Determination of Forward and Futures Prices

Chapter 5

Outline

- Investments vs consumption assets
- Short selling securities
- Determinants of forward contract prices
  - Assets with no income and no storage cost
  - Assets with known income
  - Assets with known income yield

Outline

- Valuing forward contracts
- Forward prices and futures prices
- Futures prices of stock indices
- Forward and futures contracts on currencies
- Futures on consumption assets
  - Assets with storage costs
- The cost of carry
- Futures prices and expected spot prices

Consumption vs Investment Assets

- It is important to distinguish assets that are primarily consumption assets from investment assets for forward and futures contracts pricing
- Investment assets:
  - Held primarily for investments
  - Held for investments by a large number of investors
- Examples:
  - Stocks and bonds
  - Foreign currency
  - Gold and silver (may not be held exclusively for investments)

Consumption vs Investment Assets

- Consumption assets are assets held primarily for consumption
- Examples:
  - Oil
  - Pork belly
  - Corn
  - lumber

Short Selling

- Short selling involves selling securities you do not own
- Your broker borrows the securities from another client and sells them in the market in the usual way
- At some stage you must buy the securities back so they can be replaced in the account of the client
- You must pay dividends and other benefits the owner of the securities receives

Short Selling

A Broker B (owns 100 IBM)

Jan 06: A short sells 100 IBM stocks

A ← Broker ← B

100 IBM 100 IBM

A sells 100 IBM stocks at 110 for 11K

Mar 06: IBM pays $3 dividend/stock

A → Broker → B

$300 $300

May 06: A buys 100 IBM at 100

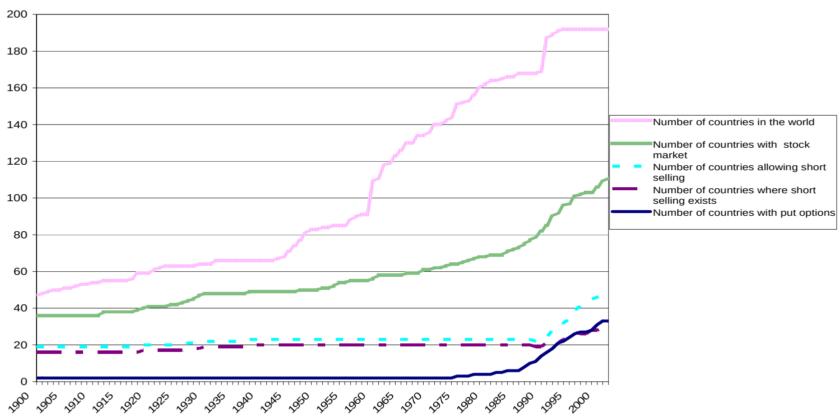
A → Broker → B

100 IBM 100 IBM

A Makes 11K-10K-300 = 700

Short Selling

Figure 1. Short Selling Regulations & Put Options in the Twentieth Century



Notation

S_0: Spot price today

F_0: Futures or forward price today

T: Time until delivery date

r: Risk-free interest rate for maturity T

Gold: An Arbitrage Opportunity?

- Suppose that:
  - The spot price of gold is US$390
  - The quoted 1-year forward price of gold is US$425
  - The 1-year US$ interest rate is 5% per annum (continuous compounding)
  - No income or storage costs for gold
- Is there an arbitrage opportunity?

Gold: An Arbitrage Opportunity?

- Today:
  - Enter into a short forward contract at 425 (cash flow 0)
  - Borrow $390 and buy 1 oz of gold
- At expiration/delivery
  - Deliver 1 oz of gold for 425.00
  - Return borrowed money with interest $\$390 * e^{0.05*1} = 410$
  - Make 425-410 = 15

Gold: Another Arbitrage Opportunity?

- Suppose that:
  - The spot price of gold is US$390
  - The quoted 1-year forward price of gold is US$390
  - The 1-year US$ interest rate is 5% per annum
  - No income or storage costs for gold
- Is there an arbitrage opportunity?

Gold: An Arbitrage Opportunity?

- Today:
  - Enter into a long forward contract at $390 (cash flow 0)
  - Sell short 1 oz of gold and get $390
  - Deposit cash in an account to get 5%
- At expiration/delivery
  - Get delivery of 1 oz of gold and pay $390
  - Close out the short position in gold by returning gold
  - Cash with interest $390*$e^{0.05*1} = 410$
  - Make 410-390 = 20

The forward price of assets that provides no income and has no storage costs

- If the spot price of gold is S_0, the forward price is for a contract deliverable in T years is F_0, then

$$F_0 = S_0 e^{rT}$$

where r is the 1-year (domestic currency) risk-free rate of interest

- In our examples, S_0=390, T=1, and r=0.05 so

$$F_0 = 390 * e^{0.05*1} = 410$$

- For annual compounding interest rates

$$F = S\,(1+r\,)^T$$

Concept check

- What is the forward price of a 3-month forward contract on MSFT stock?
- Assume MSFT pays no dividend during this 3 months
- MSFT is currently trading at $50
- Annual interest rates are 6% (continuous compounding)

  G) 50.76

  Y) 53.09

  R) 51.01

Concept check

- Is there an arbitrage opportunity if the forward price on the 3-month forward contract on MSFT is $52.00?
- If there is, what should be your arbitrage strategy and how much money can you make per 1 contract (assume 1 contract is for 1 stock?

  G) Short the forward contract; Borrow $50 for 3 months and buy MSFT now at $50. Make $1.24

  Y) Long the forward contract; Sell short MSFT and get $50. Money earns interest over the 3 months. Make 1.24

  R) Long the forward contract; Borrow $50 for 3 months and buy MSFT now at $50. Make $1.09

Concept check

- What if it was not possible to short sell the underlying asset and $F_0 < S_0e^{rT}$?

G) We only have an upper bound for the forward price

Y) The lower bound for the forward price is determined by the CAPM

R) We can still use $F_0 = S_0e^{rT}$

Arbitrage

- What if it was not possible to short sell the underlying asset and $F_0 < S_0e^{rT}$?
- If large number of investors hold assets for investment
- These investors who already hold the underlying asset for investments find it profitable to sell their assets and invest the money in the risk free rate obtaining S_0e^{rT} at T. They also enter into a forward contract to buy the asset back at T for which they will pay F_0 . These investors make S_0e^{rT}-F_0.
- As more investors enter into the trading strategy above, forward price increases and asset price decreases such that $F_0 = S_0e^{rT}$

When an Investment Asset Provides a Known Dollar Income

- Consider the case where the underlying asset is a coupon bond with 9 month maturity and provides a coupon of $40 in 4 months
- The current price of this bond is $900
- Assume the four-month and nine-month risk-free rate is 3% and 4% per annum.

When an Investment Asset Provides a Known Dollar Income

- If the forward price is $910
- Today:
  - Borrow $900: 39.60 for four months and 860.4 for 9 months
  - Buy one bond
  - Enter into a forward contract to sell the bond for 910 in nine months
- In four months:
  - Receive $40 coupon and repay first loan with interest
- In nine months
  - Sell bond for 910
  - Use $860.4e^{9/12*0.04}$=886.6 to repay the second loan with interest
- Total profit = 910-886.6=23.40

When an Investment Asset Provides a Known Dollar Income

- If the forward price is \$870
- Today:
  - Sell short one bond and get \$900
  - Invest 39.60 for four months and 860.4 for 9 months
  - Enter into a forward contract to buy the bond for 870 in nine months
- In four months:
  - Receive \$40 from risk-free investment and pay coupon on the short bond
- In nine months
  - Use $860.4e^{9/12*0.04}$=886.6 to buy bond for 870
- Total profit = 886.6-870 =16.60

When an Investment Asset Provides a Known Dollar Income

- Forward price of this bond is

$$(900-39.6)\ e^{3/4*0.04}$$

- In general the forward price of an asset that pays know income is

$$F_0 = (S_0 - I\,)e^{rT}$$

where I is the present value of the income

When an Investment Asset Provides a Known Yield

- Consider the a six months forward contract on an asset that is expected to provide income equal to 4% (annual rate) of the asset price during these six months
- The income = 4% annual rate = 3.96% continuous rate
- The risk-free rate is 10% per annum
- The current asset price is $25

When an Investment Asset Provides a Known Yield

- If the forward price is $30
- Today:
  - Borrow $25 at the rate of 10%
  - Buy one asset at 25
  - Enter into a forward contract to sell the asset at 30 in 6 months
- Over the six months:
  - Use the 3.96% income to continuously pay down your loan. The total loan at the end of 6 months=$25e^{6/12*0.10}25e^{-6/12*0.0396}$=25.77
- In six months
  - Deliver the asset and obtain $30. Repay the loan 25.77 =
- Total profit = 30-25.77=4.23

When an Investment Asset Provides a Known Yield

- If the forward price is $20
- Today:
  - Sell short one asset at 25 and invest this money at 10% annum for 6 months
  - Enter into a forward contract to buy the asset for 20 in six months
- Over the six months:
  - Use the initial 25 investment to pay income to the short position at 3.96%. The $25 grows to=$25e^{6/12*0.10}25e^{-6/12*0.0396}$=25.77
- In six months
  - Use 25.77 to buy asset at 20
- Total profit = 25.77-20 =5.77

When an Investment Asset Provides a Known Yield

- Forward price of this asset is

$$25^{6/12*0.10}25^{-6/12*0.0396}=25.77$$

- In general the forward price of an asset that pays know income is

$$F_0 = S_0\, e^{(r-q)T}$$

where q is the yield from the asset (expressed with continuous compounding per annum)

Concept check

- The CBOT list electricity futures for trading. Trading unit: Megawatt per hour, Price: dollar/megawatt hour; Delivery area: commmonwealth Edison's control area.
- Assume for now that futures price are the same (close enough) to forward price
- The electricity price today is $30 kw/hr
- Can you use the forward price equations we just learned to price a 3-month electricity futures? Why?

G) Yes

R) No

Forward pricing

- Assumptions in forward pricing of investment assets
  - Investment assets
  - Assets are traded in the market and we observe current prices
  - We can short sell the asset or there are a large pool of investors who hold assets as investment assets
  - The assets can be stored

Valuing a Forward Contract

- The value of the forward contract at the time it is first entered into is zero
- After a period of time, the price of underlying asset changes, time changes, and the value of the forward contract changes as well
- The value of the contract may increase or decrease
- Notation:
  - K: Delivery price in the original contract
  - f: Value of forward contract

Valuing a Forward Contract

- Consider the following case: a long forward contact on a non-dividend-paying stock was entered into some time ago. It currently has six months to maturity. The risk-free rate is 10% per annum, the stock price is currently trading at $25, and the delivery price of the original contract is $24
- If you enter into a six-month long forward contract today, the forward price would be

  $F_0 = S_0 e^{rT}$ = \$ 25$e^{0.1*0.5}$=26.28
- Therefore compared to the new contract, the original contract has a positive cash flow at maturity of

  26.28-24 = 2.17

Valuing a Forward Contract

- Any investor should be willing to pay $2.17e^{-0.1*0.5}$ for the original contract
- In general the value of a forward contract (to the long position) can be positive or negative
- The value of a long forward contract, f, is

$$f = (F_0 - K)e^{-rT}$$

Forward vs Futures Prices

- Forward and futures prices are the same when interest rates in the future are known
- Forward and futures prices are usually assumed to be the same
- When interest rates are uncertain they are, in theory, slightly different

Forward vs Futures Prices

- A strong positive correlation between interest rates and the asset price implies the futures price is slightly higher than the forward price
- A strong negative correlation implies the reverse
- Futures prices can deviate significantly from forward prices when the maturity is long, such as some currency futures with 10 years maturity

Stock Index

- Can be viewed as an investment asset paying a dividend yield
- The futures price and spot price relationship is therefore

$$F_0 = S_0 e^{(r-q)T}$$

where q is the dividend yield on the portfolio represented by the index

Stock Index

- For the formula to be true it is important that the index represent an investment asset
- In other words, changes in the index must correspond to changes in the value of a tradable portfolio

Stock Index

- The Nikkei index futures contract traded on the CME pays $US 5S, where S is the level of the Nikkei index
- We cannot invest in the Nikkei index to obtain a payoff of 5S dollars
- We can only invest 5S in Yen or 5QS in dollar where Q is the dollar value of one Yen
- We cannot use the equation derived earlier to price this Nikkei index futures
- Need more complex pricing equation

Index Arbitrage

- When $F_0 > S_0 e^{(r-q)T}$ an arbitrageur buys the stocks underlying the index and sells futures
- When $F_0 < S_0 e^{(r-q)T}$ an arbitrageur buys futures and shorts or sells the stocks underlying the index

Index Arbitrage

- Index arbitrage involves simultaneous trades in futures and many different stocks
- Very often a computer is used to generate the trades
- Occasionally (e.g., on Black Monday) simultaneous trades are not possible and the theoretical no-arbitrage relationship between F_0 and S_0 does not hold
- Example: index arbitrage on Mini DJ industrial average which pays cash delivery of $5 times the index (traded on CME).

When an Investment Asset Provides a Known Yield

- Assume the index is at 11,000, interest rate is 5% and stock in the index pay a dividend yield of 4%
- The six-month futures price should be $11000e^{6/12*0.01} = 11055.14$
- If the futures price is at 11500, then
- Today
  - Borrow $5*$11000 at the rate of 5%
  - Buy the 30 stocks in the index in the same proportion as the index
  - Enter into a short futures contract

When an Investment Asset Provides a Known Yield

- Over the six months:
  - Use the 4% income from dividend on the stocks purchased to continuously pay down your loan. The total loan at the end of 6 months= $\$5*11000e^{6/12*0.05}e^{-6/12*0.04} = \$5*11055.14$
- In six months
  - Sell the stocks at $S and deliver $5*($S-11500) to the long position
  - We are left with $S- $5*($S-11500)- $5*11055.14 = $5*444.86

Futures and Forwards on Currencies

- A foreign currency is analogous to a security providing a dividend yield
- The continuous dividend yield is the foreign risk-free interest rate
- It follows that if r_f is the foreign risk-free interest rate

$$F_0 = S_0 e^{(r - r_f)T}$$

Why the relation must be true

1000 units of foreign currency at time zero

$1000 e^{r_f T}$ units of foreign currency at time T

$1000 S_0$ dollars at time zero

$1000 F_0 e^{r_f T}$ dollars at time T

$1000 S_0 e^{rT}$ dollars at time T

$$F_0 = S_0 e^{(r - r_f)T}$$

Futures and Forwards on Currencies

- Suppose that the two-year interest rates in the US and UK are 6% and 5%
- The current exchange rate is 1.6 dollars per pound
- A 2-year forward exchange rate should be

$$F_0 = 1.6e^{(0.06-0.05)2} = 1.6323$$

- What arbitrage strategies exist if the futures prices are 1.55 and 1.7?

Futures and Forwards on Currencies

- If futures price = 1.7
- Today:
  - Borrow $US1000 at 6%
  - Convert $US1000 into pounds = 1000/1.6 pounds and invest in the risk-free rate in UK for 2 years
  - Enter into a short futures contract to sell pounds in 2 years
- After 2 years
  - UK investment grows to 1000/1.6 * $e^{2*0.05}$
  - Sell these pounds using the short futures position
  - Get 1000/1.6 * $e^{2*0.05}$ *1.7 = $US 1174.24
  - Pay back the loan at 6% = 1000 $e^{2*0.06}$ =1127.5
- Total profit =$US 46.75

Futures and Forwards on Currencies

- If futures price = 1.55
- Today:
  - Borrow 1000 pound at 5%
  - Convert into pounds dollars = $US 1600 and invest in the risk-free rate in US for 2 years
  - Enter into a long futures contract to buy pounds in 2 years
- After 2 years
  - US investment grows to 1600 * $e^{2*0.06}$
  - Use money to buy pounds using long futures position
  - Get 1600/1.55 * $e^{2*0.06}$ = 1163.86 pounds
  - Pay back the loan at 5% = 1000 $e^{2*0.05}$ =1105.17
- Total profit =58.69 pounds

Concept check

- According to Table 5 which lists a number of currency futures for Feb 5, 2004 (5th column is the settle price)
- Futures on the Japanese Yen increases with maturity of the contract
- What does this mean about interest rates in the US and Japan during the life of this futures ?

G) The risk-free rate in Japan is probably lower than in the US

Y) The risk-free rate in Japan is Probably higher than in the US

R) The risk-free rate in the two countries are about the same

Concept check

- Futures on the British Pound decreases with maturity of the contract
- What does this mean about interest rates in the US and UK during the life of this futures ?

G) The risk-free rate in UK is probably lower than in the US

Y) The risk-free rate in UK is Probably higher than in the US

R) The risk-free rate in the two countries are about the same

Futures on Consumption Assets

$$F_0 \leq S_0 \, e^{(r+u)T}$$

where u is the storage cost per unit time as a percent of the asset value.

Alternatively,

$$F_0 \leq (S_0 + U)e^{rT}$$

where U is the present value of the storage costs.

Futures on Consumption Assets

- Why $F_0 \leq S_0 \, e^{(r+u)T}$?
- To take an arbitrage position in this case, the owner of the consumption asset has to:
  - Sell the commodity, save the storage cost, and invest in the risk-free rate
  - Take a long position in a forward contract
- Investors who have the consumption asset are reluctant to sell the asset because forwards contracts cannot be consumed
- Since investors are reluctant to execute the arbitrage, futures price can be $F_0 \leq S_0 \, e^{(r+u)T}$

Futures on Consumption Assets

- The owner of the consumption asset holds the asset for production and needs to keep some inventory or for immediate consumption
- Example
  - Crude oil
  - Pork belly
  - Corn

Futures on Consumption Assets

- The rate of return that is high enough for the owners of consumption assets to sell their assets and buy the forward contract is termed the convenience yield (y).
- When we include the convenience yield, we have $F_0 = S_0\, e^{(r+u-y)T}$

The Cost of Carry

- The cost of carry, c, is the storage cost plus the interest costs less the income earned
- For an investment asset $F_0 = S_0 e^{cT}$
- For a consumption asset $F_0 \leq S_0 e^{cT}$
- The convenience yield on the consumption asset, y, is defined so that $F_0 = S_0\, e^{(c-y)T}$

Futures Prices & Expected Future Spot Prices

- Suppose k is the expected return required by investors on an asset
- We can invest $F_0 e^{-rT}$ now to get S_T back at maturity of the futures contract
- This shows that

$$F_0 = E(S_T)e^{(r-k)T}$$

Futures Prices & Expected Future Spot Prices

- If the asset has
  - no systematic risk, then $k = r$ and F_0 is an unbiased estimate of S_T
  - positive systematic risk, then $k > r$ and $F_0 < E(S_T)$
  - negative systematic risk, then $k < r$ and $F_0 > E(S_T)$

Futures: an example of futures on non-trading variable

- **Futures on the weather:** http://www.cmegroup.com/trading/weather/
- Heating engineers who wanted a way to relate each day's temperatures to the demand for fuel to heat and cool buildings developed the concept of degree days.
- Cooling degree days are based on the day's average temperature minus 65. They relate the day's temperature to the energy demands of air conditioning. For example, if the day's high is 90 degrees and the day's low is 70 degrees, the day's average is 80 degrees. 80 minus 65 is 15 cooling degree days.
- To calculate the heating degree days for a particular day, find the day's average temperature by adding the day's high and low temperatures and dividing by two. If the number is above 65, there are no heating degree days that day. If the number is less than 65, subtract it from 65 to find the number of heating degree days.

Futures: an example of futures on non-trading variable

- **CME Cooling Degree Day Futures Trade**
- **Unit** $20 times the Cooling Degree Day Index
- **Contract Listing** Apr, May, Jun, Jul, Aug, Sep and Oct.
- **Product Code** Clearing + Ticker:
  K1=Atlanta
  K2=Chicago
  K3=Cincinnati
  K4=New York

- Can we use our forward pricing equation to price weather futures?

G) Yes

R) No

Forward pricing (caveats)

- Assumptions in forward pricing of investment assets or consumption asset
  - Investment assets or consumption asset (using the concept of convenience yield)
  - Assets are traded in the market and we observe current prices
  - We can short sell the asset or there are a large pool of investors who hold assets as investment assets
  - The assets can be stored

Summary

- Investments vs consumption assets
- Short selling securities
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