Answer HW 2 EE 325

1. If X_1, X_2, X_3 is a random sample from a population with mean μ and variance σ^2.

   X_1, X_2, X_3 are not independent $Cov(X_1, X_2) = Cov(X_1, X_3) = Cov(X_2, X_3) = \frac{1}{2}\sigma^2$

   $\bar{X}$ is estimator used to estimate mean value. $\bar{X} = \frac{1}{3}(X_1 + X_2 + X_3)$

   Find $E(\bar{X})$ and $\text{var}(\bar{X})$

$$
\begin{aligned}
E(\bar{X}) &= E\left(\frac{1}{3}(X_1 + X_2 + X_3)\right) \\
&= \frac{1}{3}\big(E(X_1) + E(X_2) + E(X_3)\big) \\
&= \frac{1}{3}3\mu = \mu
\end{aligned}
$$

$$
\begin{aligned}
\text{var}(\bar{X}) &= \text{var}\left(\frac{1}{3}(X_1 + X_2 + X_3)\right) \\
&= \frac{1}{9}\text{var}(X_1 + X_2 + X_3) \\
&= \frac{1}{9}(\text{var}(X_1) + \text{var}(X_2) + \text{var}(X_3) + 2(\text{cov}(X_1X_2) + \text{cov}(X_1X_3) + \text{cov}(X_2X_3)) \\
&= \frac{1}{9}(\sigma^2 + \sigma^2 + \sigma^2 + 2(\frac{1}{2}\sigma^2 + \frac{1}{2}\sigma^2 + \frac{1}{2}\sigma^2)) = \frac{6}{9}\sigma^2
\end{aligned}
$$

2. Given X_1, X_2, X_3, X_4 are independent identically distributed random variables from population with mean μ and variance σ^2. $\bar{X}$ is estimator used to estimate mean value. $\bar{X} = \frac{1}{3}(X_1 + X_2 + X_3 + X_4)$

   2.1 Find $E(\bar{X})$ and $\text{var}(\bar{X})$ in term of μ and σ

$$
\begin{aligned}
E(\bar{X}) &= E\left(\frac{1}{3}(X_1 + X_2 + X_3 + X_4)\right) \\
&= \frac{1}{3}(E(X_1) + E(X_2) + E(X_3) + E(X_4)) = \frac{4}{3}\mu
\end{aligned}
$$

$$\begin{aligned}\operatorname{var}(\bar{X}) &= \operatorname{var}(\frac{1}{3}(X_1 + X_2 + X_3 + X_4)) \\ &= \frac{1}{9}\operatorname{var}(X_1 + X_2 + X_3 + X_4) \\ &= \frac{1}{9}(\operatorname{var}(X_1) + \operatorname{var}(X_2) + \operatorname{var}(X_3) + \operatorname{var}(X_4)) = \frac{4}{9}\sigma^2\end{aligned}$$

2.2 Given $\tilde{X} = \frac{1}{8}X_1 + \frac{1}{4}X_2 + \frac{1}{8}X_3 + \frac{1}{2}X_4$ is another estimator of μ. Show that $\tilde{X}$ is an unbiased estimator of μ

$$\begin{aligned}E(\tilde{X}) &= E\left(\frac{1}{8}X_1 + \frac{1}{4}X_2 + \frac{1}{8}X_3 + \frac{1}{2}X_4\right) \\ &= \left(\frac{1}{8} + \frac{1}{4} + \frac{1}{8} + \frac{1}{2}\right)\mu = \mu\end{aligned}$$

2.3 Between $\bar{X}$ and $\tilde{X}$, which one is the better estimator for μ? Why?

$\tilde{X}$ are unbiased estimator of μ.

3.

3.1 From the simple regression model $Y_i = \beta_1 + \beta_2 X_i + u_i, u_i \sim NIID(0,\sigma^2)$ Find estimators of β_1 and β_2 from the OLS method and interpret the meaning

$$\begin{aligned}\hat{\beta}_2 &= \frac{n\sum X_i Y_i - \sum X_i \sum Y_i}{n\sum X_i^2 - (\sum X_i)^2} \\ &= \frac{\sum (X_i - \bar{X})(Y_i - \bar{Y})}{\sum (X_i - \bar{X})^2} \\ &= \frac{\sum x_i y_i}{\sum x_i^2}\end{aligned}$$

$$\begin{aligned}\hat{\beta}_1 &= \frac{\sum X_i^2 \sum Y_i - \sum X_i \sum X_i Y_i}{n\sum X_i^2 - (\sum X_i)^2} \\ &= \bar{Y} - \hat{\beta}_2 \bar{X}\end{aligned}$$

$\hat{\beta}_2 = 1$

$\hat{\beta}_1 = \bar{Y} - \hat{\beta}_2 \bar{X} = 10 - 1(20) = -10$

When X equals to zero, Y will equal -10.

When X increases by 1 unit, Y will increase 1 unit.

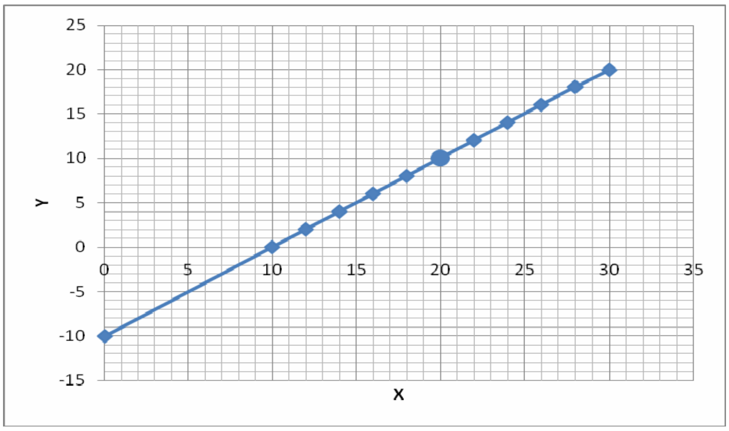
3.2 Find the value of $\hat{Y}_i$ and $\hat{u}_i$. Show that $\sum \hat{u}_i \approx 0$

X_i	Y_i	$\hat{Y}_i$	$\hat{u}_i$
10	0	0	0
12	2	2	0
14	4	4	0
16	6	6	0
18	8	8	0
22	12	12	0
24	14	14	0
26	16	16	0
28	18	18	0
30	20	20	0

$$\begin{aligned}Y_i &= \hat{\beta}_1 + \hat{\beta}_2 X_i + \hat{u}_i \\ &= \hat{Y}_i + \hat{u}_i\end{aligned}$$

$\sum \hat{u}_i \approx 0$

3.3 Plot graph and draw regression line. Does the line pass $(\bar{X}, \bar{Y})$?



3.4 If $X_i = 25$, what is the predicted Y?

$\hat{Y}_i = -10 + 25 = 15$

3.5 Find $\text{var}(\hat{u}_i)$, $\text{var}(\hat{\beta}_1), \text{var}(\hat{\beta}_2)$

$$\text{var}\left(\hat{u}_i\right) = \frac{\sum_{i=1}^{n} \hat{u}_i^2}{n-2} = 0$$

$$\text{var}\left(\hat{\beta}_1\right) = \frac{\sum_{i=1}^{n} X_i^2}{n\sum_{i=1}^{n} x_i^2}\sigma^2 = 0$$

$$\text{var}\left(\hat{\beta}_2\right) = \frac{\sigma^2}{\sum_{i=1}^{n} x_i^2} = 0$$