

CHAPTER 7

Derivatives of More-Than-One Independent Variable Function

Topics:

- First-order partial derivatives
- Second-order partial derivatives
- Differential
- Total differential
- Total derivatives
- Implicit function and its derivative
- Examples in economics
 - Partial market equilibrium
 - Multipliers in macroeconomic models
 - Utility function
 - Production function
 - Etc.



In comparative-static analysis, we are likely to encounter the situation in which several parameters/independent variables appear in a model. Many functions in economics, such as Production function, Cost function, Profit function, Utility function, Demand function, deal with many independent variables. Therefore, we must learn how to find the derivative of a function of more than one variable (Multivariable calculus).



“Transition of idea”

Single variable calculus/ Optimization with one choice variable	Multivariable calculus/ Optimization with more-than-one choice variable
Function: ch 5, 6 $y = f(x)$	Function: ch 7, 8, 9 $y = f(x_1, x_2, \dots, x_n)$
Derivatives: $\frac{dy}{dx}$ “derivatives” “slope of function f” differential	Partial Derivatives: $\frac{\partial y}{\partial x_1}, \frac{\partial y}{\partial x_2}, \dots, \frac{\partial y}{\partial x_n}$: Hold other variables as constants Total Derivatives: $\frac{dy}{dx_1}, \frac{dy}{dx_2}, \dots, \frac{dy}{dx_n}$: take postulated relationships between arguments into account total differential
Example $\pi(Q), Q(L), U(x)$	Example $\pi(Q_1, Q_2), Q(K, L), U(x_1, x_2)$
Optimization problem $\max_Q \pi(Q)$ FOC : $\pi'(Q) = 0$ SOC : $\pi''(Q) < 0$	Optimization problem $\max_{Q_1, Q_2} \pi(Q_1, Q_2)$ FOC : Jacobian matrix SOC : Hessian matrix



Examples of multivariable function in economics

The constant elasticity demand function

$$q_1 = f(p_1, p_2, y) = k p_1^\alpha p_2^\beta y^\gamma$$

E_{p_1}, E_{p_2}, E_y are constants & equal to α, β, γ respectively

The firm's production function

Linear:

$$q = a_1 x_1 + a_2 x_2$$

Cobb-Douglas:

$$q = k x_1^\alpha x_2^\beta$$

Leontief:

$$q = \min \left\{ \frac{x_1}{c_1}, \frac{x_2}{c_2} \right\}$$

fixed proportion production function

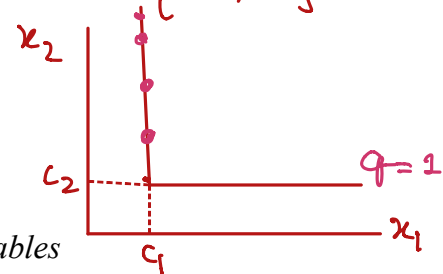
$$\rightarrow q = \min \left\{ \frac{c_1}{c_1}, \frac{c_2}{c_2} \right\} = \min \{ 1, 1 \} = 1$$

$$q = \min \left\{ \frac{2c_1}{c_1}, \frac{2c_2}{c_2} \right\} = \min \{ 2, 2 \} = 2$$

$$q = \min \left\{ \frac{c_1}{c_1}, \frac{2c_2}{c_2} \right\} = \min \{ 1, 2 \} = 1$$

Constant elasticity of substitution:

$$q = C \left[\pi x_1^{\frac{\sigma-1}{\sigma}} + (1-\pi) x_2^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}}$$

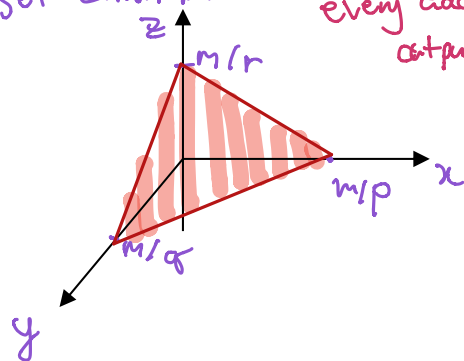
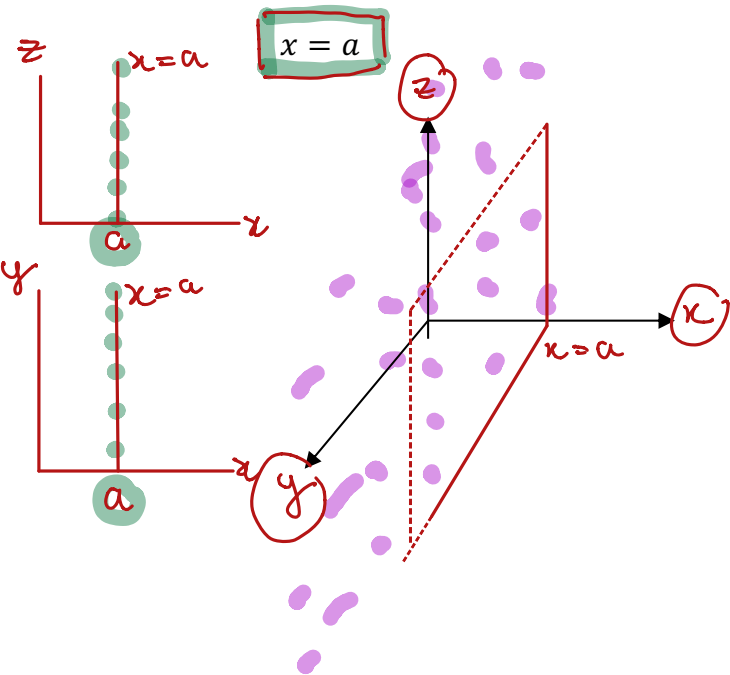


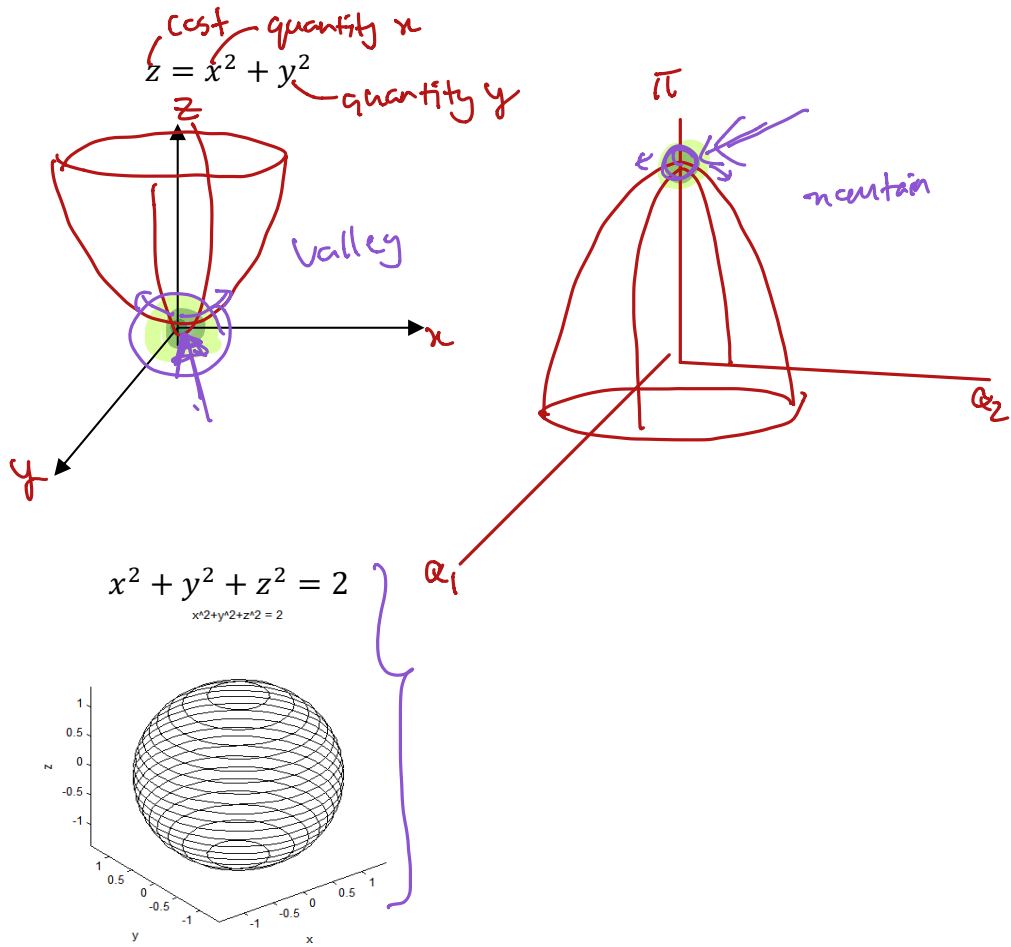
Geometric Representations of Functions of Several Variables

$$px + qy + rz = m$$

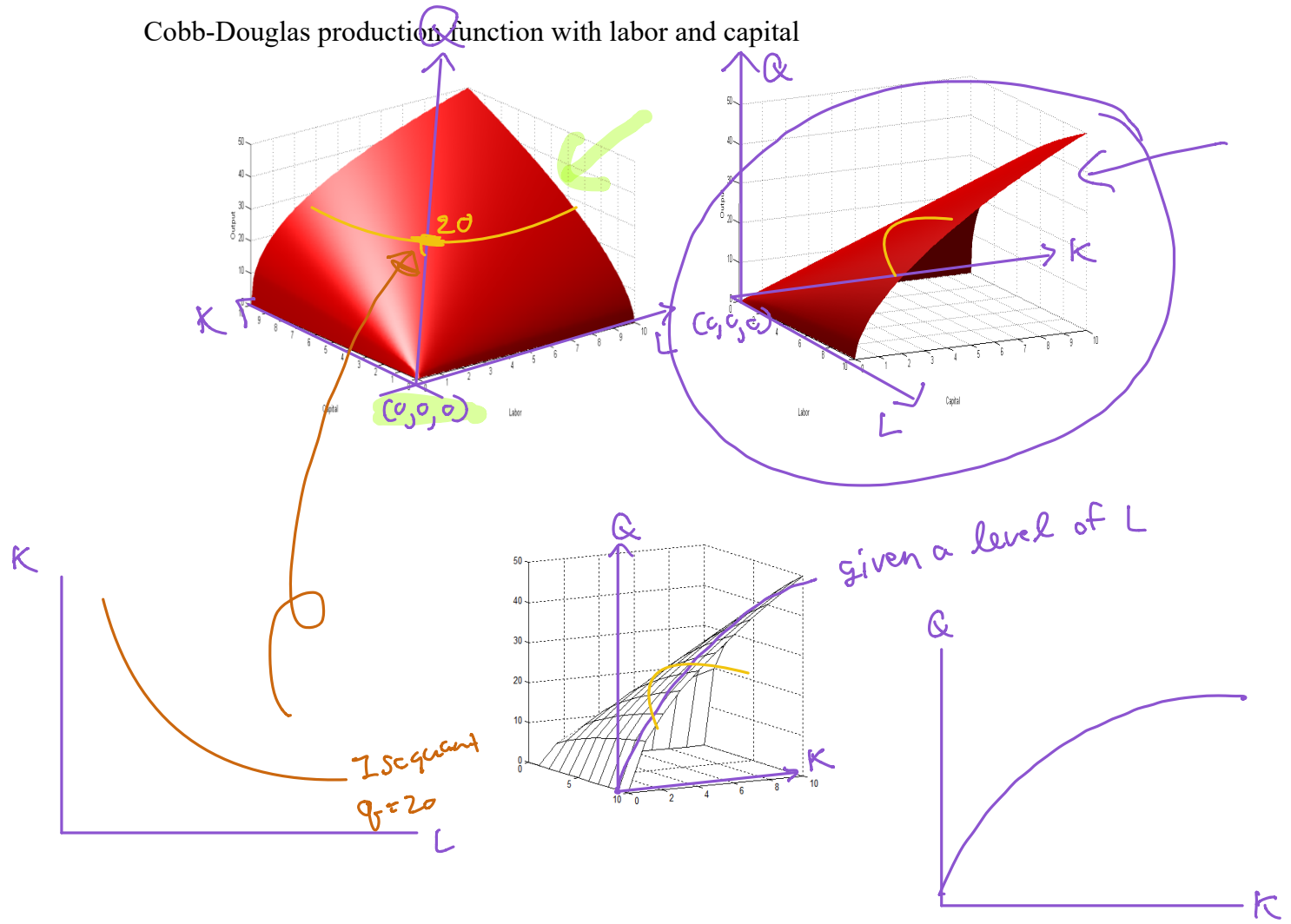
e.g. budget constraint

c_1 units of x_1 , c_2 units of x_2 are required to produce every additional unit of output.





Cobb-Douglas production function with labor and capital

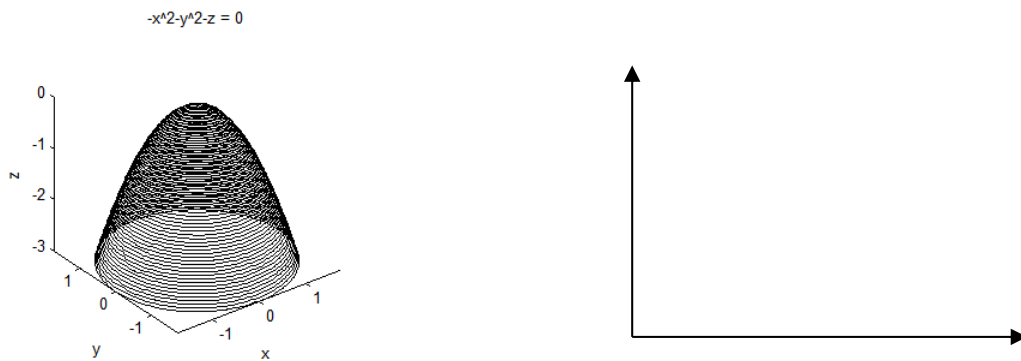




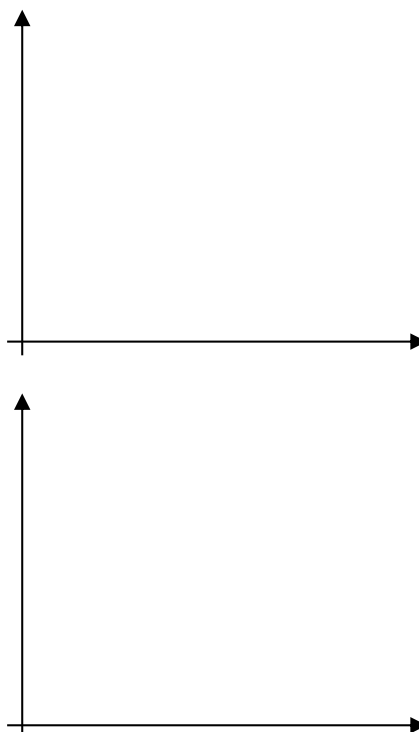
Level Curves for $z = f(x, y)$

From above three-dimensional graphs, we can find the relationship between x and y at each level of function f . That is, we can find x and y such that $f(x, y) = c$. Looking at each level of f and plot the relevant x and y , then we will get “the level curve” at that level of f .

Draw the level curve of $z = f(x, y) = -x^2 - y^2$



A level curve of production function is an isoquant curve.





Partial Derivative

Let

$$y = f(x_1, x_2, \dots, x_n),$$

where x_1, \dots, x_n are independent variable and all independent of one another, so that each can vary by itself without affecting the others. If the variable x_1 changes by Δx_1 , while x_2, \dots, x_n all remain fixed, there will be a corresponding change in y , Δy . The difference quotient is:

$$\frac{\Delta y}{\Delta x_1} = \frac{f(x_1 + \Delta x_1, x_2, \dots, x_n) - f(x_1, x_2, \dots, x_n)}{\Delta x_1}$$

The partial derivative of y with respect to x_1 is:

$$\lim_{\Delta x_1 \rightarrow 0} \frac{\Delta y}{\Delta x_1} = \lim_{\Delta x_1 \rightarrow 0} \frac{f(x_1 + \Delta x_1, x_2, \dots, x_n) - f(x_1, x_2, \dots, x_n)}{\Delta x_1} \equiv \frac{\partial y}{\partial x_1} \equiv f_1$$

The key :

We must hold $n - 1$ independent variables constant while allowing one variable to vary.

For $y = f(x_1, x_2)$, we find:

$\frac{\partial y}{\partial x_1}$ by assuming x_2 to be a constant.

$\frac{\partial y}{\partial x_2}$ by assuming x_1 to be a constant.

Example: $y = f(x_1, x_2) = 3x_1^2 + x_1x_2 + 4x_2^2$, what are $\frac{\partial y}{\partial x_1}$, $\frac{\partial y}{\partial x_2}$?

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Example: $y = f(u, v) = (u + 4)(3u + 2v)$, what are $\frac{\partial y}{\partial u}, \frac{\partial y}{\partial v}$?

$\frac{\partial y}{\partial u}$ by viewing $f(u, v) = g(u)h(u)$

$\frac{\partial y}{\partial v}$ by viewing $f(u, v) = k_1(k_2 + 2v)$

Example: $y = f(u, v) = (3u - 2v)/(u^2 + 3v)$, what are $\frac{\partial y}{\partial u}, \frac{\partial y}{\partial v}$?

H.W.

- Compute all the partial derivatives of the following function

a) $4x^2y - 3xy^3 + 6x$

b) xy

c) xy^2

d) e^{2x+3y}

e) $\frac{x+y}{x-y}$

f) $3x^2y - 7x\sqrt{y}$

- Compute the partial derivative of the Cobb-Douglas function

$q = k_1x_1^{a_1}x_2^{a_2}$ and of the Constant Elasticity of Substitution (CES)

production function $q = C \left[\pi x_1^{\frac{\sigma-1}{\sigma}} + (1 - \pi)x_2^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}}$

Economic Interpretation

$F(K, L)$

$\frac{\partial F(K, L)}{\partial K}$ is Marginal product of capital, MP_K , given labor as a constant

$\frac{\partial F(K, L)}{\partial L}$ is Marginal product of labor, MP_L , " — " capital " — "

Example: Find MP_K and MP_L of Cobb-Douglas production function $Q = 4K^{\frac{3}{4}}L^{\frac{1}{4}}$, at the current level of factors at $L = 625, K = 10,000$

$$MP_L = \frac{\partial Q(K, L)}{\partial L} = \frac{\partial (4K^{\frac{3}{4}}L^{\frac{1}{4}})}{\partial L} = K^{\frac{3}{4}}L^{-\frac{3}{4}} = \left(\frac{K}{L}\right)^{\frac{3}{4}}$$

as if constants

$$MP_K = \frac{\partial Q(K, L)}{\partial K} = \frac{\partial (4K^{\frac{3}{4}}L^{\frac{1}{4}})}{\partial K} = 3K^{-\frac{1}{4}}L^{\frac{1}{4}} = 3\left(\frac{L}{K}\right)^{\frac{1}{4}}$$

as if constant

$$MP_L \Big|_{L=625, K=10000} = \left(\frac{10,000}{625}\right)^{\frac{3}{4}} = 8$$

$$MP_K \Big|_{L=625, K=10000} = \frac{3}{2}$$


Example: Find MU_{x_1}, MU_{x_2} of the utility function $U(x_1, x_2) = x_1 \log x_2$

$$MU_{x_1} = \frac{\partial U(x_1, x_2)}{\partial x_1} = \log x_2$$

$$MU_{x_2} = \frac{\partial U(x_1, x_2)}{\partial x_2} = \frac{\partial x_1 \log x_2}{\partial x_2} = \frac{x_1}{x_2 \ln 10}$$

as if a constant

$$y = \log_a x \rightarrow \frac{dy}{dx} = \frac{1}{x \ln a}$$


Higher-Order Partial Derivative

Since $\frac{\partial f}{\partial x_1}$ is a function of x_1, \dots, x_n , we can also find partial derivative of $\frac{\partial f}{\partial x_1}$

The second order partial derivative of f is:

$$\textcircled{1} \rightarrow \frac{\partial}{\partial x_j} \left(\frac{\partial f}{\partial x_i} \right) = \frac{\partial^2 f}{\partial x_j \partial x_i}$$

$$\textcircled{2} \rightarrow \frac{\partial^2 f}{\partial x_i \partial x_i} = \frac{\partial^2 f}{\partial x_i^2}$$

$$y = f(x_1, x_2, \dots, x_n)$$

$$x_j \neq x_i$$

$\frac{\partial^2 f}{\partial x_j \partial x_i}, i \neq j$, is called cross partial derivatives or mixed partial derivatives.

Second-order Derivative and Hessians

Example: Find all second derivatives of $Q = 4K^{\frac{3}{4}}L^{\frac{1}{4}}$

1st order partial derivative :

$$Q_K = \frac{\partial Q}{\partial K} = 3K^{-\frac{1}{4}}L^{\frac{1}{4}}$$

$$Q_L = \frac{\partial Q}{\partial L} = K^{\frac{3}{4}}L^{-\frac{3}{4}}$$

2nd order " " " " :

$$Q_{KK} = \frac{\partial Q_K}{\partial K} = \frac{\partial^2 Q}{\partial K^2}$$

$$Q_{KL} = \frac{\partial Q_L}{\partial K} = \frac{\partial^2 Q}{\partial K \partial L}$$

$$Q_{LK} = \frac{\partial Q_K}{\partial L} = \frac{\partial^2 Q}{\partial L \partial K}$$

$$Q_{LL} = \frac{\partial Q_L}{\partial L} = \frac{\partial^2 Q}{\partial L^2}$$

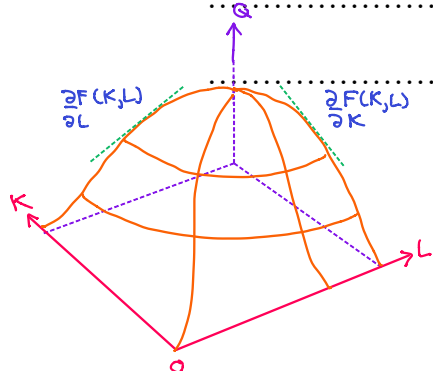
H =
Hessian
matrix

]

2x2

$Q_{LK} = Q_{KL}$ Young's thm.

$H^t = H$ H is symmetric



$f(x, y)$ $f_{xy} = f_{yx}$

Function with n independent variables will n^2 second order partial derivatives, from which we can write out the $n \times n$ matrix. Row i , Column j corresponds to $\frac{\partial^2 f}{\partial x_i \partial x_j}$. This matrix is called Hessian Matrix. The Hessian matrix is a symmetric matrix (Young's theorem).

$f(x_1, x_2, \dots, x_n) \Rightarrow \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n}$

$$H = D^2 f_x = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \dots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \dots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \dots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}_{n \times n}$$

$H^T = H$

H.W.

- 1) Let $y = f(x_1, x_2) = x_1 e^{x_1 + x_2^2}$ find f_1, f_2 , and Hessian Matrix
- 2) Let $Q = f(K, L, T) = AK^\alpha L^\beta T^\gamma$, find marginal products, and Hessian matrix



The Total Differential of A Function of Several Variables

Derivatives vs. Differentials

$$y = f(x)$$

$$\frac{dy}{dx} = f'(x)$$

$dy = f'(x) dx$
 ↑
 how much y changes when x changes 1 unit

The symbol $\frac{dy}{dx}$ for the derivative of the function $y = f(x)$ has been regarded as a single entity.

We shall now reinterpret $\frac{dy}{dx}$ as a ratio of two quantities, dy and dx , the differentials of y and x , respectively.

differentials in y single entity

$$\frac{dy}{dx} = f'(x) \quad \text{derivative}$$

differential in x

$$dy = f'(x) dx \quad \text{differential}$$

$f'(x)$ is a "converter" translating a given independent change dx into a counterpart change in dependent change dy .

The process of finding dy from a given function $y = f(x)$ is called differentiation.

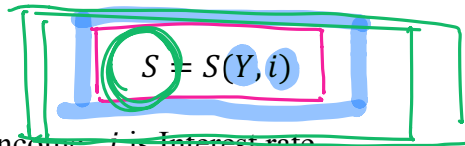
The process of finding derivative $\frac{dy}{dx}$ from a given function $y = f(x)$ is called differentiation with respect to x .

H.W. Given that $y = 3x^2 + 7x - 5$, find dy

$y = f(x_1, \dots, x_n) \dots dy ?$

Total differentials

The concept of differential can easily be extended to a function of two or more independent variable. Consider a saving function:



S is Saving, Y is national Income, i is Interest rate.

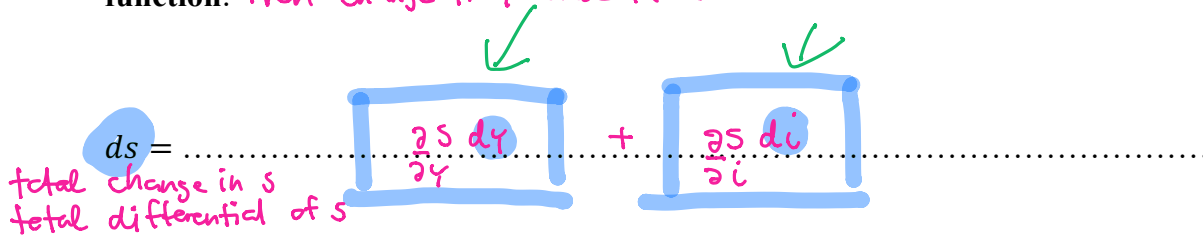
The partial derivative of S with respect to Y : $\frac{\partial S}{\partial Y}$ marginal propensity to save
change in S when Y changes 1 unit, given i constant

For any change in Y , dy , the resulting change in S can be approximated by the partial differential from Y
quantity: $\frac{\partial S}{\partial Y} dy \Rightarrow$ change in S when Y changes by dy units, given i constant

The partial derivative of S with respect to i : $\frac{\partial S}{\partial i}$
change in S when i changes 1 unit, given Y constant

For any change in i , di , the resulting change in S can be approximated by the partial differential from i
quantity: $\frac{\partial S}{\partial i} di \Rightarrow$ change in S when i changes by di units, given Y constant

The total change in S is then approximated by the total differential of saving function: from change in Y and in i



The partial differential of the saving function:

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Rule of Differentials

For a constant k , and function $U(X_1, X_2)$ and $V(X_1, X_2)$

Rule 1: $dk = 0$

Rule 2: $d(cU^n) = cnU^{n-1}dU$

Rule 3: $d(U \pm V) = dU \pm dV$

Rule 4: $d(UV) = UdV + VdU$

Rule 5: $d\left(\frac{U}{V}\right) = \frac{VdU - UdV}{V^2}$

H.W. Find dy for

1) $y = 5x_1^2 + 3x_2$

2) $y = 3x_1^2 + x_1x_2^2$

3) $y = \frac{x_1+x_2}{2x_1^2}$

4) $y = 3x_1(2x_2 - 1)(x_3 + 5)$

5) $y = -5 + 30x_1 - 3x_1^2 + 25x_2 - 5x_2^2 + x_1x_2$, given that $x_1 = 5$, $x_2 = 2$, $dx_1 = 0.02$, $dx_2 = 0$

$$dy = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2$$

$$dy = 10x_1 dx_1 + 3 dx_2$$

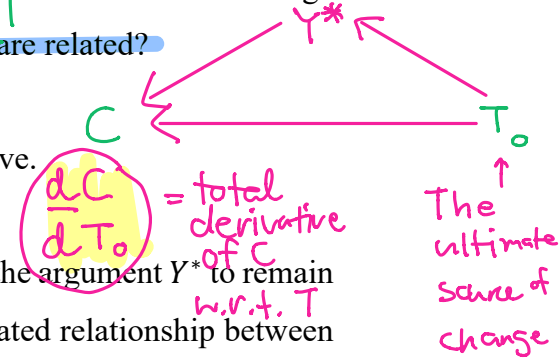


Total Derivatives

Question: Consider equilibrium consumption function which is a function of equilibrium national income and exogenous tax, $C(Y^*, T_0)$. What is the rate of change of the function $C(Y^*, T_0)$ with respect to T_0 , when Y^* and T_0 are related?

To answer this question, we need to learn about total derivative.

Unlike a partial derivative, a total derivative does not require the argument Y^* to remain constant as T_0 varies. A total derivative allows for the postulated relationship between the two arguments.



$C = f(T_0)$
 $\frac{dC}{dT_0}$... derivative of C w.r.t. T_0

Finding the Total Derivative

Consider $y = f(x, w)$, where $x = g(w)$

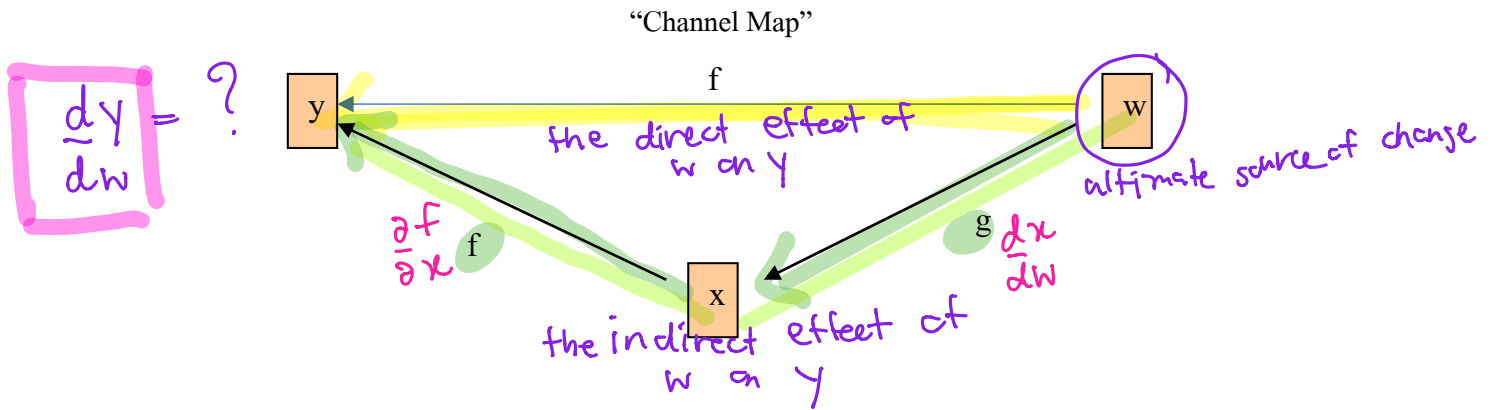
Question: What is the total derivative of y with respect to w? How to do the total differentiation of y with respect to w?

Note that: the two functions f and g can be combined into a composite function $y = f(g(w), w)$

The three variables y, x, w are related to one another. In the following *channel map*, w, the ultimate source of change, can affect y through two separate channels:

- (1) Directly, via the function f
- (2) Indirectly, via the function g, then f

$$y = f(x, w), \quad x = g(w)$$



The direct effect can be represented by the partial derivative

$\frac{\partial f}{\partial w}$, i.e. f_w how much y change when w changes, holding x constant (as if turning off the x channel)

The indirect effect can be represented by a product of two derivatives:

$\frac{\partial f}{\partial x} \frac{dx}{dw}$, i.e. $f_x \frac{dx}{dw}$ how much y change when x changes, holding w constant

, by the chain rule for a composite function **BUT** x changes when w changes

Adding up the two effects give us the total derivative of y with respect to w :

$$\frac{dy}{dw} = f_w + f_x \frac{dx}{dw}$$

direct effect indirect effect

Alternatively, we can also find the total derivative by:

(a.) Total differentiating the function $y = f(x, w)$

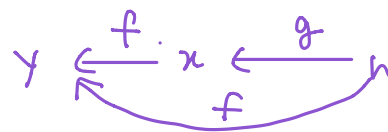
$$dy = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial w} dw$$

$$dy = f_x dx + f_w dw$$

(b.) Dividing through the total differential by dw

$$\frac{dy}{dw} = f_x \frac{dx}{dw} + f_w$$

Example 1: Find Total derivative $\frac{dy}{dw}$, when
 $y = f(x, w) = 3x - w^2$, $x = g(w) = 2w^2 + w + 4$

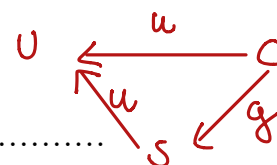


$$\frac{dy}{dw} = f_w + f_x \frac{dx}{dw}$$

$$= (-2w) + 3(4w + 1)$$

or you can directly substitute $x = g(w)$ into $y = f(x, w)$

Example 2: $U = u(c, s)$, and $s = g(c)$, c is quantity of coffee, s is quantity of sugar, which depends on quantity of coffee consumed. How much will total utility change as quantity of coffee consumed changes?

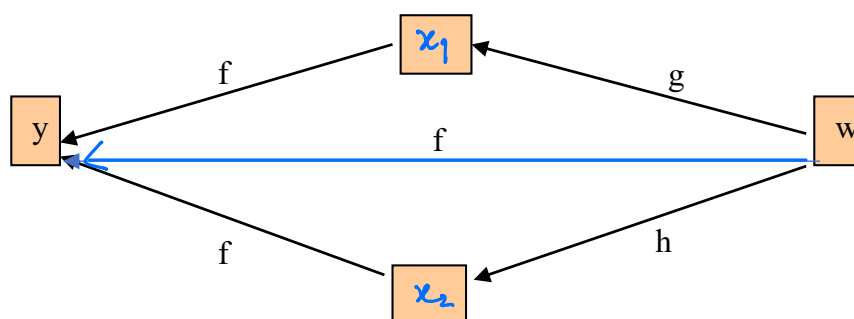


$$\frac{dU}{dc} = u_c + u_s \frac{ds}{dc}$$

Three channels:

$y = f(x_1, x_2, w)$, $x_1 = g(w)$, $x_2 = h(w)$ What is $\frac{dy}{dw}$?

$$\frac{dy}{dw} = f_w + f_{x_1} \frac{dx_1}{dw} + f_{x_2} \frac{dx_2}{dw}$$



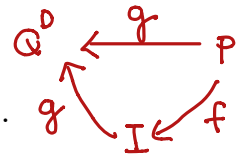
Example 3: $Q = Q(K, L, t)$, $K = K(t)$, $L = L(t)$

$\frac{dQ}{dt} = ?$

Example 4: $Q^D = g(P, I)$, $I = f(P)$, Q^D is quantity demanded, I is income, and P is price.

$\frac{dQ^D}{dP} = \frac{\partial Q^D}{\partial P} + \frac{\partial Q^D}{\partial I} \frac{dI}{dP}$

Total effect of change in price = Substitution effect + income effect



H.W. Find total derivative of

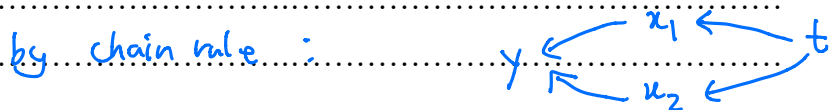
(1.) $z = f(x, y, t)$, $x = a + bt$, $y = c + dt$

(2.) $z = f(x, y) = 2x + xy - y^2$, $x = g(y) = 3y^2$

Chain rule:

Example 5: Let $y = \ln(x_1 + x_2)$, $x_1 = t$, and $x_2 = t^2$. Find $\frac{dy}{dt}$ by direct substitution, draw channel diagram, and also find $\frac{dy}{dt}$ by chain rule.

$\frac{dy}{dt}$: by direct substitution $y = \ln(t + t^2) \rightarrow \frac{dy}{dt}$



$\frac{dy}{dt} = \frac{\partial y}{\partial x_1} \frac{dx_1}{dt} + \frac{\partial y}{\partial x_2} \frac{dx_2}{dt}$

$= \frac{1}{x_1 + x_2} \cdot 1 + \frac{1}{x_1 + x_2} \cdot 2t$

H.W. Find $\frac{dz}{dt}$ at $t = 0$, $z = \frac{5t^2 + 3xy}{2w^2y}$, $x = t^2 + 1$, $y = \sqrt{t^2 + 1}$, $w = e^t + 1$



Implicit Functions

$y = a + bx$
 $y = \log x \dots$

A function given in the form of $y = f(x)$ is called an explicit function, because the variable y is explicitly expressed as a function of x . For example, $y = f(x) = 3x^4$.
 $\frac{dy}{dx} = 12x^3$

Consider a given $F(y, x) = 0$. For $F(y, x) = 0$, the left hand side is a function of the two variables y and x .

$F(y, x) = 0$

For example, $F(y, x) = 0$
 $y^3 - x^2y - \frac{1}{y} + 5xy = 0$

$y = f(x)$ implicit function
of an implicit function f
 $\frac{dy}{dx}$

$F(y, x) = 0$ can imply the function $y = f(x)$, in which case the function f is called an implicit function.

Derivatives of Implicit function

If the equation $F(y, x) = 0$ can be solved for y , we can write out the function $y = f(x)$ and find its derivatives by the methods learned before.

But if the equation $F(y, x) = 0$ cannot be solved for y , the derivatives of implicit function can be found by applying the concept of total differentiation.

$F(y, x) = 0$
 $dF(y, x) = d0$
 $dF(y, x) = 0$
 $dF = F_y dy + F_x dx = 0$
 $F_y dy + F_x dx = 0$

Thus, the implicit-function rule:

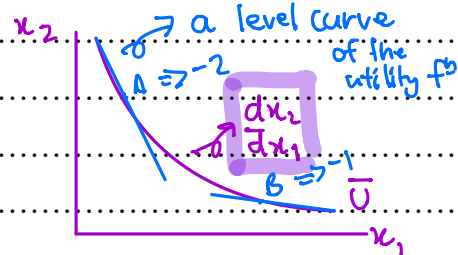
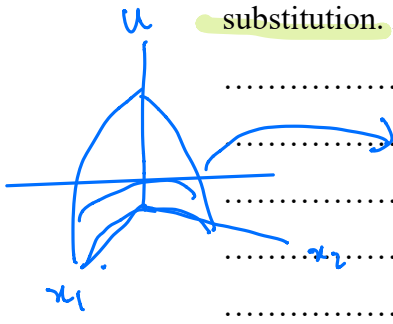
derivative of y w.r.t. x of the implicit $f(x)$
 $\frac{dy}{dx} = -\frac{F_x}{F_y}$
 $\frac{dy}{dx} = -\frac{F_x dx}{F_y}$

For $F(y, x_1, \dots, x_m) = 0$, the partial derivative f_i of the implicit function $y = f(x_1, \dots, x_m)$ is:

$F(y, x_1, \dots, x_m) = 0$
 $y = f(x_1, \dots, x_m)$
 $\frac{\partial f}{\partial x_i} = f_i = \frac{\partial y}{\partial x_i} = -\frac{F_{x_i}}{F_y}$

Example: Consider Utility function $U = U(x_1, x_2)$, find the marginal rate of

substitution. And, if $U = U(x_1, x_2) = x_1^\alpha x_2^\beta$, Is the indifferent curve convex?



$MRS_{x_1, x_2} = -\frac{dx_2}{dx_1}$
 = the amount of x_2 which a consumer can exchange for one unit of x_1

Ic curve is convex if $\frac{d^2 x_2}{dx_1^2} > 0$

We find MRS_{x_1, x_2} from IC curve by using the understanding that indifferent curve is (x_1, x_2) that gives equal level of utility.

Let $U(x_1, x_2) = \bar{U}$; IC curve is an implicit fn from $U(x_1, x_2) = \bar{U}$
 $(x_2 = f(x_1))$ implicitly

$F(x_1, x_2) = U(x_1, x_2) - \bar{U} \equiv 0$

total differential of function F: $dF = F_{x_1} dx_1 + F_{x_2} dx_2 = d0 = 0$

$F_{x_1} dx_1 + F_{x_2} dx_2 = 0$; $F_{x_1} = \frac{\partial F}{\partial x_1} = \frac{\partial (U(x_1, x_2) - \bar{U})}{\partial x_1} = \frac{\partial U}{\partial x_1} = MU_{x_1}$

$MRS_{x_1, x_2} = -\frac{dx_2}{dx_1} = \frac{MU_{x_1}}{MU_{x_2}}$

$\frac{dx_2}{dx_1} = -\frac{F_{x_1}}{F_{x_2}} = -\frac{U_{x_1}}{U_{x_2}} = -\frac{MU_{x_1}}{MU_{x_2}}$

if $U(x_1, x_2) = x_1^\alpha x_2^\beta \Rightarrow \frac{dx_2}{dx_1} = -\frac{\alpha x_1^{\alpha-1} x_2^\beta}{\beta x_1^\alpha x_2^{\beta-1}} = -\frac{\alpha}{\beta} \frac{x_2}{x_1} < 0$

IC curve is convex if $\frac{d^2 x_2}{dx_1^2} > 0$; $\frac{d^2 x_2}{dx_1^2} = \frac{d}{dx_1} \left(-\frac{\alpha}{\beta} \frac{x_2}{x_1} \right) > 0 \therefore$ IC curve is convex

$\frac{dx_2}{dx_1} = -\frac{\alpha}{\beta} \frac{x_2}{x_1}$ is a function of x_1 & x_2 such that x_1 & x_2 are related to each other in a way that $U(x_1, x_2) = \bar{U}$

Take the total derivative of function $\frac{dx_2}{dx_1}$ w.r.t. x_1



Homework: Find $\frac{dy}{dx}$ for when: $\frac{d^2 x_2}{dx_1^2} = \frac{d}{dx_1} \left(\frac{dx_2}{dx_1} \right) = \frac{\partial}{\partial x_1} \left(\frac{dx_2}{dx_1} \right) + \frac{\partial}{\partial x_2} \left(\frac{dx_2}{dx_1} \right) \frac{dx_2}{dx_1}$

- 1.) $y^3 - x^2 y - \frac{1}{y} + 5xy = 0$
- 2.) $x^3 + 2x^2 y - 3xy^2 - y^3 = 0$
- 3.) $x^2 y^2 + e^x - e^y = 0$
- 4.) $3 \ln(x + y) = y^2 - 4x^2$

$= \frac{\partial}{\partial x_1} \left(-\frac{\alpha}{\beta} \frac{x_2}{x_1} \right) + \frac{\partial}{\partial x_2} \left(-\frac{\alpha}{\beta} \frac{x_2}{x_1} \right) \frac{dx_2}{dx_1}$
 $= \frac{\alpha}{\beta} \frac{x_2}{x_1^2} + \left(-\frac{\alpha}{\beta x_1} \right) \left(-\frac{\alpha}{\beta} \frac{x_2}{x_1} \right) \frac{dx_2}{dx_1}$

$\frac{d^2 x_2}{dx_1^2} = \frac{\alpha}{\beta} \frac{x_2}{x_1^2} + \frac{\alpha^2 x_2}{\beta^2 x_1^2}$
 $= \frac{\alpha}{\beta} \left(1 + \frac{\alpha}{\beta} \right) \frac{x_2}{x_1^2} > 0 \therefore$ IC curve is convex!

$F(x, y) = x^3 + 2x^2 y - 3xy^2 - y^3 = 0$
 $\frac{\partial F}{\partial x} = 3x^2 + 4xy - 3y^2 = 0$
 $\frac{\partial F}{\partial y} = 2x^2 - 6xy - 3y^2 = 0$
 $\frac{dy}{dx} = \frac{-\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = \frac{-(3x^2 + 4xy - 3y^2)}{2x^2 - 6xy - 3y^2}$



Applications in Economics

😊 Utility function



Consider Utility function $U = U(x_1, x_2)$. Find marginal utility of each good.

What are the conditions for law of diminishing marginal utility to hold?

$$MU_{x_1} = \frac{\partial U(x_1, x_2)}{\partial x_1} = U_{x_1}$$

$$\frac{\partial MU_{x_1}}{\partial x_1} = \frac{\partial^2 U}{\partial x_1^2} < 0 \text{ for the law of diminishing MU}$$

$$MU_{x_2} = \dots$$

$$\frac{\partial MU_{x_2}}{\partial x_2} = \dots$$



Let $U(x_1, x_2) = (x_1 + 2)^2(x_2 + 3)^2$. Compare marginal utility of good 1 and good 2 when $x_1 = x_2 = 3$ units

$$MU_{x_1} = \frac{\partial U}{\partial x_1} = 2(x_1 + 2)^{\overset{3}{1}}(x_2 + 3)^{\overset{3}{2}}$$

$$MU_{x_2} = \frac{\partial U}{\partial x_2} = 2(x_1 + 2)^{\overset{3}{2}}(x_2 + 3)^{\overset{3}{1}}$$

$$MU_{x_1} \Big|_{x_1=x_2=3} = 2(3+2)(3+3)^2 = 360$$

$$MU_{x_2} \Big|_{x_1=x_2=3} = 300$$

 **Production function**



Consider $Q = f(K, L) = AK^\alpha L^\beta, 0 < \alpha, \beta < 1$

Find marginal product of each factor of production. What are the conditions for law of diminishing marginal product to hold?

$$MP_L = \frac{\partial Q}{\partial L} = \beta AK^\alpha L^{\beta-1}$$

$$\frac{\partial MP_L}{\partial L} = \frac{\partial^2 Q}{\partial L^2} = \beta(\beta-1)AK^\alpha L^{\beta-2} < 0 \text{ if } \beta < 1$$

$L \uparrow, MP_L \downarrow$

$$MP_K =$$

$$\frac{\partial MP_K}{\partial K} = \alpha(\alpha-1)AK^{\alpha-2}L^\beta < 0 \text{ if } \alpha < 1$$



Consider $Q = f(K, L)$. Whether a production is constant, increasing and decreasing return to scale can be considered as follows.

c rts i rts

c rts : increase K, L t times \longrightarrow Q increase t times

i rts : " " K, L t " " \longrightarrow Q " " more than t times

d rts : " " K, L t " " \longrightarrow Q " " less than t times

when increase K, L t times, we get $f(tK, tL)$

when Q increase t times, $t f(K, L)$

When Q increase t times, $t f(K, L)$

c rts : $f(tK, tL) = t f(K, L)$

i rts : $f(tK, tL) > t f(K, L)$

d rts : $f(tK, tL) < t f(K, L)$

If we let $Q = AK^\alpha L^\beta$, and let K, L increase t times, we will get $Q(tK, tL) = A(tK)^\alpha (tL)^\beta$
 $= t^{\alpha+\beta} AK^\alpha L^\beta$
 $= t^{\alpha+\beta} Q(K, L)$

c rts : $t^{\alpha+\beta} Q(K, L) = t Q(K, L) \Rightarrow \alpha+\beta = 1$

i rts : $t^{\alpha+\beta} Q(K, L) > t Q(K, L) \Rightarrow \alpha+\beta > 1$

d rts : $t^{\alpha+\beta} Q(K, L) < t Q(K, L) \Rightarrow \alpha+\beta < 1$

e.g. $Q = AK^{0.7} L^{0.5} \Rightarrow$ i rts

Note: If we can write any function in the form of $y = f(x_1, x_2)$
 $y = f(tx_1, tx_2) = t^k f(x_1, x_2)$

we can call the $f^k f(x_1, x_2)$ as "homogeneous f^k of degree k ".

\Rightarrow the degree of $Q = AK^\alpha L^\beta$ is $\alpha+\beta$

📌 Elasticity

📌📌 Elasticity of output with respect to factor of production

$$Q = f(K, L) = AK^\alpha L^\beta, 0 < \alpha, \beta < 1$$

$$E_L = \frac{\partial Q}{\partial L} \cdot \frac{L}{Q} = \frac{\frac{\partial Q}{\partial L}}{\frac{Q}{L}} = \frac{MP_L}{AP_L} = \frac{\beta AK^\alpha L^{\beta-1}}{AK^\alpha L^{\beta-1}} = \beta$$

$$E_K = \frac{\partial Q}{\partial K} \cdot \frac{K}{Q} = \alpha$$

📌📌 Elasticity of demand

Consider $Q_A^D = f(P_A, P_B, y) = 100 - 10P_A + 15P_B + 0.3y$, Find own price/cross price/income elasticity of quantity demanded. Let $P_A = 8, P_B = 5, I = 500, Q_A = 60$

Own price elasticity:

$$E_{P_A} = \frac{\partial Q_A^D}{\partial P_A} \cdot \frac{P_A}{Q_A^D} = -10 \cdot \frac{P_A}{Q_A} = -10 \cdot \frac{8}{60} < 0$$

law of demand

Cross price elasticity:

$$E_{P_B} = \frac{\partial Q_A^D}{\partial P_B} \cdot \frac{P_B}{Q_A^D} = 15 \cdot \frac{P_B}{Q_A^D} = 15 \times \frac{5}{60} > 0$$

$E_{P_B} > 0$ if substitute
 $E_{P_B} < 0$ complement

Income elasticity:

$$E_I = \frac{\partial Q_A^D}{\partial I} \cdot \frac{1}{Q_A^D} = 0.3 \cdot \frac{1}{Q_A^D} = 0.3 \times \frac{500}{60} > 0$$

$E_I > 0$ if normal goods
 $E_I < 0$ inferior

HOMEWORK

1.) Market equilibrium

$$Q_d = Q_s$$

$$Q_d = a - bP \quad (a, b > 0)$$

$$Q_s = -c + dP \quad (c, d > 0)$$

$$P^* = \frac{a+c}{b+d}, \quad Q^* = \frac{ad-bc}{b+d}$$

What are $\frac{\partial P^*}{\partial a}, \frac{\partial Q^*}{\partial a}, \frac{\partial P^*}{\partial c}, \frac{\partial Q^*}{\partial c}, \frac{\partial P^*}{\partial b}, \frac{\partial Q^*}{\partial b}, \frac{\partial P^*}{\partial d}, \frac{\partial Q^*}{\partial d}$?

2.) Multipliers in Keynesian crossing model

$$Y = C + I + G + X - M$$

$$Y = C_0 + C_1(Y - T) + I_0 + iY + G_0 + X_0 + \gamma_0 Y^* - M_0 - \lambda_0 Y$$

$$Y = C_0 + C_1(Y - t_0 - t_1 Y) + I_0 + iY + G_0 + X_0 + \gamma_0 Y^* - M_0 - \lambda_0 Y$$

$$Y_E = \frac{C_0 - t_0 C_1 + I_0 + G_0 + X_0 + \gamma_0 Y^* - M_0}{1 - C_1(1 - t_1) - i + \lambda_0}$$

What are $\frac{\partial Y_E}{\partial Y^*}, \frac{\partial Y_E}{\partial I_0}, \frac{\partial Y_E}{\partial C_0}, \frac{\partial Y_E}{\partial C_1}$?