Second order sufficient condition.

$$\bar{H} = \begin{bmatrix} 0 & g_x & g_y & \\ g_x & \mathcal{L}_{x,x} & \mathcal{L}_{x,y} & \\ g_y & \mathcal{L}_{y,x} & \mathcal{L}_{y,y} & \\ & & & \end{bmatrix}$$

$$= \begin{bmatrix} 0 & & & \\ & & & \\ & & & \\ & & & \end{bmatrix}$$

$$0$$

$$\left|\bar{H}_2\right| = \begin{vmatrix} 0 & & \\ & & \\ & & \end{vmatrix}$$

$$=$$

$$\text{....} \qquad 0$$

$$\left|\bar{H}_3\right| = \left| \begin{matrix} 0 & & \\ & & \\ & & \\ & & \end{matrix} \right|$$

$$=$$

$$.... \qquad 0$$

5.2 Optimization with Equality Constraints : n-variables and two equality constraints

5.2.1 The objective function

$$z = f(x_1, x_2, x_{3,...}, x_n)$$

subject to the contraint
$g^1(x_1, x_2, x_3, ..., x_n) = c_1$.
$g^2(x_1, x_2, x_3, ..., x_n) = c_2$.

The Lagrangian function :
$\mathcal{L}(x_1, x_2, x_3, ...x_n, \lambda) = f(x_1, x_{2,}x_3, ..., x_n) + \lambda_1[c_1 - g^1(x_1, x_2, x_3, ..., x_n) + \lambda_2[c_2 - g^2(x_1, x_2, x_3, ..., x_n)]$

5.2.2 FONC.

$$\left.\begin{array}{lclr} \mathcal{L}_1 & = & f_1(x_1, x_2, x_3, ..., x_n) - \lambda_1 g_1^1(x_1, x_2, x_3, ..., x_n) - \lambda_2 g_1^2(x_1, x_2, x_3, ..., x_n) & = 0 \\ \mathcal{L}_2 & = & f_{...}(x_1, x_2, x_3, ..., x_n) - \lambda_1 g_{....}^1(x_1, x_2, x_3, ..., x_n) - \lambda_2 g_{....}^2(x_1, x_2, x_3, ..., x_n) & = 0 \\ \vdots & & \vdots & \vdots \\ \mathcal{L}_n & = & f_{....}(x_1, x_2, x_3, ..., x_n) - \lambda_1 g_{.....}^1(x_1, x_2, x_3, ..., x_n) - \lambda_2 g_{.....}^2(x_1, x_2, x_3, ..., x_n) & = 0 \end{array}\right\} \mathcal{L}_i = 0; \text{for } i = 1, 2, 3..., n$$

$$\boxed{\begin{array}{lclrl} \mathcal{L}_i & = & f_{....}(x_1, x_2, x_3, ..., x_n) - \lambda_1 g_{.....}^1(x_1, x_2, x_3, ..., x_n) - \lambda_2 g_{.....}^2(x_1, x_2, x_3, ..., x_n) & = 0 & \text{for} \quad i = 1, 2, ..., n \\ \mathcal{L}_{\lambda_1} & = & c_{...} - g^{.....}(x_1, x_2, x_3, ..., x_n) & = 0 & \\ \mathcal{L}_{\lambda_2} & = & c_{...} - g^{.....}(x_1, x_2, x_3, ..., x_n) & = 0 & \end{array}}$$

5.3 Optimization with Equality Constraints : n-variables and k equality constraints

5.3.1 The objective function

$$z = f(x_1, x_2, x_{3,...}, x_n)$$

subject to the contraint
$g^1(x_1, x_2, x_3, ..., x_n) = c_1$.
$g^2(x_1, x_2, x_3, ..., x_n) = c_2$.
$\vdots$
$g^{...}(x_1, x_2, x_3, ..., x_n) = c_{...}$

The Lagrangian function :

$$\begin{array}{rcl} \mathcal{L}(x_1, x_2, x_3, ...x_n, \lambda) & = & f(x_1, x_{2,}x_3, ..., x_n) + \lambda_1[c_1 - g^1(x_1, x_2, x_3, ..., x_n) + \lambda_2[c_2 - g^2(x_1, x_2, x_3, ..., x_n)] \\ & & + \ldots + \lambda_{....}[c_{....} - g^{.....}(x_1, x_2, x_3, ..., x_n)] \end{array}$$

5.3.2 FONC.

$$\left.\begin{array}{lcll}
\mathcal{L}_1 & = & f_1(x_1,x_2,x_3,...,x_n)-\lambda_1 g_1^1(x_1,x_2,x_3,...,x_n)-\lambda_2 g_1^2(x_1,x_2,x_3,...,x_n)-...-\lambda_{....}g_{.....}^{....}(x_1,x_2,x_3,...,x_n) & =0\\
\mathcal{L}_2 & = & f_{...}(x_1,x_2,x_3,...,x_n)-\lambda_1 g_{....}^1(x_1,x_2,x_3,...,x_n)-\lambda_2 g_{....}^2(x_1,x_2,x_3,...,x_n)-...-\lambda_{....}g_{.....}^{....}(x_1,x_2,x_3,...,x_n) & =0\\
\vdots & & \vdots & \vdots\\
\mathcal{L}_n & = & f_{....}(x_1,x_2,x_3,...,x_n)-\lambda_1 g_{.....}^1(x_1,x_2,x_3,...,x_n)-\lambda_2 g_{.....}^2(x_1,x_2,x_3,...,x_n)-...-\lambda_{....}g_{.....}^{....}(x_1,x_2,x_3,...,x_n) & =0
\end{array}\right\}$$

$\mathcal{L}_i=0$; for $i=1,2,3...,n$

$$\boxed{\begin{array}{ccc}
\mathcal{L}_i & = & f_{...}(x_1,x_2,x_3,..,x_n)-\lambda_1 g_{.....}^1(x_1,x_2,x_3,..,x_n)-\lambda_2 g_{...}^2(x_1,x_2,x_3,..,x_n)-...-\lambda_{..}g_{...}^{...}(x_1,x_2,x_3,..,x_n)\\
 & & \text{for } i=1,2,...,n\\
\mathcal{L}_{\lambda_1} & = & c_{...}-g^{.....}(x_1,x_2,x_3,...,x_n)\\
\mathcal{L}_{\lambda_2} & = & c_{...}-g^{.....}(x_1,x_2,x_3,...,x_n)\\
\vdots & & \vdots\\
\mathcal{L}_{\lambda_{....}} & = & c_{...}-g^{.....}(x_1,x_2,x_3,...,x_n)
\end{array}}
\begin{array}{c}=0\\ \\ =0\\ =0\\ \\ =0\end{array}$$

5.3.3 Bordered Hessian

$$\bar{H}=\begin{bmatrix}
0 & 0 & \cdots & 0 & g_1^1 & g_2^1 & \cdots & g_n^1\\
0 & 0 & \cdots & 0 & g_1^2 & g_2^2 & \cdots & g_{...}^2\\
\vdots & & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\
0 & 0 & \cdots & 0 & g_1^k & g_2^k & \cdots & g_{...}^k\\
g_1^1 & g_1^2 & \cdots & g_1^{....} & \mathcal{L}_{11} & \mathcal{L}_{12} & \cdots & \mathcal{L}_{1....}\\
g_2^1 & g_2^1 & \cdots & g_2^{....} & \mathcal{L}_{21} & \mathcal{L}_{22} & \cdots & \mathcal{L}_{2....}\\
\vdots & \vdots & \ddots & \vdots & \vdots & & \ddots & \vdots\\
g_n^{....} & g_n^{....} & \cdots & g_n^{....} & \mathcal{L}_{....} & \mathcal{L}_{....} & \cdots & \mathcal{L}_{n....}
\end{bmatrix}$$

5.3.4 Second order condition

- The second order sufficient conditions are stated in terms of the signs of the following $(n-k)$ bordered leading principal minors:

$$\left|\bar{H}_{k+1}\right|,\left|\bar{H}_{k+2}\right|,...,\left|\bar{H}_n\right|;\left|\bar{H}_n\right|=\left|\bar{H}\right|$$

- For a **maximum** of z, the bordered leading principal minors **alternate in sign** and the sign of $\left|\bar{H}_{k+1}\right|=(-1)^{k+1}$.
- For a **minimum** of z, the bordered leading principal minors take **the same sign** of $(-1)^k$.

5.3.5 Example :

- The objective function

$$z=f(x_1,x_2,x_3)$$

subject to the contraint
$g(x_1,x_2,x_3)=c$.

$h(x_1, x_2, x_3) = d.$

The Lagrangian function :
$\mathcal{L}(x_1, x_2, x_3, \lambda) = f(x_1, x_2, x_3) + \lambda_1[c - g(x_1, x_2, x_3) + \lambda_2[d - h(x_1, x_2, x_3)]$

- FONC.

$$\left.\begin{array}{lcll} \mathcal{L}_1 & = & f_1(x_1, x_2, x_3, ..., x_n) - \lambda_1 g_1(x_1, x_2, x_3) - \lambda_2 h_1(x_1, x_2, x_3) & = 0 \\ \mathcal{L}_2 & = & f_{....}(x_1, x_2, x_3, ..., x_n) - \lambda_{....} g_{.....}(x_1, x_2, x_3) - \lambda_{....} h_{....}(x_1, x_2, x_3) & = 0 \\ \mathcal{L}_3 & = & f_{....}(x_1, x_2, x_3, ..., x_n) - \lambda_{....} g_{.....}(x_1, x_2, x_3) - \lambda_{....} h_{....}(x_1, x_2, x_3) & = 0 \end{array}\right\} \quad \mathcal{L}_i = 0; \text{for } i = 1, 2, 3..., n$$

- Hessian Matrix

$$\bar{H} = \begin{bmatrix} 0 & 0 & g_1 & g_2 & g_3 \\ 0 & 0 & h_1 & h_2 & h_{...} \\ g_1 & h_1 & \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{1....} \\ g_2 & h_2 & \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{2....} \\ g_3 & h_3 & \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{2....} \end{bmatrix}$$

- For a **maximum** of z, the bordered leading principal minors alternate in sign and the sign of $\left|\bar{H}_{k+1}\right| = (-1)^{k+1}$.

  $\left|\bar{H}_{.....}\right| = (-1)^{...+1}......0$

- For a **minimum** of z, the bordered leading principal minors take the same sign of $(-1)^k$.

  $\left|\bar{H}\right| = (-1)^{.....}......\ 0$