Quiz 1: Date: April 19, 2022 from 11.00-12.30

Question 1 (10 Points) Score......................

Consider the one-period model of consumption and portfolio choice. Let an individual in this economy has the utility function as follow:

$$U(C) = ln(C)$$

Also, let $\frac{C_1}{C_0}$ is distributed as log-normal with mean equals μ_c and its variance is σ_c.
Please read and answer the following questions carefully and completely.

Score......................

Question 1.1 (10 marks) Calculate the risk free rate R_f in terms of the individual's consumption, C_0 and C_1. Then,explain the relationship between the level of consumption and the risk free rate in this economy.

$u(c_0) + \delta E[u(\tilde{c}_1)]$

$\max_{c_0,(w_i)} \; u(c_0) + \delta E[u(c_1)]$

$\mathcal{L} = u(\ln(c_0)) + \delta E[u(c_1)] + \lambda'[1 - \sum_{i=1}^{n} w_i]$

$\frac{\partial \mathcal{L}}{\partial c_0} = u'(c_0) + \delta E[u'(c_1) \sum_{i=1}^{n} w_i R_i] = 0$

$\frac{\partial \mathcal{L}}{\partial w_i} = \delta E[u'(\tilde{c}_1) R_i] - \lambda' = 0$

$\therefore E[u'(c_1)R_i] = E[u'(c_1)R_j]$ $\rightarrow c_1 \uparrow$ $u'(c_1) \downarrow$

so that this means diversification

$E[u'(c_1)R_i] = E[u'(c_1)]E(R_i) + cov(u'(c_1), R_i)$

$u'(c_0) = \delta E[u'(c_1)\sum_{i=1}^{n} w_i R_i]$

$= \sum_{i=1}^{n} w_i \delta E[u'(c_1)R_i]$

$= \sum_{i=1}^{n} w_i \lambda = \lambda$

then, $\delta E[u'(c_1)R_i] = u'(c_0)$

$P_i u'(c_0) = \delta E[u'(c_1)x_i]$

$u'(c_0) = R_f \delta E[u'(c_1)]$

CRRA $u(c) = \ln(c)$

$u'(c_1) = \frac{1}{c_1}$ $u'(c_0) = \frac{1}{c_0}$

$\frac{1}{R_f} = \delta E\left[\left(\frac{c_0}{c_1}\right)\right]$

$$\boxed{R_f = \frac{1}{\delta}\left(\frac{c_1}{c_0}\right)}$$

implying that when the interest rate is high, the expected growth in level of consumption will be high too.

Score....................

Question 1.2 (10 marks) Calculate the elasticity of intertemporal substitution in this setting. If in the next year, the interest rate is falling, Will the individual's consumption level increase or decrease? Why? To support your answer, use the concepts of income effect and substitution effect.

$$\frac{\partial R_f}{\partial \frac{C_1}{C_0}} = \frac{1-\gamma}{\delta}\left(\frac{C_1}{C_0}\right)^{-\gamma}$$

$$= (1-\gamma)\frac{R_f}{\frac{C_1}{C_0}}$$

$$\varepsilon = \frac{\% \Delta \frac{C_1}{C_0}}{\% \Delta R_f} \quad ; \quad \varepsilon = \frac{R_f}{\frac{C_1}{C_0}} \frac{\partial \frac{C_1}{C_0}}{\partial R_f} = \frac{\partial \ln(C_1/C_0)}{\partial \ln(R_f)}$$

$$= \frac{1}{1-\gamma}$$

For the concept of substitution effect, the decrease in interest rate will down the return from transforming current consumption into future consumption, providing incentive to save less.

And for income effect from the lower return will make the individual worst off, it would decrease the individual's consumption level.

To sum up, if the interest rate is falling, it will decrease the second-period consumption. So the initial level of individual consumption will decrease.

Score....................

Question 1.3 (10 marks) Solve for the pricing kernel P_i of any risky asset i in this economy. Then explain the meaning of this pricing kernel.

$$P_i = E\left[\frac{\delta u'(c_1)}{u'(c_0)} x_i\right]$$

$$= E[m_{01} x_i]$$

$$\frac{P_i^N}{CPI_0} = E\left[\frac{\delta u'(c_1)}{u'(c_0)} \frac{x_i^N}{CPI_1}\right] \quad ; \quad I_{ts} = CPI_s / CPI_t$$

$$P_i^N = E\left[\frac{1}{I_{01}} \frac{\delta u'(c_1)}{u'(c_0)} x_i^N\right]$$

$$= E[M_{01} x_i^N]$$

The price kernel or stochastic discount factor (SDF) is used for summarization the investor preferences for payoffs over different countries despite the arbitrage. For the pricing kernel, all asset prices can be expressed as expected value of product and asset payoff.

Score....................

Question 1.4 (10 marks) Calculate Hansen-Jaganathan Bound and explain the meaning.

bound on risk premia

$$m_{01} \equiv \delta u'(c_1)/u'(c_0)$$

$$E(R_i) = R_f - \rho_{m_{01}, R_i} \frac{\sigma_{m_{01}} \sigma_{R_i}}{E[m_{01}]}$$

$$\frac{E(R_i) - R_f}{\sigma R_i} = -\rho_{m_{01}, R_i} \frac{\sigma_{m_{01}}}{E[m_{01}]}$$

Hensen-Jagannathan bounds

$$-1 \leq \rho_{m_{01}, R_i} \leq 1$$

$$\left| \frac{E[R_i] - R_f}{\sigma R_i} \right| \leq \frac{\sigma_{m_{01}}}{E[m_{01}]} = \sigma_{m_{01}} R_f \quad \text{—— } (*)$$

The H-J bound is a type of mean-variance frontier.

if the portfolio's return is perfectly negatively correlated with m_{01} then $(*)$ holds with equity. The CAPM implies slope (sharp ratio) of capital market line, $S_e \equiv \dfrac{E(R_m) - R_f}{\sigma R_m} = \sigma_{m_{01}} R_f$