5.6 Using the Poisson Distribution to Approximate the Binomial Distribution

Use the Poisson distribution to approximate the binomial distribution when n is large and π is very small. The approximation gets better as n gets larger and π gets smaller. The following mathematical expression for the Poisson model approximates the true (binomial) result:

$$P(X = x|\lambda) \cong \frac{e^{-n\pi}(n\pi)^X}{X!} \qquad \textbf{(5.18)}$$

where

$P(X = x|\lambda)$ = probability of $X = x$ events of interest given the parameters n and π

n = sample size

π = probability of an event of interest

e = mathematical constant approximated by 2.71828

x = number of events of interest in the sample ($x = 0, 1, 2, \ldots n$)

The Poisson random variable theoretically ranges from 0 to ∞. However, when the Poisson distribution is used as an approximation to the binomial distribution, the Poisson random variable—the number of events of interest out of n observations—cannot be greater than the sample size n. With large n and small π, Equation (5.18) implies that the probability of observing a large number of events of interest becomes small and approaches zero rapidly.

As Section 5.3 mentions, in the Poisson distribution, the mean μ and the variance σ^2 are each equal to λ. Thus, when using the Poisson distribution to approximate the binomial distribution, use Equation (5.19) to approximate the mean.

$$\mu = E(X) = \lambda = n\pi \qquad \textbf{(5.19)}$$

Equation (5.20) approximates the standard deviation.

$$\sigma = \sqrt{\lambda} = \sqrt{n\pi} \qquad \textbf{(5.20)}$$

The standard deviation given by Equation (5.20) closely approximates the standard deviation for the binomial model [Equation (5.7)] when π is close to zero so that $(1 - \pi)$ is close to one.

Suppose 8% of the tires manufactured at a particular plant are defective. To illustrate the use of the Poisson approximation for the binomial, calculate the probability of exactly one defective tire from a sample of 20 using Equation (5.18) as follows:

$$P(X = 1) = \frac{e^{-(20)(0.08)}[(20)(0.08)]^1}{1!} = \frac{e^{-1.6}(1.6)^1}{1!} = 0.3230$$

Alternatively, one can use tables of the Poisson distribution (see Table E.7) to get the Poisson approximation or software, including Excel, JMP, and Minitab. When using the Poisson tables, only the parameter λ and the desired number of events of interest x are needed. Because in the above example $\lambda = 1.6$ and $X = 1$,

$$P(X = 1) = 0.3230$$

Table 5.6 shows this probability.

TABLE 5.6
Finding a Poisson Probability

	λ									
X	**1.1**	**1.2**	**1.3**	**1.4**	**1.5**	$\boxed{\textbf{1.6}}$	**1.7**	**1.8**	**1.9**	**2.0**
0	.3329	.3012	.2725	.2466	.2231	.2019	.1827	.1653	.1496	.1353
$\boxed{1}$	.3662	.3614	.3543	.3452	.3347	$\boxed{.3230}$	.3106	.2975	.2842	.2707
2	.2014	.2169	.2303	.2417	.2510	.2584	.2640	.2678	.2700	.2707
3	.0738	.0867	.0998	.1128	.1255	.1378	.1496	.1607	.1710	.1804
4	.0203	.0260	.0324	.0395	.0471	.0551	.0636	.0723	.0812	.0902

Source: Extracted from Table E.7.

Had the true distribution, the binomial, been used instead of the approximation,

$$P(X = 1) = \frac{20!}{1!19!}(0.08)^1(0.92)^{19} = 0.3282$$

However, that calculation is tedious. With software tools available today, one can compute directly the binomial probability for $n = 20$, $\pi = 0.08$, and $X = 1$, thus making the Poisson approximation less important to know.

PROBLEMS FOR SECTION 5.6

LEARNING THE BASICS

5.63 When should you use the Poisson distribution to approximate the binomial distribution?

5.64 When given the parameters of a binomial distribution, n and π, what are the mean and variance of the Poisson distribution used for approximating the binomial?

5.65 Given a binomial distribution with $n = 100$ and $\pi = 0.01$, use the Poisson distribution to approximate the following:
a. $P(X = 0)$
b. $P(X = 1)$
c. $P(X = 2)$
d. $P(X \le 2)$
e. $P(X > 2)$

5.66 Given a binomial distribution with $n = 50$ and $\pi = 0.004$, use the Poisson distribution to approximate the following:
a. $P(X = 0)$
b. $P(X = 1)$
c. $P(X = 2)$
d. $P(X \le 2)$
e. $P(X > 2)$

APPLYING THE CONCEPTS

5.67 Based upon past experience, 1% of the telephone bills mailed to households are incorrect. If a sample of 20 bills is selected, using the binomial distribution and the Poisson approximation to the binomial distribution, find the probability that at least one bill is incorrect. Briefly compare and explain your results.

5.68 A computer manufacturing company samples incoming computer chips. After receiving a large shipment of computer chips, the company randomly selects 800 chips. If three or fewer nonconforming chips are found, the entire lot is accepted without inspecting the remaining chips in the lot. If four or more chips are nonconforming, every chip in the entire lot is carefully inspected at the supplier's expense. Assume that the true proportion of nonconforming computer chips being supplied is 0.001. Approximate the probability the lot will be accepted.

5.69 Last month your company sold 10,000 new watches. Past experience indicates that the probability that a new watch will need repair during its warranty period is 0.002. Approximate the probability that:
a. zero watches will need warranty work.
b. no more than 5 watches will need warranty work.
c. no more than 10 watches will need warranty work.
d. no more than 20 watches will need warranty work.