LECTURE NOTE EE 325 : INTRODUCTORY ECONOMETRICS

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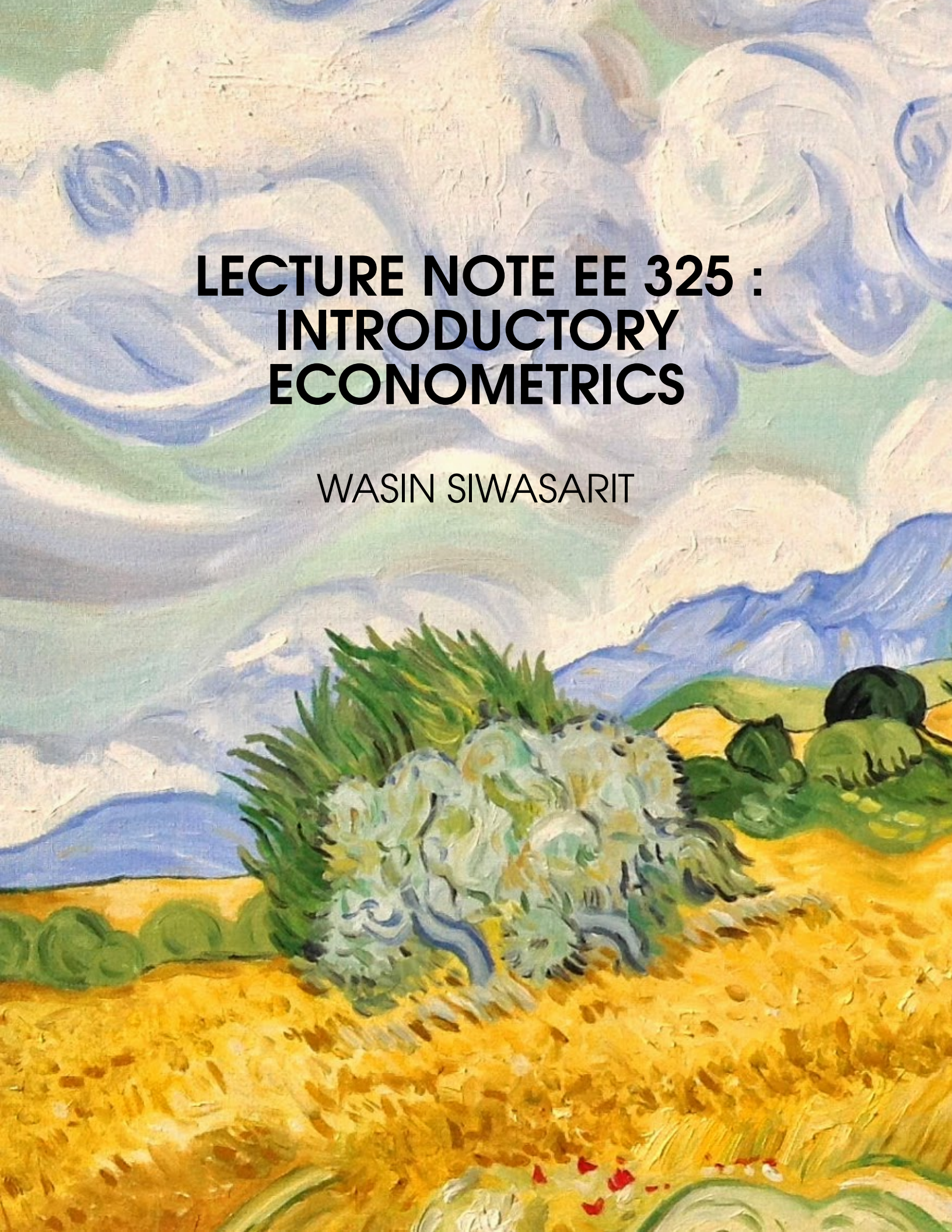

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1. Introduction

1.1 Review of Some Statistical Concepts

The notation $\sum$ (sigma), in mathematical term, denotes the **summation**

$$\sum_{i=1}^{n} x_i = x_1 + x_2 + \cdots + x_n \tag{1.1}$$

The noteworthy properties of summation include:

1. $\sum_{i=1}^{n} k = nk$

2. $\sum_{i=1}^{n} kx_i = k\sum_{i=1}^{n} x_i$, where k is a constant term.

3. $\sum_{i=1}^{n} (a + bx_i) = na + b\sum_{i=1}^{n} x_i$, where a and b are constants.

4. $\sum_{i=1}^{n} (X_i + Y_i) = \sum_{i=1}^{n} X_i + \sum_{i=1}^{n} Y_i$.

Multiple summation is the summation of variable that is in the form of matrix, shown as,

$$\sum_{i=1}^{n}\sum_{j=1}^{m} x_{ij} = \sum_{i=1}^{n}(x_{i1} + x_{i2} + \ldots + x_{im}) = (x_{11} + x_{21} + \ldots + x_{n1}) + (x_{12} + x_{22} + \ldots + x_{n2}) + \ldots (x_{1m} + x_{2m} + \ldots + x_{nm}) \tag{1.2}$$

where

$$\mathbf{X} = \begin{bmatrix} x_{11} & x_{12} & \ldots & x_{1m} \\ x_{21} & x_{22} & \ldots & x_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \ldots & x_{nm} \end{bmatrix}_{nXm}$$

The significant properties of multiple summations are:

1. $\sum_{i=1}^{n}\sum_{j=1}^{m} X_{ij} = \sum_{j=1}^{m}\sum_{i=1}^{n} X_{ij}$

2. $\sum_{i=1}^{n}\sum_{j=1}^{m} X_i Y_j = \sum_{i=1}^{n} X_i \times \sum_{j=1}^{m} Y_j$

3. $\sum_{i=1}^{n}\sum_{j=1}^{m}(X_{ij} + Y_{ij}) = \sum_{i=1}^{n}\sum_{j=1}^{m} X_{ij} + \sum_{i=1}^{n}\sum_{j=1}^{m} Y_{ij}$

4. $(\sum_{i=1}^{n} X_i)^2 = \sum_{i=1}^{n} X_i^2 + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} X_i X_j$

The product operator $\prod$ is defined as:

$$\prod_{i=1}^{n} x_i = x_1 * x_2 \ldots * x_n \tag{1.3}$$

1.2 Experiment

Sample space is the set of all possible results of an experiment. For example, if you toss the coin twice, all feasible outcomes are composed of head twice, head followed by tail, tail followed by head, and tail twice. Let H and T denotes head and tail, respectively. The sample space can be written as,

$$SS = \{HH, HT, TH, TT\}$$

Sample Point is the member of sample space, eg. the event that head occurs twice from tossing a coin twice. Specifically, sample point is,

$$SP = HH \text{ or } HT \text{ or } TH \text{ or } TT$$

Events are the set of specific consequences of the experiment such as the events that head occurs twice. Events are the subset of sample space.

$$A = \text{the event that head occurs twice} = \{HH\}$$

Events are **mutually exclusive**, if the occurrence of one event makes no other events in sample space possible. As an illustration, for the experiment of tossing two coins once, let C be the event that both turn head and D be the event that both turn tail. Since C and D cannot happen at the same time, these two events are said to be mutually exclusive. Another example is the experiment of drawing one card from the standard 52-card deck, let E be the event that the rank of card is King and F be the event that suit of card is Clubs. As the event E and F can occur simultaneously, namely the King of Clubs, the two events are not mutually exclusive.

Events are **collectively exhaustive** if they cover all possible outcomes in the sample space. With the experiment of tossing the coin twice, let A be the event that head appears twice, B be the event that

tail appears twice, and C be the event that head and tail each appear once. In this case, A, B and C are collectively exhaustive since all events cover all possible results from sample space; that is, HH, HT, TH and TT.

1.3 Probability and Random Variable

Probability is the possibility that any event will occur, given some specific sample space.

Let A be the event occurring in the given sample space and $P(A)$ be the probability that A will happen. Then, $P(A)$ is defined as;

$$P(A) = \frac{\text{the number of times the event A will occur}}{\text{the number of all possible outcomes in sample space}} \tag{1.4}$$

For instance, to draw one card from the standard 52-card deck, let A be the event that the rank of card is 2. Times the event will occur is 4 and the amount of all possible outcomes is 52; hence, the probability of A is $\frac{4}{52}$ or $\frac{1}{13}$.

Some properties of probability are;

1. $0 \leq P(A) \leq 1$

2. If A, B and C are exhaustive set, then,

   $P(A)+P(B)+P(C) = 1$

3. If A, B and C are mutually exclusive, then,

   $P(A+B+C) = P(A)+P(B)+P(C)$

Suppose that the results of an experiment are in the form of value, the variable, whose value is determined by one of those results, is known as **Random Variable**. Random variable can be either **discrete** or **continuous value**.

For discrete random variable, the example is the sum of the values on the face of two dice, when rolling two dice once. In other word, the obtained sum will range from 2 to 12, and it is impossible to get 2.5 or 3.5.

For continuous random variable, the example is the height of the high-school student, constricted to the range from 160 to 180 centimetres. It can be seen that the value of the height need not be the integers and can take the value of 160.5 or 160.52 centimetres.

These two distinct characteristics of random variable enable us to classify them into different probability density functions, which would be stated in section 1.4.

1.4 Probability Density Function

As the value of random variable depends on an experiment, the **probability density function** would portray the overall image of possible random results. The type of the probability density function relies on the characteristics of the random variable. In this section, many important types are discussed.

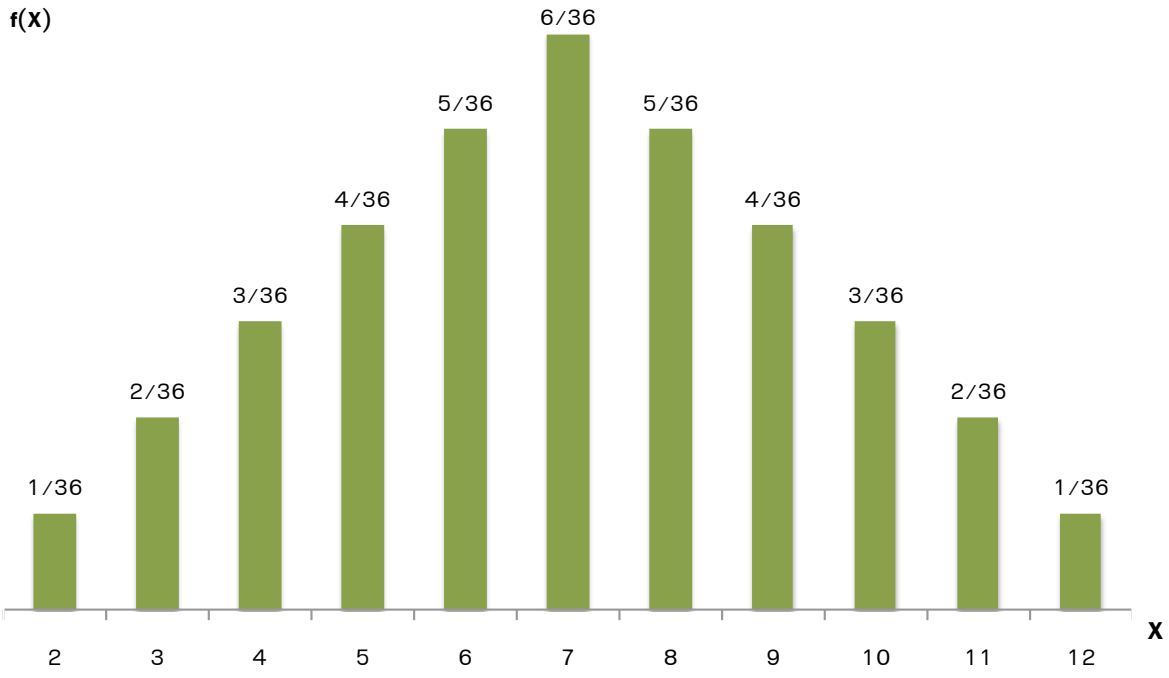
1.4.1 Probability Density Function for Discrete Random Variable

Let X be the discrete random variable with the value $x_1, x_2, \ldots, x_n$ and we get,

$$\begin{array}{rclcrcl} f(x) & = & P(X = x_i) & \text{for} & i & = & 1,2,...,n \\ f(x) & = & 0 & \text{for} & x & \neq & x_i \end{array}$$

Example: Let X be random variable of the sum of values on the face of two dices. The value might be 2 or 12, that is the value from both rolling round is 1 or 6, respectively. The Figure 1 summarizes all possible results#

Figure 1: Probability Density function of the Sum of Values on the Side of the Dice, Obtained from Rolling the Dice Twice



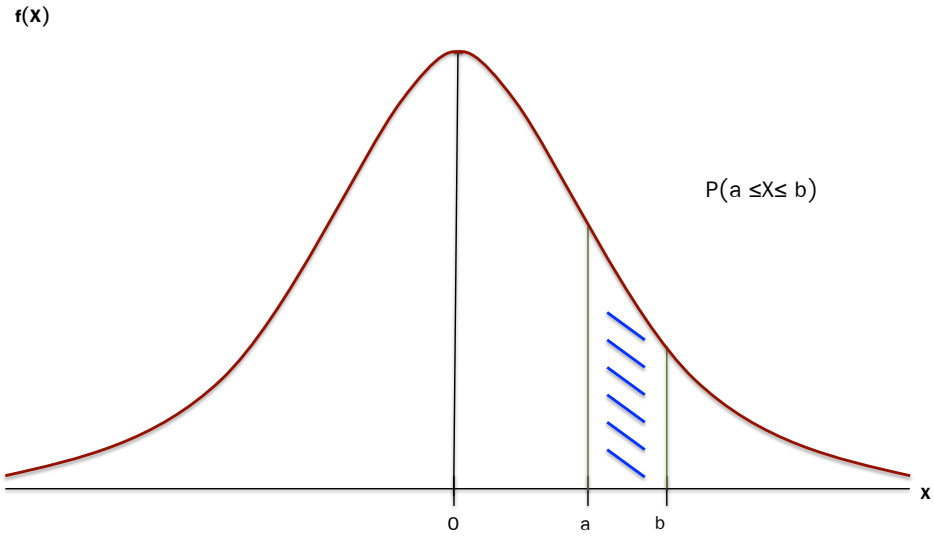
1.4.2 Probability Density Function for Continuous Random Variable

Let X be the continuous random variable. The probability density function of X satisfies the three following conditions.

1. $f(x) \geq 0$
2. $\int_{-\infty}^{\infty} f(x)dx = 1$
3. $\int_{a}^{b} f(x)dx = P(a \leq x \leq b)$

Figure 2 exhibits the probability density function for the continuous random variable, where the area under the curve represents the probability that the variable will lay on that range. Specifically, $P(a \leq X \leq b)$ means the probability that X will take the value between a and b.

Figure 2: Probability Density Function for Continuous Random Variable



1.4.3 Joint Probability Density Function

In this section, only **joint probability density function** for discrete variable is discussed. Let X and Y be discrete random variables. The joint probability density function, identifying the probability that X and Y happen simultaneously, is written as,

$$f(X,Y) = P(X = x \text{ and } Y = y)$$

Example: The following table explains the joint probability density function.

Table 1: The table illustrating the joint probability density function of X and Y

		X		
		-1	0	1
Y	1	0.11	0.08	0.05
	2	0.09	0.05	0.03
	3	0.35	0.07	0.17

According to the table 1, the probability that random variable X will be 0 and random variable Y will be 3 is 0.07 or 7 percent. In mathematical term, it can be written as $f(X = 0, Y = 3) = 0.07$.

1.4.4 Marginal Probability Density Function

The above joint probability density function $f(X,Y)$ shows the joint distribution of two variables. On the other hand, **marginal probability density function** with respect to joint probability function, displays the probability density function of single variable like $f(X)$, $f(Y)$, which can be derived from;

$$\begin{array}{lcll} f(X) & = & \sum_Y f(X,Y) & \text{called marginal PDF of X} \\ f(Y) & = & \sum_X f(X,Y) & \text{called marginal PDF of Y} \end{array}$$

where $\sum_Y$ or $\sum_X$ means the summation of probability over all values of X and Y respectively.
Example: According to Table 2 above, marginal PDF of X is obtained from

$$\begin{array}{rcl} (X=-1) & = & \\ & = & \\ & = & \\ & = & \\ f(X=0) & = & \sum_Y f(X=0,Y) \\ & = & f(X=0,Y=1)+f(X=0,Y=2)+f(X=0,Y=3) \\ & = & 0.08+0.05+0.07 \\ & = & 0.20 \\ f(X=1) & = & \sum_Y f(X=1,Y) \\ & = & f(X=1,Y=1)+f(X=1,Y=2)+f(X=1,Y=3) \\ & = & 0.05+0.03+0.17 \\ & = & 0.25 \end{array}$$

marginal PDF of Y is obtained from

$$\begin{array}{rcl} f(Y=1) & = & \\ & = & \\ & = & \\ & = & \\ f(Y=2) & = & \sum_X f(X,Y=2) \\ & = & f(X=-1,Y=2)+f(X=0,Y=2)+f(X=1,Y=2) \\ & = & 0.09+0.05+0.03 \\ & = & 0.17 \\ f(Y=3) & = & \sum_X f(X,Y=3) \\ & = & f(X=-1,Y=3)+f(X=0,Y=3)+f(X=1,Y=3) \\ & = & 0.35+0.07+0.17 \\ & = & 0.59 \end{array}$$

According to the calculation above, the result can be summarized into Table 2.

Table 2 shows joint probability of random variable X and Y

		X			
		-1	0	1	
Y	1	0.11	0.08	0.05	$f(Y=1)$ =
	2	0.09	0.05	0.03	$f(Y=2)$ =
	3	0.35	0.07	0.17	$f(Y=3)$ =
		$f(X=-1)$ =	$f(X=0)$ =	$f(X=1)$ =	$f(X)=$ $f(Y)=$

1.4.5 Conditional Probability Density Function

Conditional probability density function is the probability of one event given that some events have already occurred. The function is written as,

$$f(X|Y) = P(X = x|Y = y)$$

This function can be obtained from the joint probability density function through,

$$f(X|Y) = \frac{f(X,Y)}{f(Y)}$$

Example: According to Table 2, find $f(X=1|Y=2)$ and $f(Y=2|X=0)$

$$\begin{aligned}(X=0|Y=1) &= \\ &= \\ &= \\ &= \\ (Y=2|X=0) &= \\ &= \\ &=\end{aligned}$$

Example: Let event A be tossing the dice once and the point is odd number and B be the tossing the dice once and the point is at least 5. Find the probability that the point coming up is odd given that the point has to be at least 5.

Answer A and B will occur simultaneously if the point from tossing the dice is 5; so, the joint probability of A and B is $\frac{1}{6}$. The probability that B occurs is $\frac{2}{6}$. Hence, the conditional probability of A given B is

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)} = \frac{\frac{1}{6}}{\frac{2}{6}} = \frac{1}{2}\#$$

1.4.6 Statistical Independence

Two random variables are **independent** if the resulting value of one variable does not affect the resulting value of the other; namely,

$$f(X,Y) = f(X)f(Y)$$

Example: Consider Mr. Ake's expenditure for a meal and the Miss Somsri's expenditure for a dessert. Given that they do not know each other, the realization of Mr. Ake's expenditure does not imply the realization of Miss Somsri's expenditure. We can, thus, conclude that the expenditures of these two people are independent#

Example: Consider drawing cards sequentially from the standard 52-card deck without putting it back into the deck. Once the first card is drawn, the probability of drawing the second card will be influenced because the amount of cards in the deck is reduced. In this case, it can be concluded that drawing the first and second card are not independent#

1.5 Expectation, Variance, Covariance and Correlation

1.5.1 Mean or Expected Value

Because the value of random variable hinges on the value of random results of experiment which cannot be determined certainly, statisticians have invented the measures of central tendency of the random variable. One of them is **expected value**, indicating the mean of the random variable.

For discrete random variable, the expected value is calculated by;

$$E(X) = \sum_{i=i}^{n} x_i f(x_i) = x_1 f(x_1) + x_2 f(x_2) + \ldots + x_n f(x_n)$$

For continuous random variable, the expected value is calculated by,

$$E(X) = \int_a^b x f(x) dx$$

where;

$E(X)$ is the measure of central tendency of random variable, resulting from repeated trial of experiment.

$\sum_{i=i}^{n} x_i f(x_i)$ is the average of random variable weighted by the probability corresponding to each value.

a and b are the lowest and highest constant possible respectively.

Example: Find the expected value of rolling two dice once (Figure 1)

Example:

Crucial properties of expected value include:

1. $E(b) = b$

2. $E(aX + b) = aE(X) + b$

3. $E(XY) = E(X)E(Y)$; given that X and Y are independent

4. $E(g(X)) = \sum_x g(X)f(X)$

where a and b are constant.

Conditional expectation value is the expectation value of random variable under some conditions such as expected value of X conditional on Y or $E(X|Y = 5)$

Let $f(X,Y)$ be the joint probability function of X and Y. The expectation of X conditional on some value of Y is defined as,

For discrete random variable $E(X|Y = y) = \sum_X X_i f(X|Y = y)$

For continuous random variable $E(X|Y = y) = \int_{-\infty}^{\infty} X_i f(X|Y = y)$

Example

1.5.2 Variance

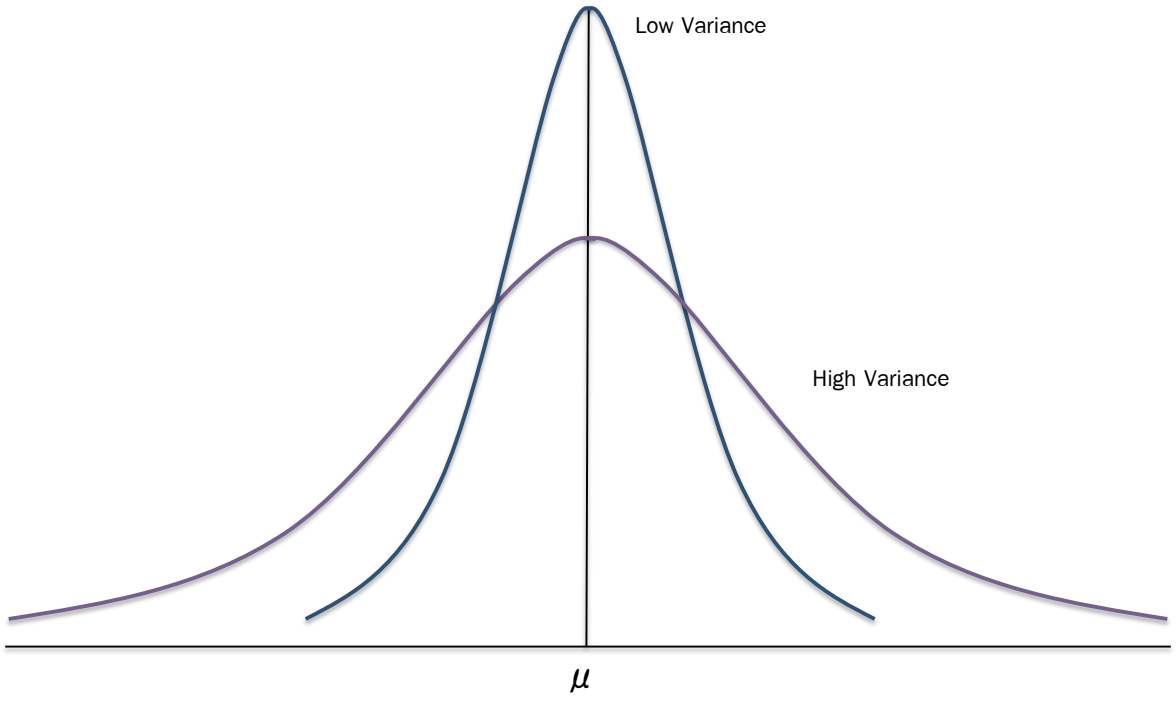
Variance is the measure of dispersion of the value of variable around the expected value. The higher the variance, the more dispersing the random variable (Figure 3). If X is the random variable with expected value μ, we get;

$$Var(X) = \sigma_X^2 = E[X - E(X)]^2 = E(X)^2 - \mu^2 \tag{1.5}$$

From,

$$\begin{aligned} Var(X) &= \sigma_X^2 \\ &= E[X - E(X)]^2 \\ &= E[X^2 - 2XE(X) + (E(X))^2] \\ &= E(X^2) - 2(E(X))^2 + (E(X))^2 \\ &= E(X)^2 - \mu^2 \end{aligned}$$

Figure 3: Distribution of Random Variables with Different Variance



Important properties of expected value include;

1. $Var(b) = 0$

2. $Var(aX + b) = a^2 Var(X)$

3. $Var(X \pm Y) = Var(X) + Var(Y)$; given that X and Y are independent

4. $Var(aX \pm bY) = a^2 Var(X) + b^2 Var(Y)$

where a and b are constant.

1.5.3 Conditional Variance

The conditional variance of X is given Y = y is defined as following:

$$\begin{aligned} var(X|Y=y) &= E\left\{[X - E(X|Y=y)]^2 | Y=y\right\} \\ &= \sum_x [X - E(X|Y=y)]^2 f(x|Y=y) \\ &= \int_{-\infty}^{\infty} [X - E(X|Y=y)]^2 f(x|Y=y)dx \end{aligned} \tag{1.6}$$

Example

Properties of conditional expectation and conditional variance

1.5.4 Covariance

Theorem. Let X and Y be two random variables with means μ_x and μ_y, respectively. Then, we can define the covariance between these two variables as following:

$$cov(X,Y) = E\{(X-\mu_x)(Y-\mu_y)\} = E(XY) - \mu_x\mu_y \tag{1.7}$$

If X and Y are continuous random variables we can calculate their cov(X,Y):

$$\begin{aligned} cov(X,Y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(X-\mu_x)(Y-\mu_y)f(x,y)dxdy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}XYf(x,y)dxdy - \mu_x\mu_y \end{aligned} \tag{1.8}$$

Properties of Covariance

1. If X and Y are independent, the covariance between X and Y is zero.

Proof:

2. $cov(a+bX, c+dY) = bd * cov(X,Y)$, where a,b,c, and d are constants.

Proof:

Example Suppose the join PDF of random variables X and Y can be represented as in the below table. What is the covariance between X and Y?

		X			
		1	2	3	
Y	1	0.25	0.25	0	$f(Y=1)$ =
	2	0	0.25	0.25	$f(Y=2)$ =
		$f(X=1)$ =	$f(X=2)$ =	$f(X=3)$ =	$f(X)=$ $f(Y)=$

Next, we will turn our attention to seeing how we can apply the covariance to calculate the correlation between the random variables X and Y

1.5.5 Correlation

When we calculate the covariance of X and Y, it reflects the units of both random variables. However, it is useful to have **a dimensionless measure of dependency** by calculating the correlation instead.

Definition Let X and Y be any two random variables (discrete or continuous) with standard deviation σ_X and σ_Y, respectively. The **correlation coefficient** of X and Y, denoted **corr(X,Y)** or ρ_{XY} (the greek letter "rho") is defined as:

$$\rho_{XY} = \frac{cov(X,Y)}{\sqrt{var(X)var(Y)}} = \frac{cov(x,y)}{\sigma_X \sigma_Y} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

Example Suppose the join PDF of random variables X and Y can be represented as in the below table. What is the correlation between X and Y?

		X			
		1	2	3	
Y	1	0.25	0.25	0	$f(Y=1)$ =0.5
	2	0	0.25	0.25	$f(Y=2)$ =0.5
		$f(X=1)$ = 0.25	$f(X=2)$ = 0.5	$f(X=3)$ =0.25	$f(X)=1$ $f(Y)=1$

From the definition, ρ_{XY} is measure of linear association between two random variables. The value of ρ lies between -1 and +1, $-1 \leq \rho_{XY} \leq +1$. We can interpret the value of correlation as:

- If $\rho_{XY} = 1$, then X and Y are perfectly, positively, linearly correlated.
- If $\rho_{XY} = -1$, then X and Y are perfectly, negatively, linearly correlated.
- If $\rho_{XY} = 0$, then X and Y are completely,un-linearly correlated. This means that X and Y may correlated in some other manner i.e. a parabolic manner., but NOT in a linear manner
- If $\rho_{XY} \leq 0$, then X and Y are positively, linearly correlated, but NOT perfectly.
- If $\rho_{XY} \geqq 0$, then X and Y are negatively, linearly correlated, but NOT perfectly.

Theorem. If X and Y are independent random variables, then:

$$corr(X,Y) = cov(X,Y) = 0$$

Example: Let X = the outcome of a fair, black, 6-sided die.
Let Y =outcome of a fair, red, 4-sided die.
What is the covariance of X and Y? What is the correlation of X and Y?

NOTE: The converse of the theorem is NOT NECESSARILY CORRECT!

Example: Let X and Y be two discrete random variables with the following join PDF:

		X			
		0	1	2	
	0	0	0.20	0.10	$f(Y=0)$ =
Y	1	0.20	0.40	0	$f(Y=1)$ =
	2	0.10	0	0	$f(Y=2)$ =
		$f(X=0)$ =	$f(X=1)$ =	$f(X=2)$ =	

What is the correlation between X and Y? And, are X and Y independent?

1.5.6 Variances of Correlated Variables

Let X and Y be two random variables, then

$$\begin{aligned} var(X+Y) &= var(X)+var(Y)+2cov(X,Y) \\ &= var(X)+var(Y)+2\rho\sigma_x\sigma_y \\ var(X-Y) &= var(X)+var(Y)-2cov(X,Y) \\ &= var(X)+var(Y)-2\rho\sigma_x\sigma_y \end{aligned} \tag{1.9}$$

The generalized result:
Let $\sum_{i=1}^{n} X_i = X_1 + X_2 + \cdots + X_n$, then the variance of the linear combination $\sum X_i$ is:

$$\begin{aligned} var\left(\sum_{i=1}^{n} X_i\right) &= \sum_{i=1}^{n} var(X_i) + 2\sum\sum_{i<j} cov(X_i,X_j) \\ &= \sum_{i=1}^{n} var(X_i) + 2\sum\sum_{i<j} \rho_{ij}\sigma_i\sigma_j \end{aligned} \tag{1.10}$$

Example:
what is the var$(X_1 + X_2 + X_3)$?

1.5.7 Higher Moments of Probability Distributions

In the previous subsection, we have already discussed about mean, variance, and covariance as the measures of the first and second moments of univariate and multivariate PDFs. Besides the first two moments, we are occasionally interested in the higher moments such as the third and fourth moments which are normally applied in studying the “Shape" of the distribution. In general, the r^{th} moments about the mean is defined as

$$r^{th}\textbf{moment} : E(X-\mu)^r$$

By the definition of r^{th} moments, we can easily define the third and fourth moments as:

Third moment:

$$E(X-\mu)^3$$

Fourth moment:

$$E(X-\mu)^4$$

We can study the shape of the distribution by calculating **skewness** and **kurtosis**.

SKEWNESS is a measure of the asymmetry of the probability distribution of a real-valued random variable about its mean.

One measure of skewness is defined as:

$$\begin{aligned} S &= \frac{E(X-\mu)^3}{\sigma^3} \\ &= \frac{\text{third moment about the mean}}{\text{cube of the standard deviation}} \end{aligned} \tag{1.11}$$

KURTOSIS is a measure of the peakedness of the probability distribution of a real-valued random variable

We can also measure kurtosis as:

$$\begin{aligned} S &= \frac{E(X-\mu)^4}{\sigma^4} \\ &= \frac{\text{fourth moment about the mean}}{\text{square of the second moment}} \end{aligned} \tag{1.12}$$

♣ **Platykurtic (fat or short-tailed)** $\Longrightarrow$ PDFs with Kurtosis < 3
♣ **Leptokurtic (slim or long-tailed)** $\Longrightarrow$ PDFs with Kurtosis > 3
♣ **Mesokurtic (which is the normal distribution)** $\Longrightarrow$ PDFs with Kurtosis $= 3$

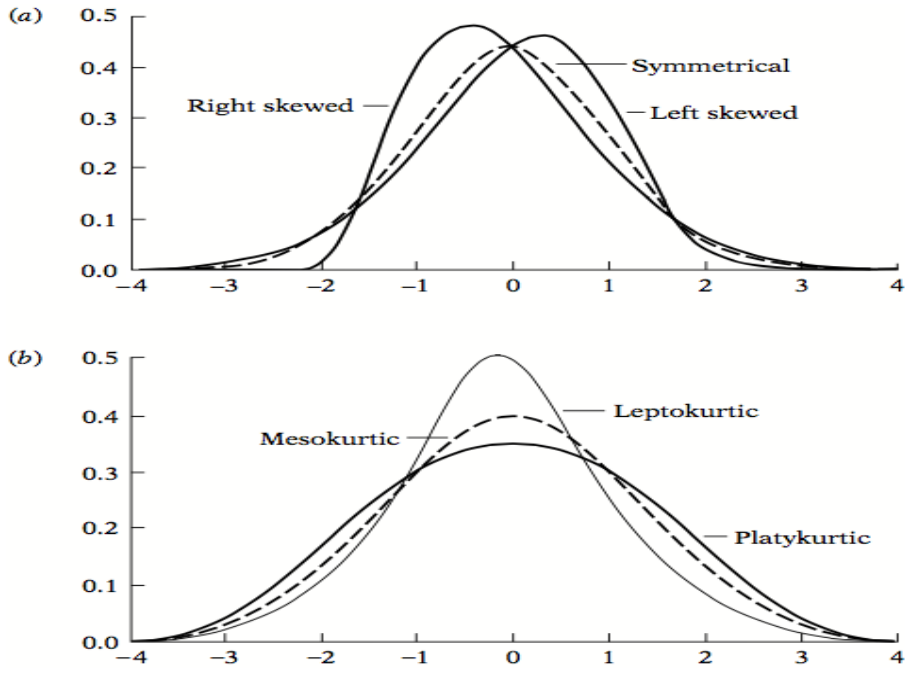


Figure 1.1: (a) Skewness; (b)Kurtosis

1.6 Some important probability distribution

1.6.1 Normal Distribution

A continuous random variable X has a normal distribution with mean μ and variance σ^2 if its probability density function (pdf) is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\right) \quad \text{where} \quad -\infty < x < \infty$$

NOTE: The normal distribution can be described by two parameters

- μ = The mean of the distribution.
- σ = The standard deviation of the distribution.

Therefore, changing the values of μ and σ alter the positions and shapes of the distributions.

If X is Normally distributed with mean μ and standard deviation σ, we can write it as:

$$X \sim N(\mu, \sigma^2)$$

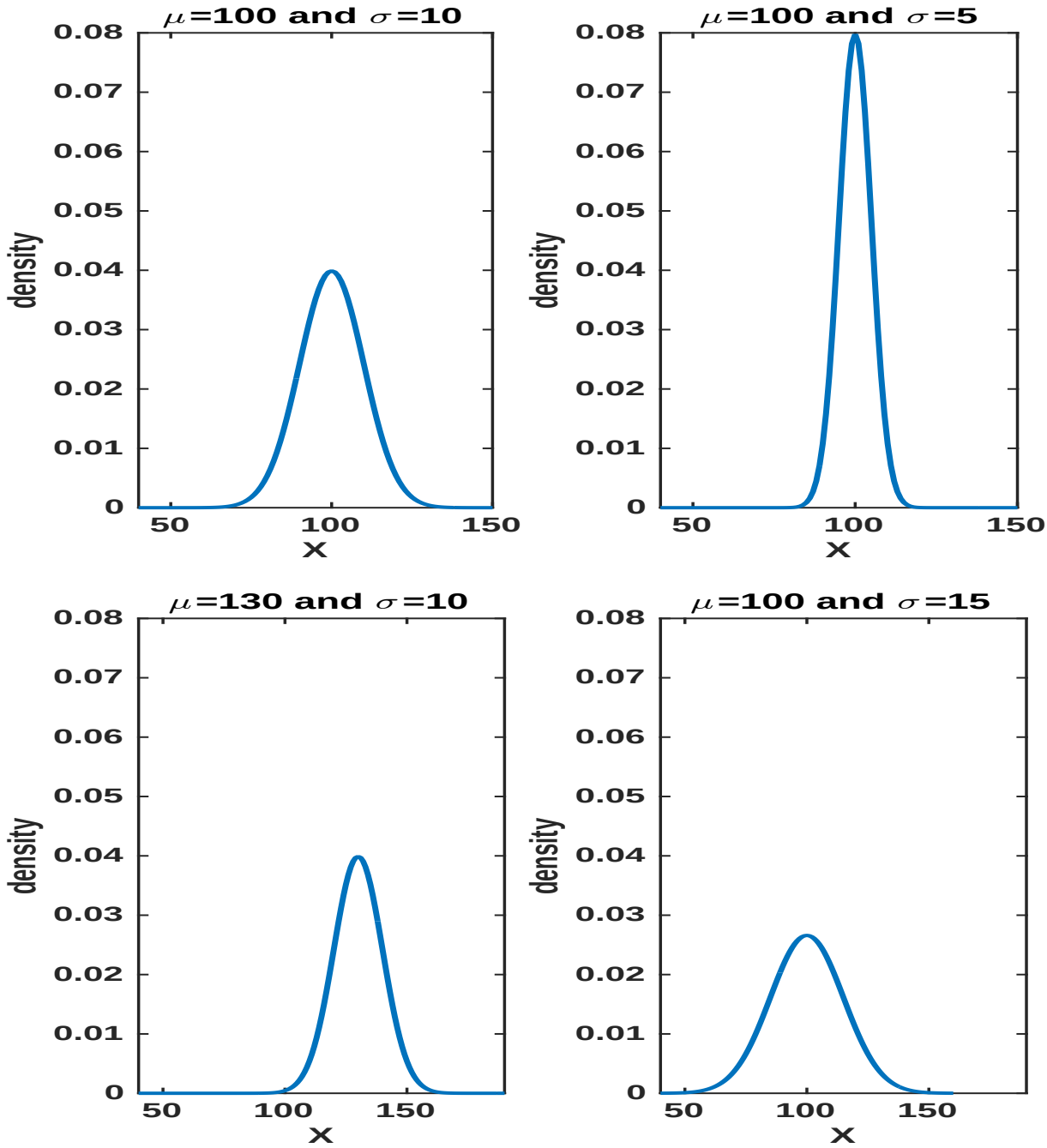


Figure 1.2: Compare the mean and standard deviation of the normal distribution

The properties of the normal distribution.

★ It is symmetrical around its mean value.

★ About 68 percent of the area under the normal distribution lies between the value $\mu \pm \sigma$

About 95 percent of the area under the normal distribution lies between the value $\mu \pm 2\sigma$

About 99.7 percent of the area under the normal distribution lies between the value $\mu \pm 3\sigma$ (as shown in figure 2)

★ We can convert the given normally distributed variable X with mean μ and σ^2 into the standardized normal variable Z by calculating Z where Z can be defined as:

$$Z =$$

With the standardized normal variable Z, we can rewrite the normal pdf as:

$$f(Z) =$$

In sum, you can see that we convert the given normally distributed variable X into the standardized normal variable by:
(i) Subtracting the mean μ
(ii)Dividing by the standard deviation σ

♡ Subtracting the mean re-centers the distribution on zero.

♡ Dividing by the standard deviation re-scales the distribution so it has standard deviation 1.

It should be remarked that it mean value is zero and its variance is unity for any standardized variable.

By convention, we can denote a normally distributed variable X with zero mean and unit variance as

$$X \sim N(0,1)$$

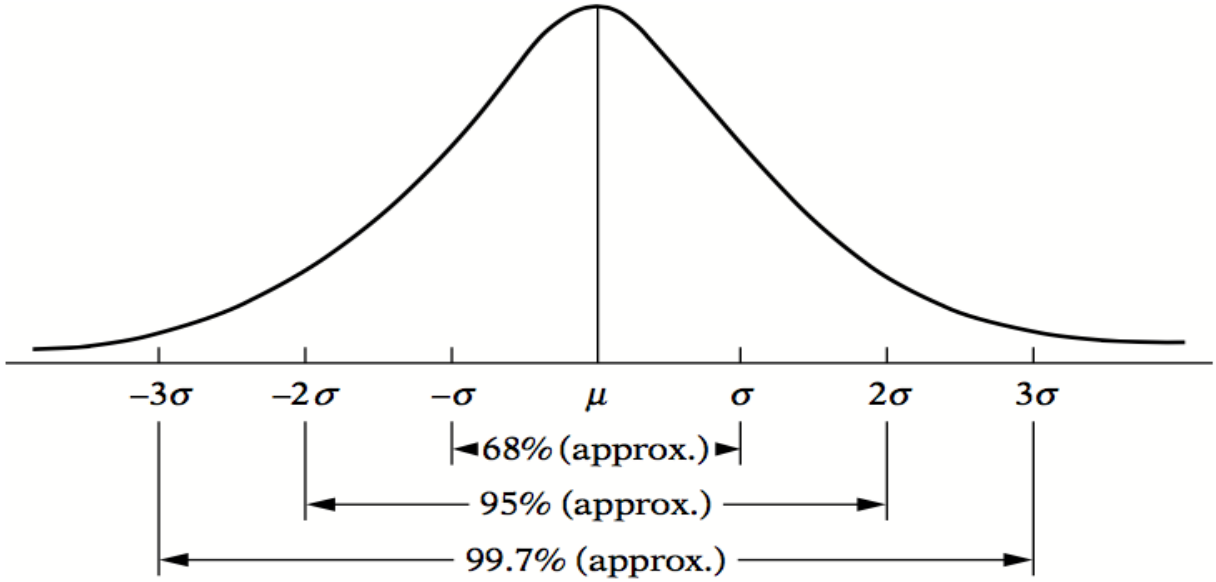


Figure 1.3: Areas under the normal distribution

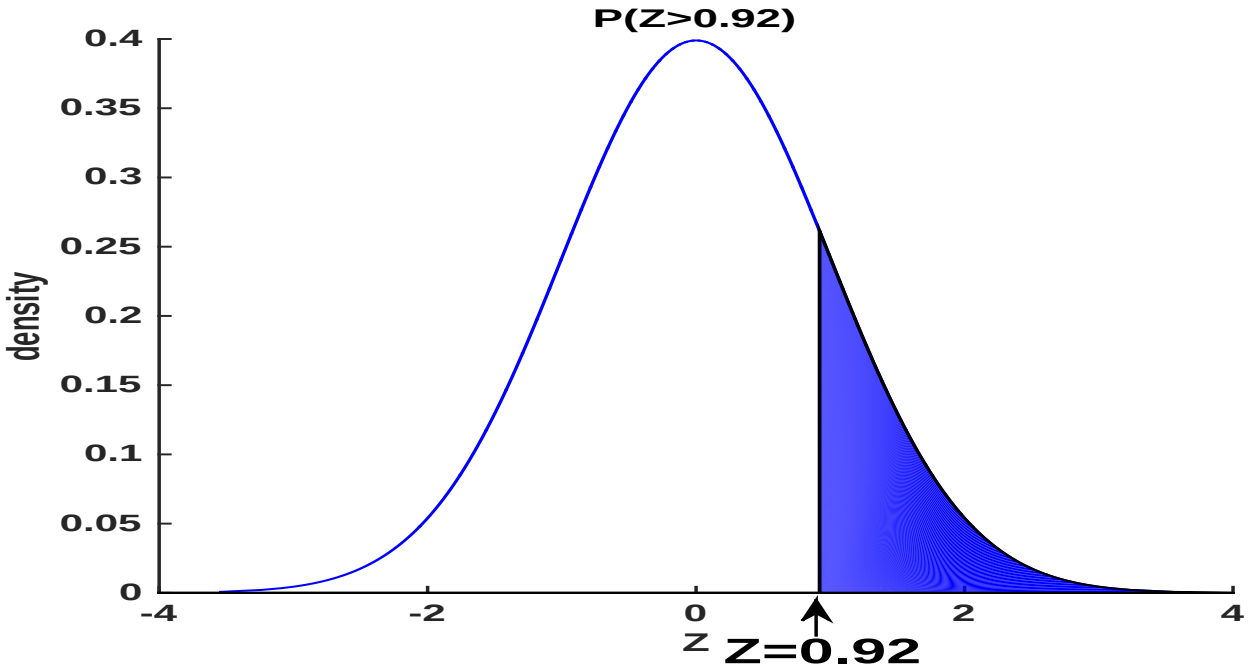


Figure 1.4: If Z ∼ N(0,1), the probability that P(Z>0.92)

Example If $Z \sim N(0,1)$ what is $P(Z>0.92)$?

Example If $Z \sim N(0,1)$ what is $P(-0.64<Z<0.43)$?

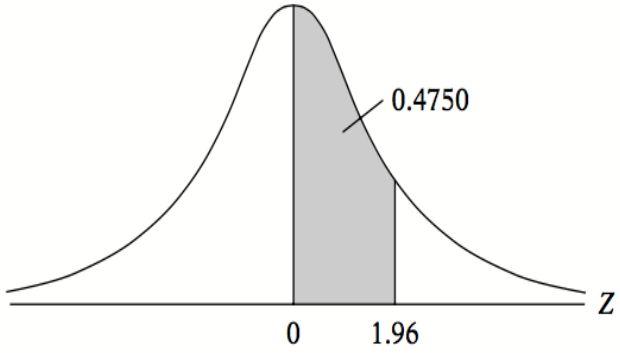
Example If $X \sim N(3500,500^2)$ what is $P(X<3100)$?

AREAS UNDER THE STANDARDIZED NORMAL DISTRIBUTION

Example

$\Pr(0 \le Z \le 1.96) = 0.4750$

$\Pr(Z \ge 1.96) = 0.5 - 0.4750 = 0.025$



Z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.0000	.0040	.0080	.0120	.0160	.0199	.0239	.0279	.0319	.0359
0.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0753
0.2	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141
0.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517
0.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879
0.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
0.6	.2257	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2517	.2549
0.7	.2580	.2611	.2642	.2673	.2704	.2734	.2764	.2794	.2823	.2852
0.8	.2881	.2910	.2939	.2967	.2995	.3023	.3051	.3078	.3106	.3133
0.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389
1.0	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621
1.1	.3643	.3665	.3686	.3708	.3729	.3749	.3770	.3790	.3810	.3830
1.2	.3849	.3869	.3888	.3907	.3925	.3944	.3962	.3980	.3997	.4015
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131	.4147	.4162	.4177
1.4	.4192	.4207	.4222	.4236	.4251	.4265	.4279	.4292	.4306	.4319
1.5	.4332	.4345	.4357	.4370	.4382	.4394	.4406	.4418	.4429	.4441
1.6	.4452	.4463	.4474	.4484	.4495	.4505	.4515	.4525	.4535	.4545
1.7	.4454	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4625	.4633
1.8	.4641	.4649	.4656	.4664	.4671	.4678	.4686	.4693	.4699	.4706
1.9	.4713	.4719	.4726	.4732	.4738	.4744	.4750	.4756	.4761	.4767
2.0	.4772	.4778	.4783	.4788	.4793	.4798	.4803	.4808	.4812	.4817
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4846	.4850	.4854	.4857
2.2	.4861	.4864	.4868	.4871	.4875	.4878	.4881	.4884	.4887	.4890
2.3	.4893	.4896	.4898	.4901	.4904	.4906	.4909	.4911	.4913	.4916
2.4	.4918	.4920	.4922	.4925	.4927	.4929	.4931	.4932	.4934	.4936
2.5	.4938	.4940	.4941	.4943	.4945	.4946	.4948	.4949	.4951	.4952
2.6	.4953	.4955	.4956	.4957	.4959	.4960	.4961	.4962	.4963	.4964
2.7	.4965	.4966	.4967	.4968	.4969	.4970	.4971	.4972	.4973	.4974
2.8	.4974	.4975	.4976	.4977	.4977	.4978	.4979	.4979	.4980	.4981
2.9	.4981	.4982	.4982	.4983	.4984	.4984	.4985	.4985	.4986	.4986
3.0	.4987	.4987	.4987	.4988	.4988	.4989	.4989	.4989	.4990	.4990

Let $X_1 \sim N(\mu_1, \sigma_1^2)$ and $X_2 \sim N(\mu_2, \sigma_2^2)$ and assume that X_1 and X_2 are independent. If we have the linear combination between X_1 and X_2 where we can write it as:

$$Y = aX_1 + bX_2,$$

where a and b are the constant terms. Then

$$Y \sim N\left[(a\mu_1 + b\mu_2), (a^2\sigma_1^2 + b^2\sigma_2^2)\right]$$

In other words, **a linear combination of normally distributed variables is itself normally distributed.**

Central limit theorem Let $X_1, X_2, ..., X_n$ denote n independent random variables and

$$X_i \sim N(\mu, \sigma)$$

Let $\bar{X}$=$\sum \frac{X_i}{n}$, then as n increases indefinitely (i.e, n $\to \infty$),

The third and fourth moments of the normal distribution:

Third moment: $E(X-\mu)^3 = 0$

Fourth moment: $E(X-\mu)^4 = 3\sigma^4$

1.6.2 The χ^2 (Chi-Square) Distribution

Let $Z_1, Z_2, ..., Z_k$ be **independent standardized normal variables**. Then the quantity

$$Z = \sum_{i=1}^{k} Z_i^2$$

is said to possess the χ^2 with k degree of freedom (df)

Properties of the χ^2 distribution are as follows:

1.The χ^2 distribution is a skewed distribution where the degree of the skewness depending on the df. As the number of df increases, the distribution becomes more symmetrical. For the df excess of 100, the variable

$$\sqrt{2\chi^2} - \sqrt{(2k-1)}$$

can be converted to a standardized normal variable, where k is the df.

2. The mean of the chi-square distribution is k, and its variance is 2k, where k is the df.

3. If Z_1 and $Z_2$are two independent chi-square variables with k_1 and k_2 df, then the sum of $Z_1 + Z_2$ is also a chi-square with $df = k_1 + k2$

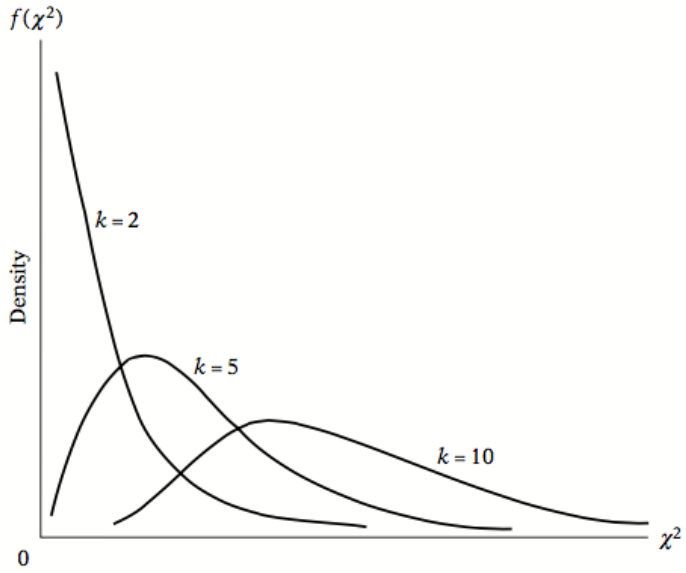


Figure 1.5: Density function of the χ^2 variable

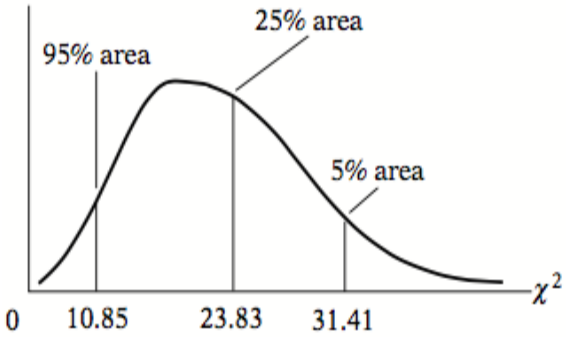
UPPER PERCENTAGE POINTS OF THE χ^2 DISTRIBUTION

Example

$\Pr(\chi^2 > 10.85) = 0.95$

$\Pr(\chi^2 > 23.83) = 0.25$ for df = 20

$\Pr(\chi^2 > 31.41) = 0.05$



Degrees of freedom \ Pr	.995	.990	.975	.950	.900
1	392704×10^{-10}	157088×10^{-9}	982069×10^{-9}	393214×10^{-8}	.0157908
2	.0100251	.0201007	.0506356	.102587	.210720
3	.0717212	.114832	.215795	.351846	.584375
4	.206990	.297110	.484419	.710721	1.063623
5	.411740	.554300	.831211	1.145476	1.61031
6	.675727	.872085	1.237347	1.63539	2.20413
7	.989265	1.239043	1.68987	2.16735	2.83311
8	1.344419	1.646482	2.17973	2.73264	3.48954
9	1.734926	2.087912	2.70039	3.32511	4.16816
10	2.15585	2.55821	3.24697	3.94030	4.86518
11	2.60321	3.05347	3.81575	4.57481	5.57779
12	3.07382	3.57056	4.40379	5.22603	6.30380
13	3.56503	4.10691	5.00874	5.89186	7.04150
14	4.07468	4.66043	5.62872	6.57063	7.78953
15	4.60094	5.22935	6.26214	7.26094	8.54675
16	5.14224	5.81221	6.90766	7.96164	9.31223
17	5.69724	6.40776	7.56418	8.67176	10.0852
18	6.26481	7.01491	8.23075	9.39046	10.8649
19	6.84398	7.63273	8.90655	10.1170	11.6509
20	7.43386	8.26040	9.59083	10.8508	12.4426
21	8.03366	8.89720	10.28293	11.5913	13.2396
22	8.64272	9.54249	10.9823	12.3380	14.0415
23	9.26042	10.19567	11.6885	13.0905	14.8479
24	9.88623	10.8564	12.4011	13.8484	15.6587
25	10.5197	11.5240	13.1197	14.6114	16.4734
26	11.1603	12.1981	13.8439	15.3791	17.2919
27	11.8076	12.8786	14.5733	16.1513	18.1138
28	12.4613	13.5648	15.3079	16.9279	18.9392
29	13.1211	14.2565	16.0471	17.7083	19.7677
30	13.7867	14.9535	16.7908	18.4926	20.5992
40	20.7065	22.1643	24.4331	26.5093	29.0505
50	27.9907	29.7067	32.3574	34.7642	37.6886
60	35.5346	37.4848	40.4817	43.1879	46.4589
70	43.2752	45.4418	48.7576	51.7393	55.3290
80	51.1720	53.5400	57.1532	60.3915	64.2778
90	59.1963	61.7541	65.6466	69.1260	73.2912
100*	67.3276	70.0648	74.2219	77.9295	82.3581

*For df greater than 100 the expression $\sqrt{2\chi^2} - \sqrt{(2k-1)} = Z$ follows the standardized normal distribution, where k represents the degrees of freedom.

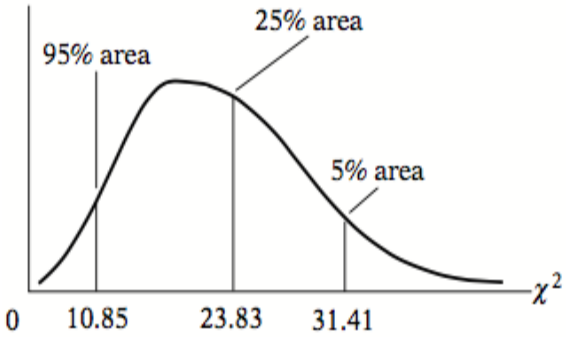
UPPER PERCENTAGE POINTS OF THE χ^2 DISTRIBUTION

Example

$\Pr(\chi^2 > 10.85) = 0.95$

$\Pr(\chi^2 > 23.83) = 0.25$ for df = 20

$\Pr(\chi^2 > 31.41) = 0.05$



.750	.500	.250	.100	.050	.025	.010	.005
.1015308	.454937	1.32330	2.70554	3.84146	5.02389	6.63490	7.87944
.575364	1.38629	2.77259	4.60517	5.99147	7.37776	9.21034	10.5966
1.212534	2.36597	4.10835	6.25139	7.81473	9.34840	11.3449	12.8381
1.92255	3.35670	5.38527	7.77944	9.48773	11.1433	13.2767	14.8602
2.67460	4.35146	6.62568	9.23635	11.0705	12.8325	15.0863	16.7496
3.45460	5.34812	7.84080	10.6446	12.5916	14.4494	16.8119	18.5476
4.25485	6.34581	9.03715	12.0170	14.0671	16.0128	18.4753	20.2777
5.07064	7.34412	10.2188	13.3616	15.5073	17.5346	20.0902	21.9550
5.89883	8.34283	11.3887	14.6837	16.9190	19.0228	21.6660	23.5893
6.73720	9.34182	12.5489	15.9871	18.3070	20.4831	23.2093	25.1882
7.58412	10.3410	13.7007	17.2750	19.6751	21.9200	24.7250	26.7569
8.43842	11.3403	14.8454	18.5494	21.0261	23.3367	26.2170	28.2995
9.29906	12.3398	15.9839	19.8119	22.3621	24.7356	27.6883	29.8194
10.1653	13.3393	17.1170	21.0642	23.6848	26.1190	29.1413	31.3193
11.0365	14.3389	18.2451	22.3072	24.9958	27.4884	30.5779	32.8013
11.9122	15.3385	19.3688	23.5418	26.2962	28.8454	31.9999	34.2672
12.7919	16.3381	20.4887	24.7690	27.5871	30.1910	33.4087	35.7185
13.6753	17.3379	21.6049	25.9894	28.8693	31.5264	34.8053	37.1564
14.5620	18.3376	22.7178	27.2036	30.1435	32.8523	36.1908	38.5822
15.4518	19.3374	23.8277	28.4120	31.4104	34.1696	37.5662	39.9968
16.3444	20.3372	24.9348	29.6151	32.6705	35.4789	38.9321	41.4010
17.2396	21.3370	26.0393	30.8133	33.9244	36.7807	40.2894	42.7956
18.1373	22.3369	27.1413	32.0069	35.1725	38.0757	41.6384	44.1813
19.0372	23.3367	28.2412	33.1963	36.4151	39.3641	42.9798	45.5585
19.9393	24.3366	29.3389	34.3816	37.6525	40.6465	44.3141	46.9278
20.8434	25.3364	30.4345	35.5631	38.8852	41.9232	45.6417	48.2899
21.7494	26.3363	31.5284	36.7412	40.1133	43.1944	46.9630	49.6449
22.6572	27.3363	32.6205	37.9159	41.3372	44.4607	48.2782	50.9933
23.5666	28.3362	33.7109	39.0875	42.5569	45.7222	49.5879	52.3356
24.4776	29.3360	34.7998	40.2560	43.7729	46.9792	50.8922	53.6720
33.6603	39.3354	45.6160	51.8050	55.7585	59.3417	63.6907	66.7659
42.9421	49.3349	56.3336	63.1671	67.5048	71.4202	76.1539	79.4900
52.2938	59.3347	66.9814	74.3970	79.0819	83.2976	88.3794	91.9517
61.6983	69.3344	77.5766	85.5271	90.5312	95.0231	100.425	104.215
71.1445	79.3343	88.1303	96.5782	101.879	106.629	112.329	116.321
80.6247	89.3342	98.6499	107.565	113.145	118.136	124.116	128.299
90.1332	99.3341	109.141	118.498	124.342	129.561	135.807	140.169

Source: Abridged from E. S. Pearson and H. O. Hartley, eds., *Biometrika Tables for Statisticians,* vol. 1, 3d ed., table 8, Cambridge University Press, New York, 1966. Reproduced by permission of the editors and trustees of *Biometrika.*

1.6.3 Student's t Distribution

If Z_1 is a standardized normal variable and Z_2 is the chi-square distribution with k degree of freedom and is distributed independently of Z_1, then the Student's t distribution (t_k) with k degree of freedom can be represented as

$$\begin{aligned} t &= \frac{Z_1}{\sqrt{(Z_2/k)}} \\ &= \frac{Z_1\sqrt{k}}{\sqrt{Z_2}} \end{aligned} \tag{1.13}$$

Properties of the Student's t distribution are as follows:

1. The t distribution is symmetrical, BUT it is flatter than the normal distribution. However, as the df increase, the t distribution is converted to the normal distribution.

2. The mean of the t distribution is zero, and the variance is $\frac{k}{k-2}$

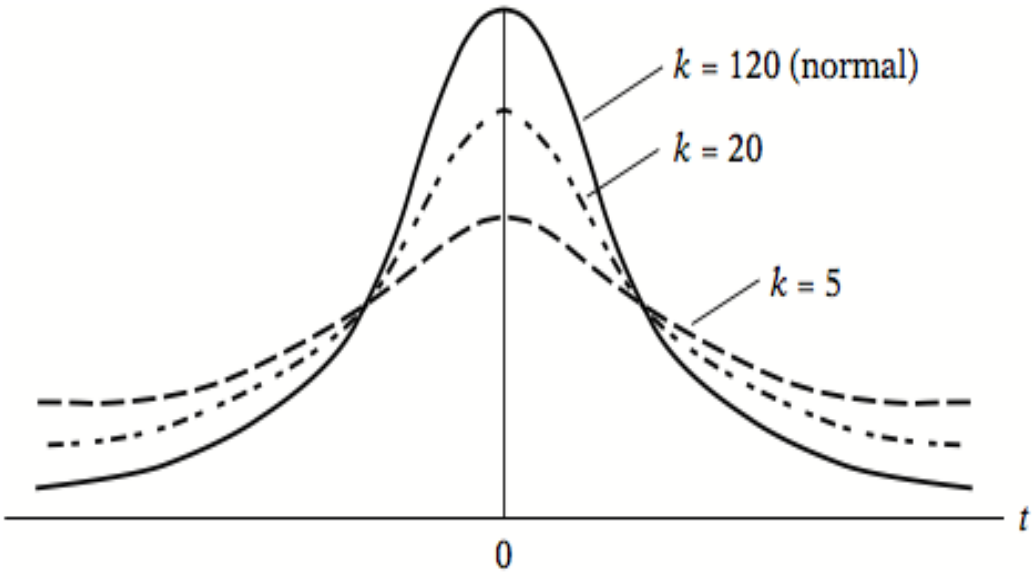


Figure 1.6: Density function of the student's t distribution

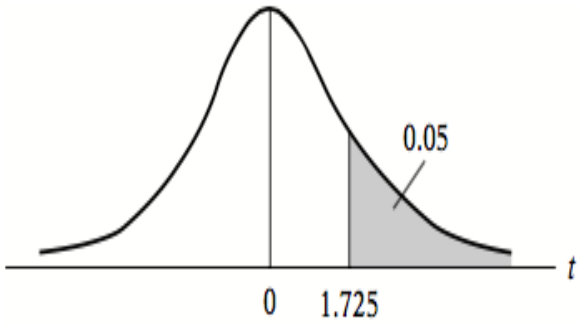
PERCENTAGE POINTS OF THE t DISTRIBUTION

Example

$\Pr(t > 2.086) = 0.025$

$\Pr(t > 1.725) = 0.05$ for df = 20

$\Pr(|t| > 1.725) = 0.10$



df \ Pr	0.25 0.50	0.10 0.20	0.05 0.10	0.025 0.05	0.01 0.02	0.005 0.010	0.001 0.002
1	1.000	3.078	6.314	12.706	31.821	63.657	318.31
2	0.816	1.886	2.920	4.303	6.965	9.925	22.327
3	0.765	1.638	2.353	3.182	4.541	5.841	10.214
4	0.741	1.533	2.132	2.776	3.747	4.604	7.173
5	0.727	1.476	2.015	2.571	3.365	4.032	5.893
6	0.718	1.440	1.943	2.447	3.143	3.707	5.208
7	0.711	1.415	1.895	2.365	2.998	3.499	4.785
8	0.706	1.397	1.860	2.306	2.896	3.355	4.501
9	0.703	1.383	1.833	2.262	2.821	3.250	4.297
10	0.700	1.372	1.812	2.228	2.764	3.169	4.144
11	0.697	1.363	1.796	2.201	2.718	3.106	4.025
12	0.695	1.356	1.782	2.179	2.681	3.055	3.930
13	0.694	1.350	1.771	2.160	2.650	3.012	3.852
14	0.692	1.345	1.761	2.145	2.624	2.977	3.787
15	0.691	1.341	1.753	2.131	2.602	2.947	3.733
16	0.690	1.337	1.746	2.120	2.583	2.921	3.686
17	0.689	1.333	1.740	2.110	2.567	2.898	3.646
18	0.688	1.330	1.734	2.101	2.552	2.878	3.610
19	0.688	1.328	1.729	2.093	2.539	2.861	3.579
20	0.687	1.325	1.725	2.086	2.528	2.845	3.552
21	0.686	1.323	1.721	2.080	2.518	2.831	3.527
22	0.686	1.321	1.717	2.074	2.508	2.819	3.505
23	0.685	1.319	1.714	2.069	2.500	2.807	3.485
24	0.685	1.318	1.711	2.064	2.492	2.797	3.467
25	0.684	1.316	1.708	2.060	2.485	2.787	3.450
26	0.684	1.315	1.706	2.056	2.479	2.779	3.435
27	0.684	1.314	1.703	2.052	2.473	2.771	3.421
28	0.683	1.313	1.701	2.048	2.467	2.763	3.408
29	0.683	1.311	1.699	2.045	2.462	2.756	3.396
30	0.683	1.310	1.697	2.042	2.457	2.750	3.385
40	0.681	1.303	1.684	2.021	2.423	2.704	3.307
60	0.679	1.296	1.671	2.000	2.390	2.660	3.232
120	0.677	1.289	1.658	1.980	2.358	2.617	3.160
∞	0.674	1.282	1.645	1.960	2.326	2.576	3.090

1.6.4 The F Distribution

If Z_1 and Z_2 are independently distributed chi-square variables with k_1 and k_2 df, respectively, the (Fisher's) F distribution with k_1 and k_2 df can be written as

$$F = \frac{Z_1/k_1}{Z_2/k_2}$$

The F distribution has the following properties:

1. The F distribution is skewed to the right, but as k_1 and k_2 become large, the F distribution is converted to normal distribution.

2.The mean value of ann F-distributed variable is $\frac{k_2}{(k_2-2)}$, and its variance is

$$\frac{2k_2^2(k_1+k_2-2)}{k_1(k_2-2)^2(k_2-4)}$$

3. The square of a t-distributed random variable with k df is equivalent to an F distribution with 1 and k df.

$$t_k^2 = F_{1,k}$$

4. If the denominator df, k_2, is fairly large, we can get the following relationship

$$k_1 F \sim \chi_{k_1}^2$$

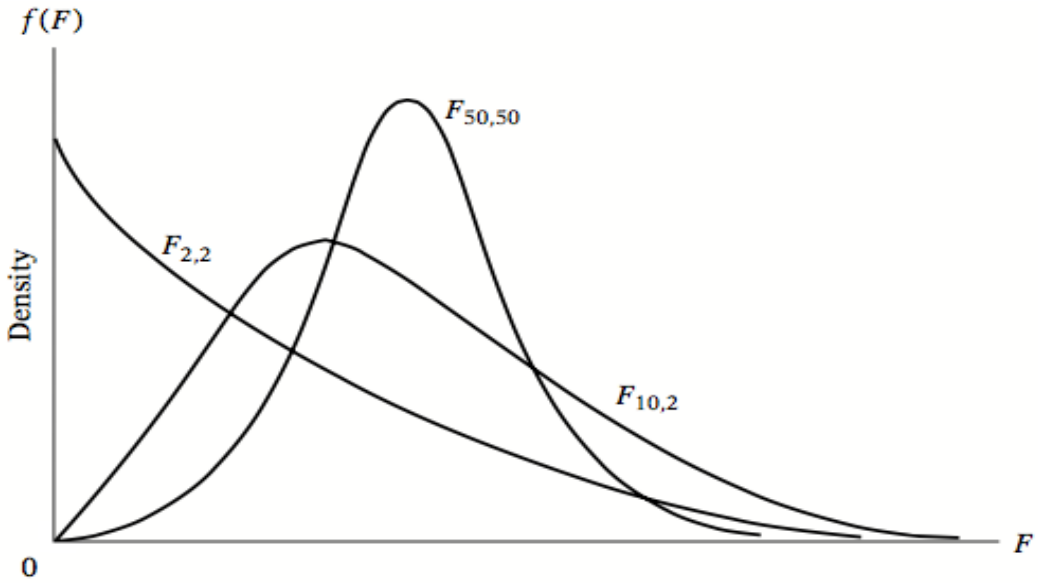


Figure 1.7: Density function of F distribution

2. TWO-VARIABLE REGRESSION ANALYSIS

In order to understand two-variable regression, consider the data given in Table 2.1.
The data in the below table refer to a total **Population** of 42 families with their weekly income (X) and weekly consumption expenditure (Y).

Table 2.1: Weekly family Expenditure (Y), Baht and Income (X), Baht

	X=Weekly family Income, Baht					
	500	600	700	800	900	1000
	360	376	458	610	600	700
	313	475	422	468	531	679
	322	380	498	575	670	730
Y= Weeky	310	382	560	542	630	591
Family Expenditure	390	390	442	588	544	550
	315	425	440	466	565	620
	390	442	-	461	-	695
	400	-	-	-	-	635
Total	2800	2870	2820	3710	3540	5200
Conditional means of Y, $E(Y\|X)$	350	410	470	530	590	650

Notes -

Table 2.2: Conditional Probabilities $p(Y|X_i)$ for the Weekly Family Income (X) and Expenditure (Y)

	X=Weekly family Income, Baht					
	500	600	700	800	900	1000
	1/8	1/7	1/6	1/7	1/6	1/8
	1/8	1/7	1/6	1/7	1/6	1/8
	1/8	1/7	1/6	1/7	1/6	1/8
Y= Weekly	1/8	1/7	1/6	1/7	1/6	1/8
Family Expenditure	1/8	1/7	1/6	1/7	1/6	1/8
	1/8	1/7	1/6	1/7	1/6	1/8
	1/8	1/7	-	1/7	-	1/8
	1/8	-	-	-	-	1/8
Conditional means of Y, $E(Y\|X)$	350	410	470	530	590	650

Notes -

Conditional expected value of weekly consumption expenditure given the income level =X , $E(Y|X)$

Unconditional expected value ,$E(Y)$

Figure 2.1: Conditional Distribution of Expenditure for Various Levels of Income

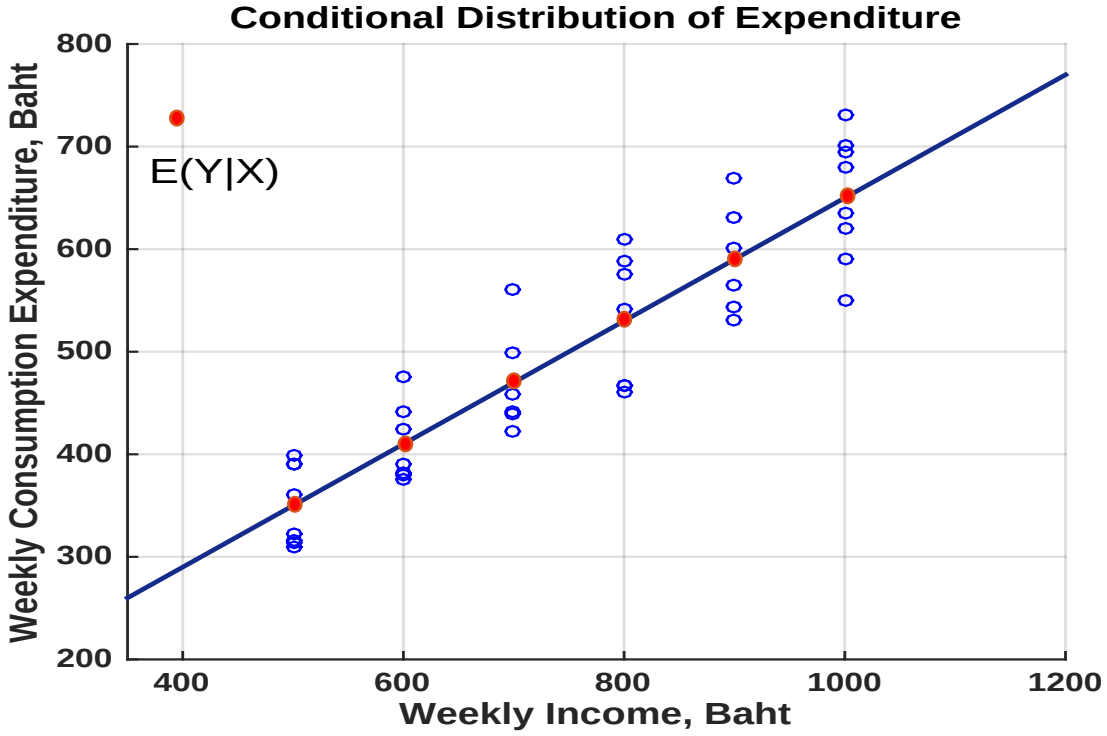
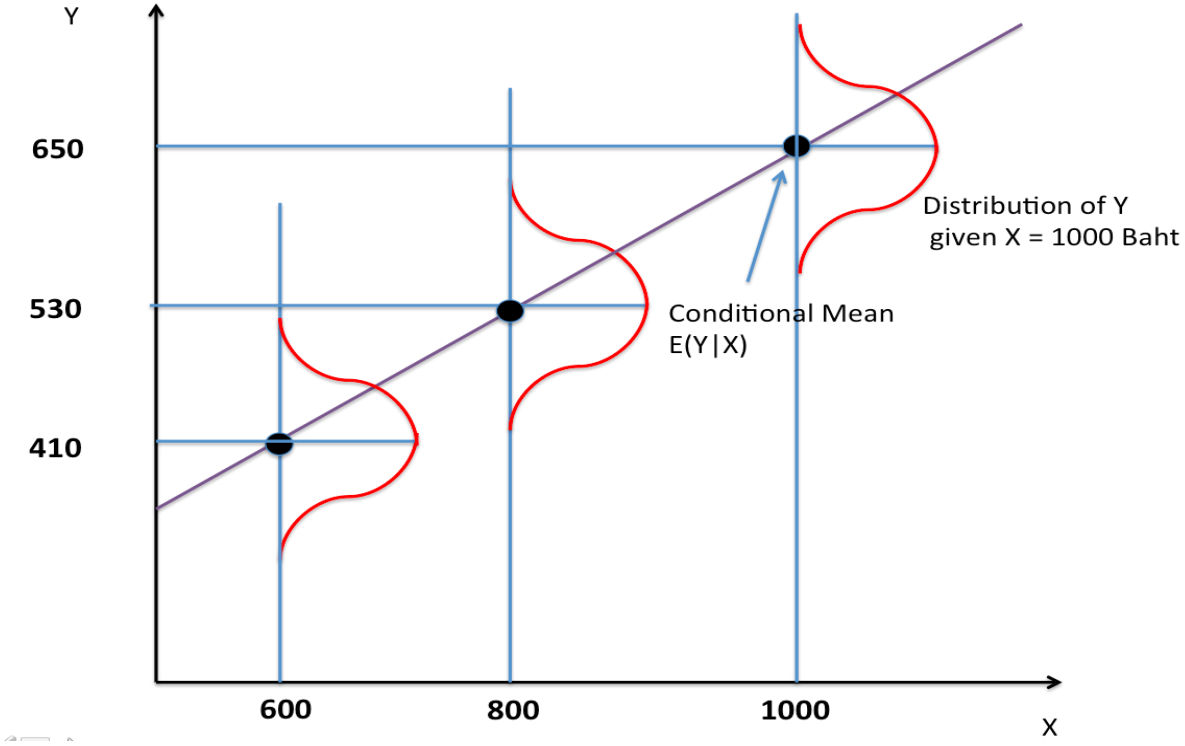


Figure 2.2: Population Regression Line (PRL)



2.1 The Concept of Population Regression Function (PRF)

The population regression function (PRF) can be written as the function of X_i:

2.1.1 What form does the function f(X) assume?

If we assume the PRF $E(Y|X_i)$ is a linear function of X_i, we get

$$E(Y|X_i) = \beta_1 + \beta_2 X_i$$

2.1.2 What is the meaning of the term LINEAR?

LINEARITY in the variables

LINEARITY in the parameters

2.2 Stochastic Specification of PRF

We can write the **deviation** of an individual Y_i around it expected value as follows:

2.2.1 The roles of the stochastic disturbance term

1. Vagueness of theory

2. Unavailability of data

3. Core variables versus peripheral variables

4. Intrinsic randomness in human behavior

5. Poor proxy variable

6. Principle of parsimony

7.Wrong functional form

2.3 The Sample Regression Function (SRF)

As mentioned, in the real situation, we cannot find out all the population of Y values corresponding to the fixed X's. We only have a sample of Y values corresponding to some fixed X's.

Therefore, our goal in this section is to estimate the population regression line (PRF) on the basis of the **SAMPLE INFORMATION.**

As a result, for the fixed X's as given in table 2.1, we only have a randomly selected sample of Y values. For example, table 2.3 and table 2.4 show a random sample from the population of table 2.1

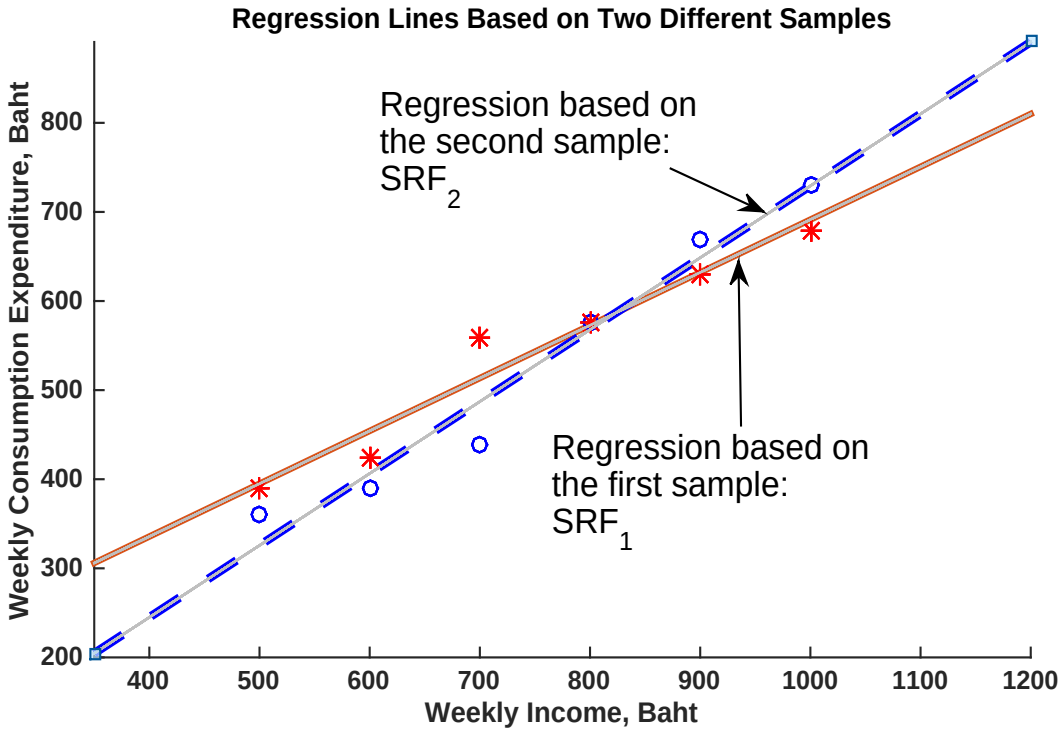
Table 2.3: A Random Sample From the Population

X	Y
500	390
600	425
700	560
800	575
900	630
1000	679

Table 2.4: Another Random Sample From the Population

X	Y
500	360
600	390
700	440
800	575
900	670
1000	730

Figure 2.3: Regression lines based on two different samples



The sample regression function (SRF) can be written as:

$$\hat{Y}_i = \hat{\beta}_1 + \hat{\beta}_2 X_i$$

where $\hat{Y}$ is read as "Y-hat"

$\hat{Y}_i$ = estimator of $E(Y|X_i)$

$\hat{\beta}_1$=estimator of β_1

$\hat{\beta}_2$=estimator of β_2

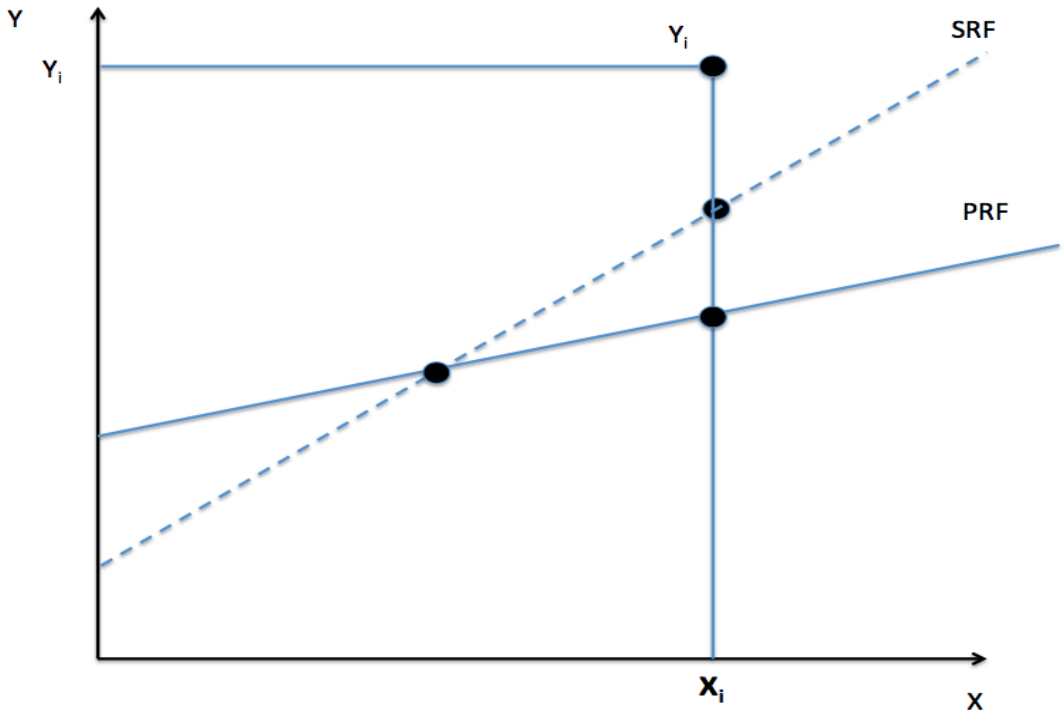
We can express the SRF in its stochastic form as follows:

$$Y_i = \hat{\beta}_1 + \hat{\beta}_2 X_i + \hat{\mu}_i$$

In sum, our ultimate goal is to estimate
the PRF

on the basis of
the SRF

Figure 2.4: Sample and Population Regression Lines



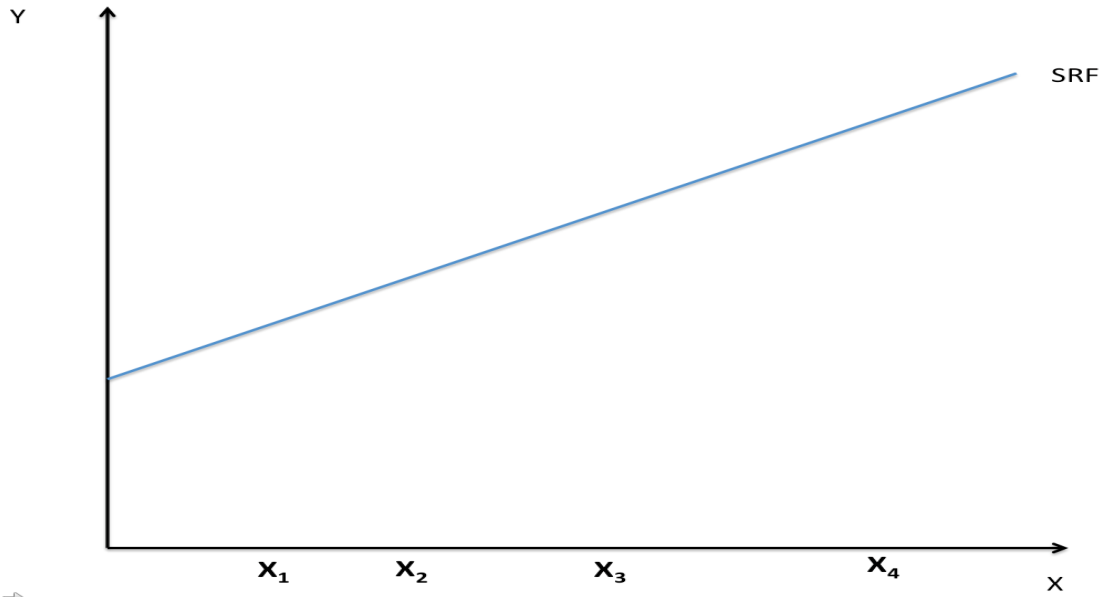
3. REGRESSION:THE PROBLEM OF ESTIMATION

As mentioned in the previous chapter, our main objective is to estimate the population regression function (PRF) based on the basis of the sample regression function (SRF) as accurately as possible.

In this chapter, we are going to discuss the method of estimation:Ordinary Least Squares (OLS)

3.1 The Method of Ordinary Least Squares (OLS)

Figure 3.1: Least-Squares Criterion



3.1.1 The Method to Find Out the Least-Squares Estimators: $\hat{\beta}_1$ and $\hat{\beta}_2$

From the SRF:

$$Y_i = \hat{\beta}_1 + \hat{\beta}_2 X_i + \hat{u}_i$$

Now, we obtain the **least-squares estimators**:

$$\begin{aligned} \hat{\beta}_1 &= \frac{\sum X_i^2 \sum Y_i - \sum X_i \sum X_i Y_i}{n\sum X_i^2 - (\sum X_i)^2} \\ &= \bar{Y} - \hat{\beta}_2 \bar{X} \end{aligned} \tag{3.1}$$

$$\hat{\beta}_2 = \frac{n\sum X_i Y_i - \sum Xi \sum Yi}{n\sum X_i^2 - (\sum X_i)^2} \tag{3.2}$$

If we define $\bar{X}$ and $\bar{Y}$ to be the sample means of X and Y. Then:

$$\begin{aligned} x_i &= (X_i - \bar{X}) \\ y_i &= (Y_i - \bar{Y}) \end{aligned} \tag{3.3}$$

We can have the alternative expressions for $\hat{\beta}_2$:

$$\begin{aligned} \hat{\beta}_2 &= \frac{\sum x_i y_i}{\sum x_i^2} \\ &= \frac{\sum x_i Y_i}{\sum X_i^2 - n\bar{X}^2} \\ &= \frac{\sum X_i y_i}{\sum X_i^2 - n\bar{X}^2} \end{aligned} \tag{3.4}$$

Show that

$$\hat{\beta}_2 = \frac{\sum x_i y_i}{\sum x_i^2}$$

EXAMPLE

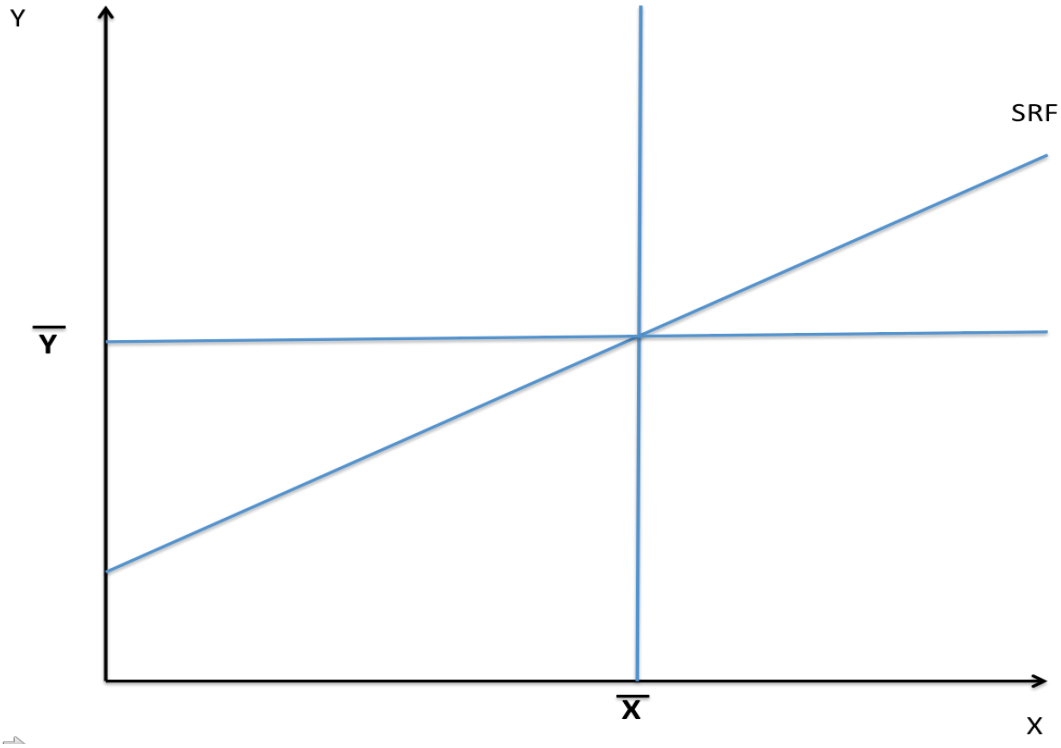
Table 3.1: A Random Sample From the Population

X	Y
500	390
600	425
700	560
800	575
900	630
1000	679

Table 3.2: Raw Data Based on the Sample Data on Table 3.1

	(1) Y_i	(2) X_i	(3) Y_iX_i	(4) X_i^2	(5) $x_i = X_i - \bar{X}$	(6) $y_i = Y_i - \bar{Y}$	(7) x_i^2	(8) x_iy_i	(9) $\hat{Y}_i$	(10) $\hat{u}_i = Y_i - \hat{Y}_i$	(11) $\hat{Y}_i\hat{u}_i$
	390	500	195,000	250,000	−250	−153.17	62,500	38,291.67			
	425	600	255,000	360,000	−150	−118.17	22,500	17,725			
	560	700	392,000	490,000	−50	16.83	2,500	−841.67			
	575	800	460,000	640,000	50	31.83	2,500	1,591.67			
	630	900	567,000	810,000	150	86.83	22,500	13,025			
	679	1,000	679,000	1,000,000	250	135.83	62,500	33,958.33			
Sum	3,259	4,500	2,548,000	3,550,000	0	0	175,000	103,750			
Mean	543.17	750	424,666.67	591,666.670	0	0	29,166.67	17,291.67			

Figure 3.2: Sample Regression Line Based on the Data of Table 3.2



3.1.2 The numerical and statistical properties of OLS estimators

1. The OLS estimators $\hat{\beta}_1$ and $\hat{\beta}_2$ are expressed solely in terms of the observable (Sample size) and quantities (i.e X and Y).

$$\begin{aligned}\hat{\beta}_1 &= \frac{\sum X_i^2 \sum Y_i - \sum X_i \sum X_i Y_i}{n\sum X_i^2 - (\sum X_i)^2} \\ &= \bar{Y} - \hat{\beta}_2 \bar{X}\end{aligned} \tag{3.5}$$

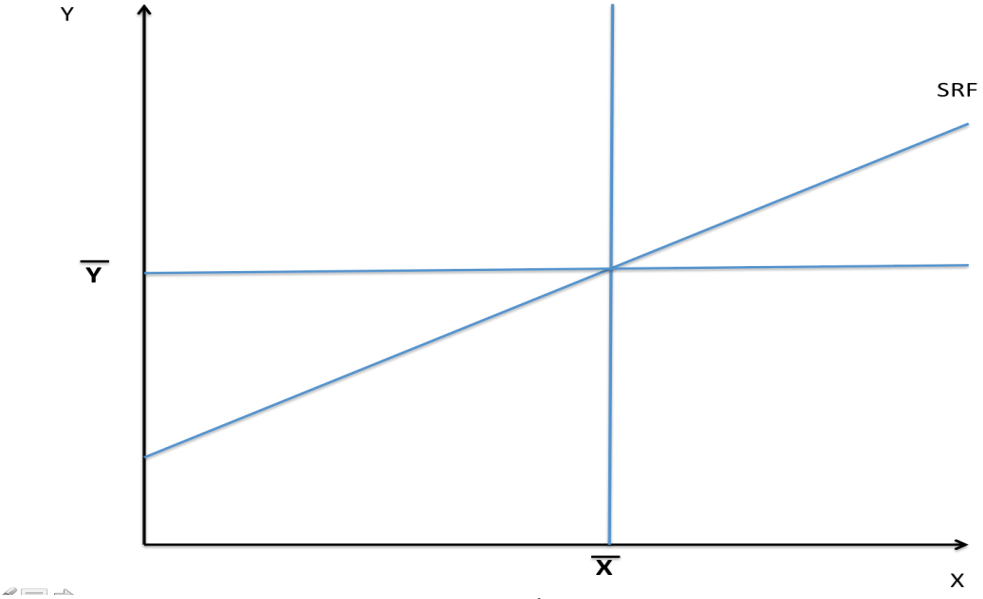
$$\hat{\beta}_2 = \frac{n\sum X_i Y_i - \sum Xi \sum Yi}{n\sum X_i^2 - (\sum X_i)^2} \tag{3.6}$$

2. They are **point estimators**.

3. The regression line has the following properties.

3.1 The sample regression function (SRF) passes through the sample means of Y and X ($\bar{Y}$ and $\bar{X}$).

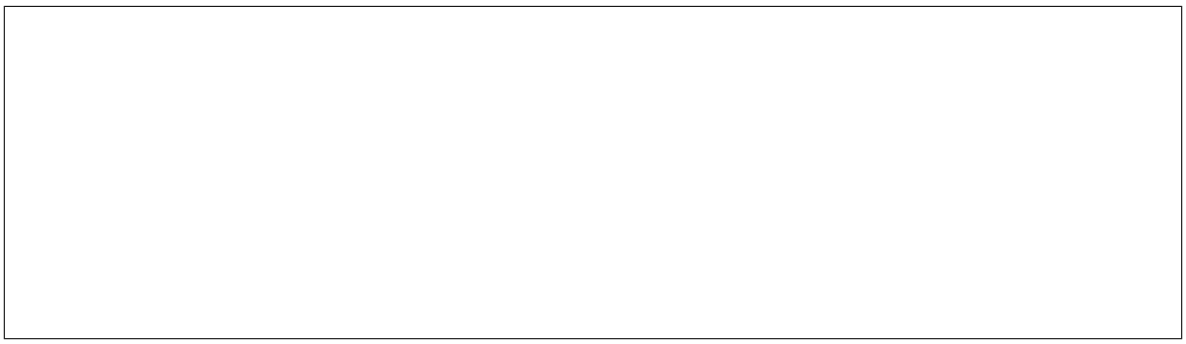
Figure 3.3: The Sample regression Line Passes through the Sample Mean Values of Y and X



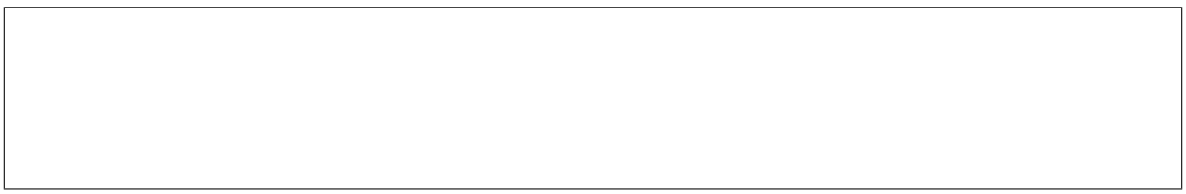
3.2 The mean value of the estimated $Y = \hat{Y}_i$ is equal to the mean value of the actual Y.

3.3. The mean value of the residuals $\hat{u}_i$ is zero.

3.4 The residuals $\hat{u}_i$ are uncorrelated with the predicted Y_i.



3.5 The residuals $\hat{u}_i$ are uncorrelated with X_i.



3.1.3 The Assumptions Underlying the Method of Least Squares

Assumption 1: Linear regression model

$$Y_i = \beta_1 + \beta_2 X_i + u_i$$

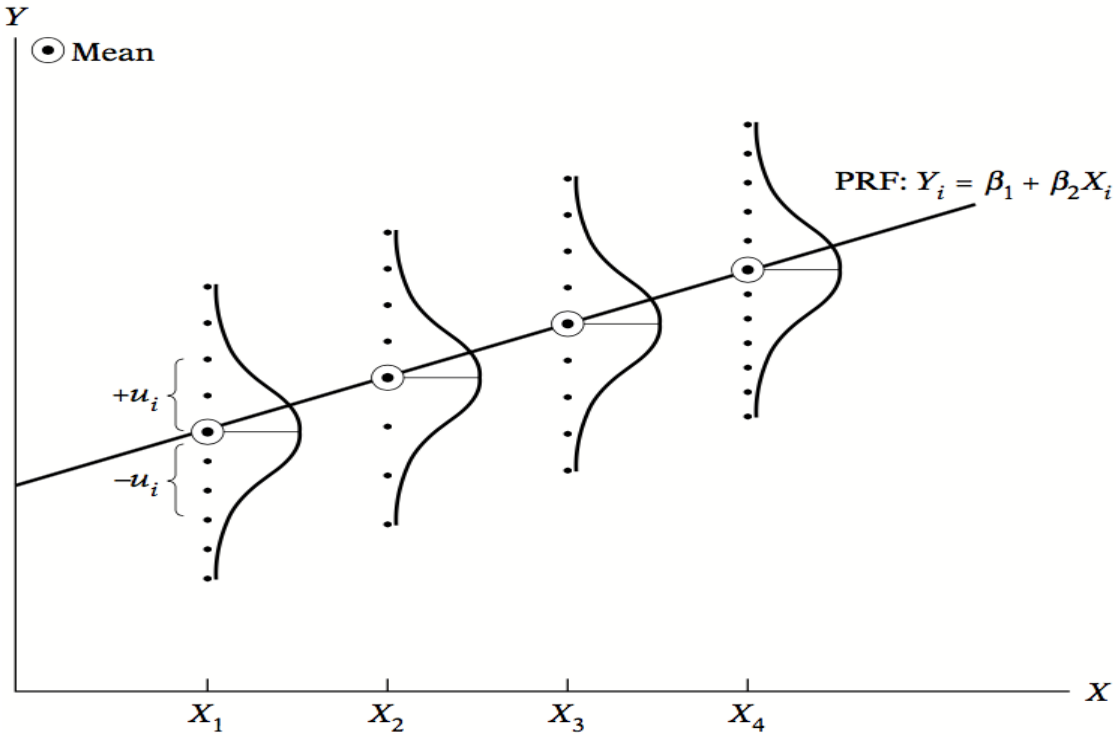
Assumption 2: X values are fixed in repeated sampling

X is assumed to be nonstochastic.

Assumption 3: Zero mean value of disturbance u_i

$$E(u_i|Xi) = 0$$

Figure 3.4: Conditional Distribution of the Disturbances u_i



Assumption 4: Homoscedasticity or Equal Variance of u_i

Figure 3.5: Homoscedasticity

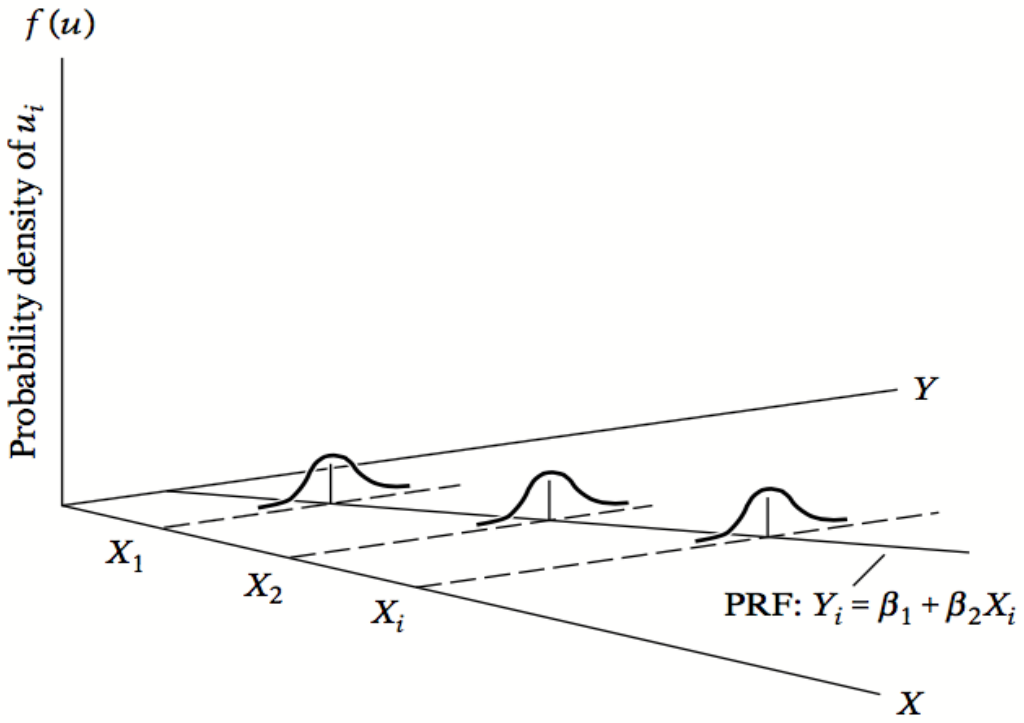
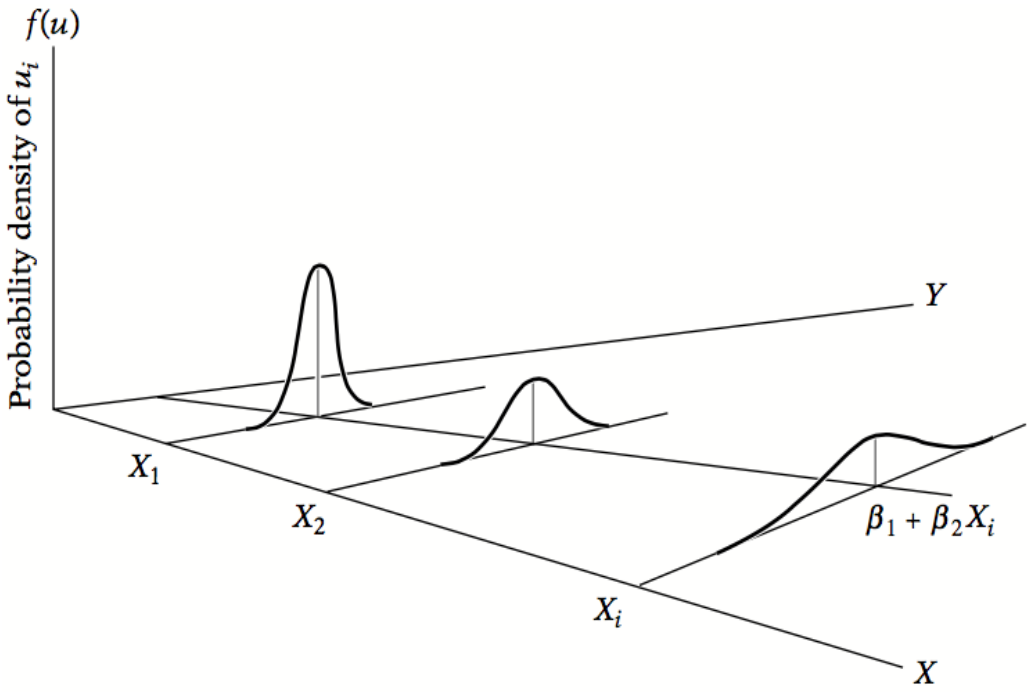


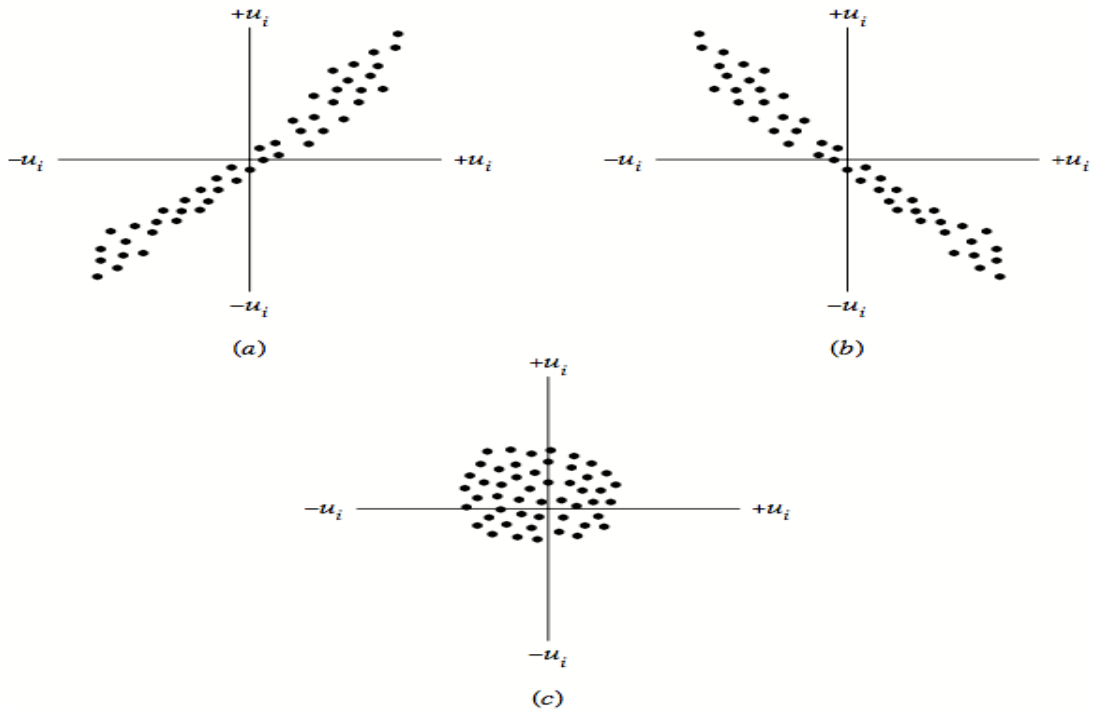
Figure 3.6: Heteroscedasticity



Assumption 5: No Autocorrelation Between the Disturbances

Assumption 6: Zero Covariance Between u_i and X_i

Figure 3.7: Patterns of Correlation Among the disturbances



Assumption 7: The number of observations n must be greater than the number of parameters to be estimated.

Assumption 8:Variability in X values.

Assumption 9: The regression model is correctly specified.

Assumption 10: There is no perfect multicollinearity.

3.1.4 Standard Errors of Least-Squares Estimates

The standard errors of the OLS estimates can be obtained as follows:
We know that

$$\hat{\beta}_2 = \frac{\sum x_i Y_i}{\sum x_i^2} = \sum k_i Yi$$

where

$$k_i = \frac{x_i}{\sum x_i^2}$$

The properties of the weights k_i

1. The k_i are nonstochastic.
2. $\sum k_i = 0$
3. $\sum k_i^2 = \frac{1}{\sum x_i^2}$
4. $\sum k_i x_i = \sum k_i Xi = 1$

Since

$$var(\hat{\beta}_2) = E[\hat{\beta}_2 - E(\hat{\beta}_2)]^2$$

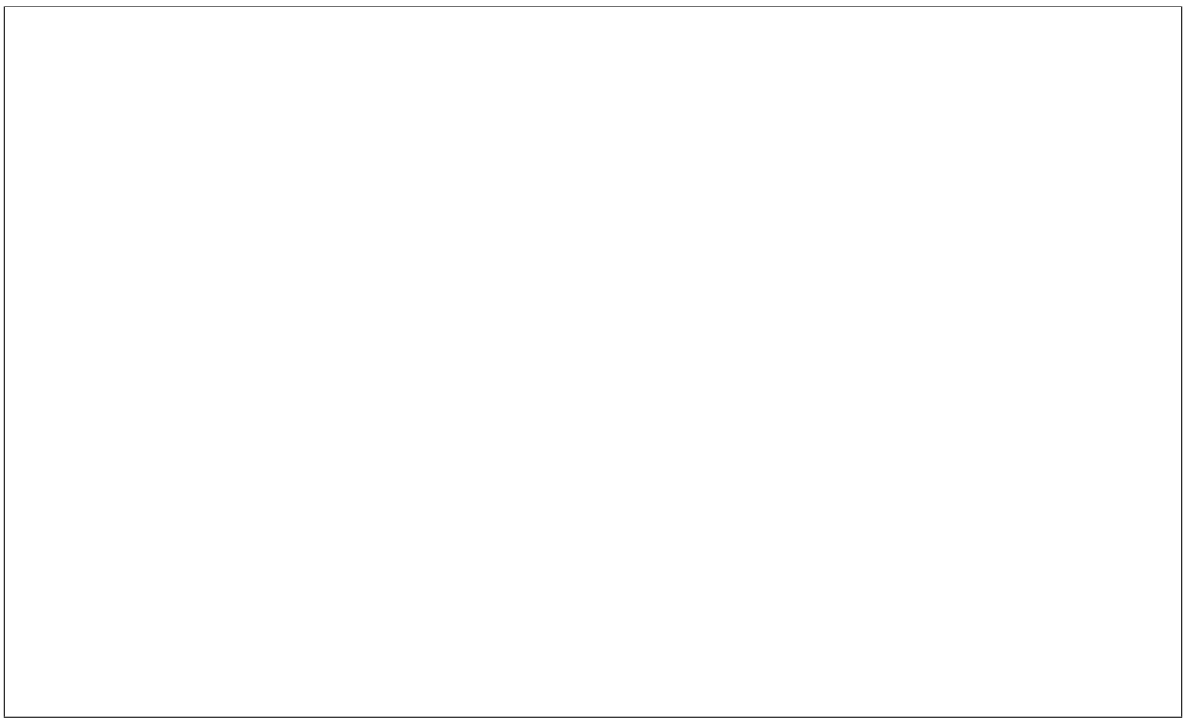
First Step
Find the $E(\hat{\beta}_2)$

Second Step

Using the definition of variance

$$var(\hat{\beta}_2) = E[\hat{\beta}_2 - E(\hat{\beta}_2)]^2$$

The covariance between $\hat{\beta}_1$ and $\hat{\beta}_2$



3.1.5 The Least-Square Estimator of σ^2

In sum, the standard errors of the OLS estimators can be obtained as follow:

$$\begin{aligned} \text{var}(\hat{\beta}_2) &= \frac{\sigma^2}{\sum x_i^2} \\ \text{se}(\hat{\beta}_2) &= \frac{\sigma}{\sqrt{\sum x_i^2}} \end{aligned} \tag{3.7}$$

$$\begin{aligned} \text{var}(\hat{\beta}_1) &= \frac{\sum X_i^2}{n\sum x_i^2}\sigma^2 \\ \text{se}(\hat{\beta}_1) &= \sqrt{\frac{\sum X_i^2}{n\sum x_i^2}}\sigma \end{aligned} \tag{3.8}$$

We can estimate the σ^2 from the data where the formula for the estimated σ^2 is following :

$$\hat{\sigma}^2 = \frac{\sum \hat{u_i}^2}{n-2}$$

where

$$\sum \hat{u_i}^2 = \sum y_i^2 - \hat{\beta_2}^2 \sum x_i^2$$

The alternative expression for computing $\sum \hat{u_i}^2$ is

$$\sum \hat{u_i}^2 = \sum y_i^2 - \frac{(\sum x_i y_i)^2}{\sum x_i^2}$$

The covariance between $\hat{\beta}_1$ and $\hat{\beta}_2$ is:

$$\begin{aligned} \text{cov}(\hat{\beta}_1, \hat{\beta}_2) &= -\bar{X}\text{var}(\hat{\beta}_2) \\ &= -\bar{X}\left(\frac{\sigma^2}{\sum x_i^2}\right) \end{aligned} \tag{3.9}$$

3.1.6 Properties of Least-Squares Estimators: The Gauss-Markov Theorem

Given the assumptions of the classical linear regression model, the least-square estimators are satisfied the optimum properties which is known as **"The Gauss- Markov Theorem."** To understand this theorem, we need to know the small-sample properties of an estimator first.

The Small-Sample Properties of An Estimator

1. Unbiasedness

An estimator $\hat{\theta}$ is said to be an unbiased estimator of θ if the expected value of $\hat{\theta}$ is equal to the true θ

$$E(\hat{\theta}) = \theta$$

Therefore, if the expected value of $\hat{\theta}$ is not equal to the true θ, then the estimator is said to be biased. We can calculate the biased as:

$$\text{bias}(\hat{\theta}) = E(\hat{\theta}) - \theta$$

Figure: Biased and Unbiased Estimators

2. Minimum Variance

$\hat{\theta}_1$ is said to be a minimum variance estimator of θ if the variance of $\hat{\theta}_1$ is smaller that or at most equal to the variance of $\hat{\theta}_2$, which is any other estimator of θ

Figure: Minimum Variance

3.Best Unbiased or Efficient Estimator = property 1+ property 2

If $\hat{\theta}_1$ and $\hat{\theta}_2$ are two unbiased estimators of θ and the variance of $\hat{\theta}_1$ is smaller that or at most equal to the variance of $\hat{\theta}_2$, then $\hat{\theta}_1$ is **a minimum-variance unbiased estimator or best unbiased estimator.**

4. Linearity

An estimator $\hat{\theta}$ is said to be a linear estimator of θ if it is a linear function of the sample observations. For example:

$$\bar{X} = \frac{1}{n}\sum X_i = \frac{1}{n}(X_1 + X_2 + ... + X_n)$$

Thus, $\bar{X}$ is a linear estimator because it is a linear function of the X values.

Best Linear Unbiased Estimators : BLUE

The estimator $\hat{\theta}$ is called as the Best Linear Unbiased Estimator **BLUE** if it is satisfied the properties 1,2,4 that is $\hat{\theta}$ is linear, is unbiased, and has the minimum variance in the class of all linear unbiased estimators of θ.

Minimum Mean-Square-Error (MSE) Estimator

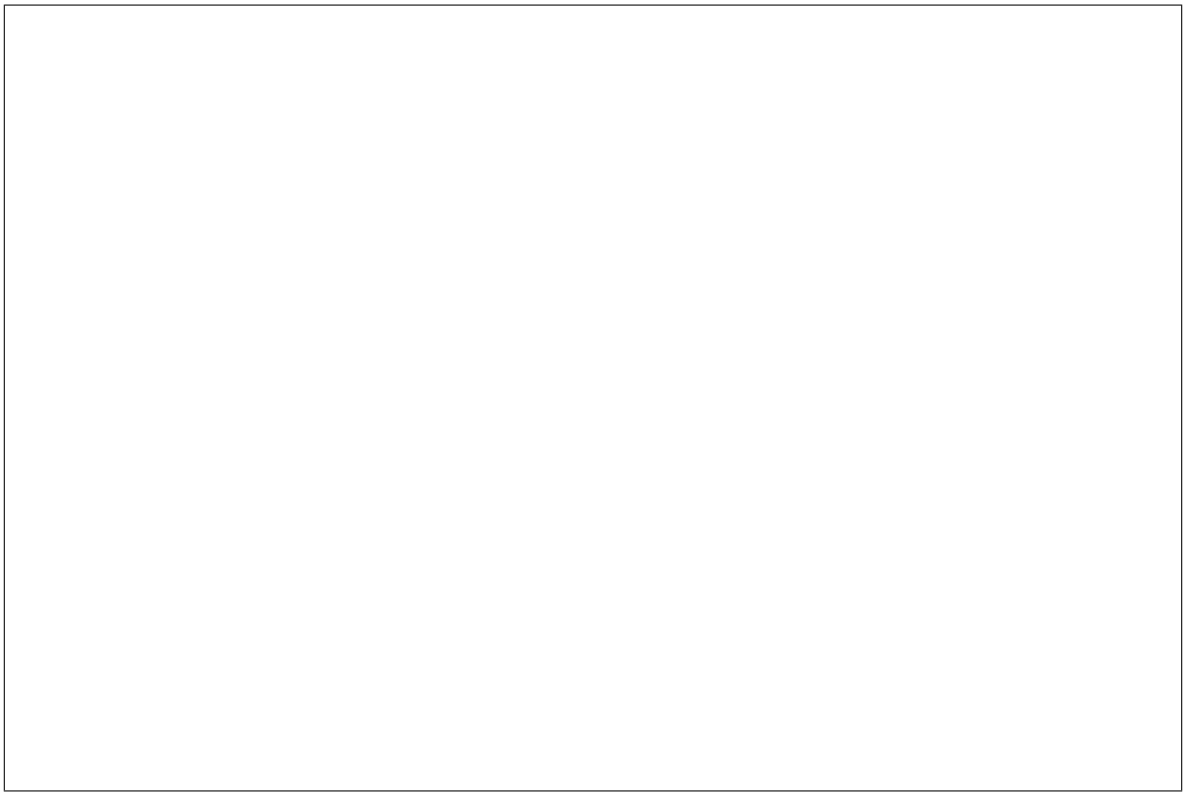
The MSE measures dispersion around the true value of the parameter. It is defined as:

$$\text{MSE}(\hat{\theta}) = E(\hat{\theta} - \theta)^2$$

However, the variance of $\hat{\theta}$ measures the dispersion of the distribution of the distribution of $\hat{\theta}$ around its mean or expected value.

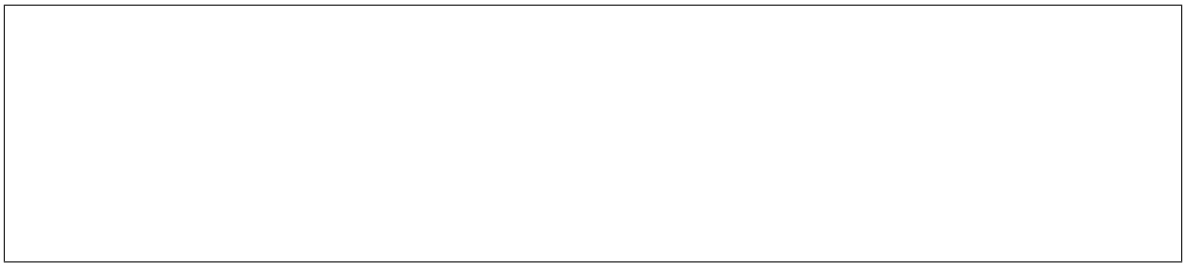
$$\text{var}(\hat{\theta}) = E(\hat{\theta} - E(\hat{\theta}))^2$$

The relationship between the MSE($\hat{\theta}$) and the var($\hat{\theta}$) is as follows:

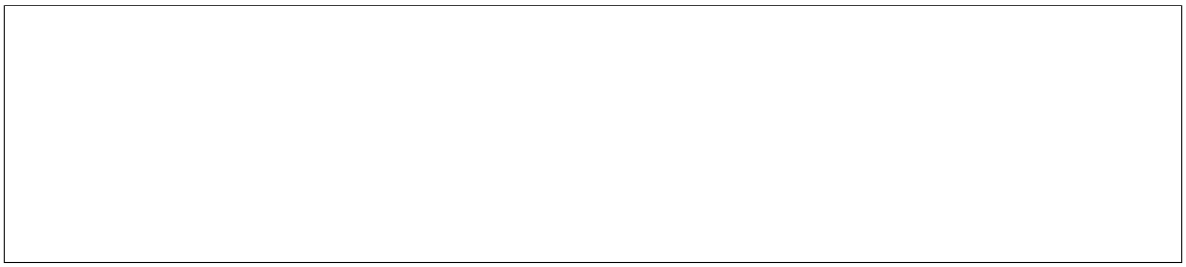


An estimator $\hat{\beta}_2$ is said to be a best linear unbiased estimator (BLUE) of β_2 if the following hold:

♣ **It is linear**. It is the linear function of a random variable.



♣ **It is unbiased**. That is $E(\hat{\beta}_2)$ is equal to the true value, β_2



♣ **It has the minimum variance in the class of all such linear unbiased estimators**.

Gauss-Markov Theorem: Given the assumptions of the classical linear regression model, the least-squares estimators, in the class of unbiased linear estimators, have minimum variance, that is , they are BLUE.

3.1.7 A measure of goodness of fit: r^2

In this section, we are going to study the goodness of fit of the fitted regression line to a set of data. Let us consider the following example:

Suppose we were to estimate the family expenditure (Y) based on our information from a random sample (as in Table 3.2).

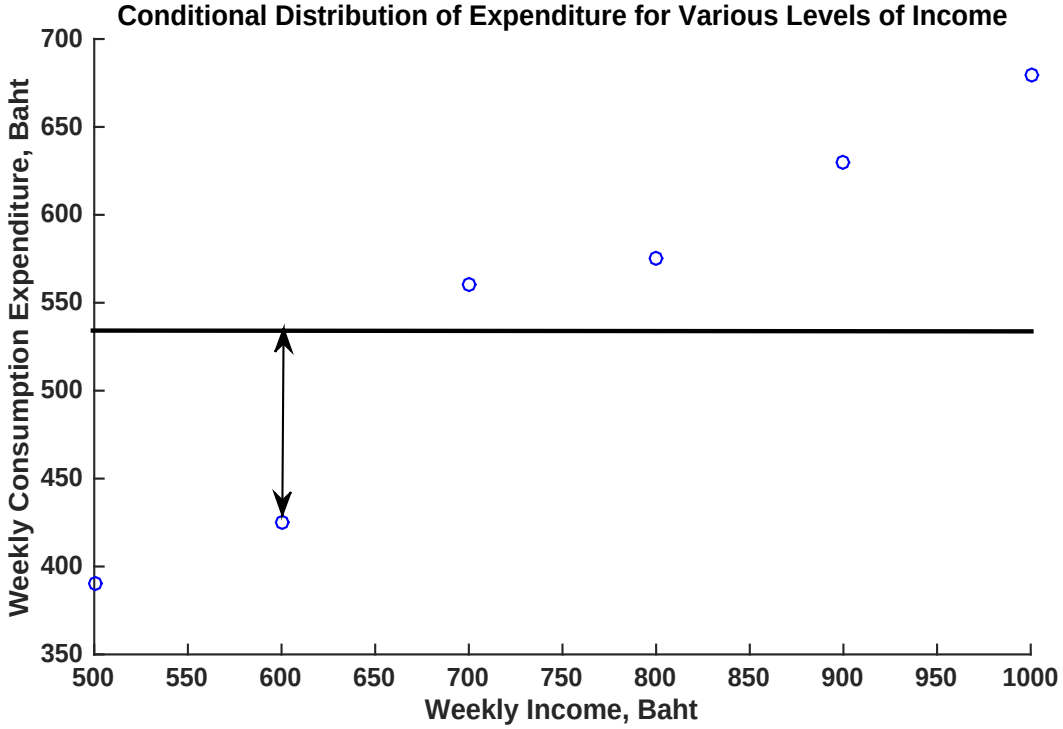
What will happen if we set the estimated Y to be $\bar{Y}$?

Table 3.3: Estimating the expenditure of the household

Family Number (i)	Actual Y_i	Estimate $\hat{Y}_i = \bar{Y}$	**Error in Estimation** $Y_i - \bar{Y}$	**Errors Squared** $(Y_i - \bar{Y})^2$
1	390	543	-153	23460.03
2	425	543	-118	13963.36
3	560	543	17	283.36
4	575	543	32	1013.36
5	630	543	87	7540.03
6	679	543	136	18450.69
Sum	3259	3259	0	64710.83

We can see all this graphically:

Figure 3.8: Graphic Representation



Question: Can we determine the total estimation error for this sample data?

Answer: Yes, we can calculate the total (combined) amount of estimation error for all observations in the sample when **using the mean as the estimate** as following:

$$TSS = \sum (Y_i - \bar{Y})^2$$

It is called the total sum of squares (TSS) which is the total variation of the actual Y values about their sample mean.

Since our objective in estimation is to minimize error (maximize precision), we need to cut down the amount of the estimation error (TSS).

We can achieve this by using information about other variables suspected to be strong predictors (strongly related to) the expenditure of the families.

We now can attempt to estimate the expenditure from the information on the income level of the family, rather than from its own mean.

Table 3.4: Estimating the expenditure of the household with income

Family (i)	Actual Y_i	Income X_i	$X - \bar{X}$	$Y - \bar{Y}$	$(X - \bar{X})(Y - \bar{Y})$	$(X - \bar{X})^2$
1	390	500	-250	-153.17	38291.67	62500
2	425	600	-150	-118.17	17725.00	22500
3	560	700	-50	16.83	-841.67	2500
4	575	800	50	31.83	1591.67	2500
5	630	900	150	86.83	13025.00	22500
6	679	1000	250	135.83	33958.33	62500
Sum	3259	4500	0	0	103750	175000

From the table 8, we can calculate the simple regression as following:

Figure 3.9: Breakdown of the variation of Y_i into two components

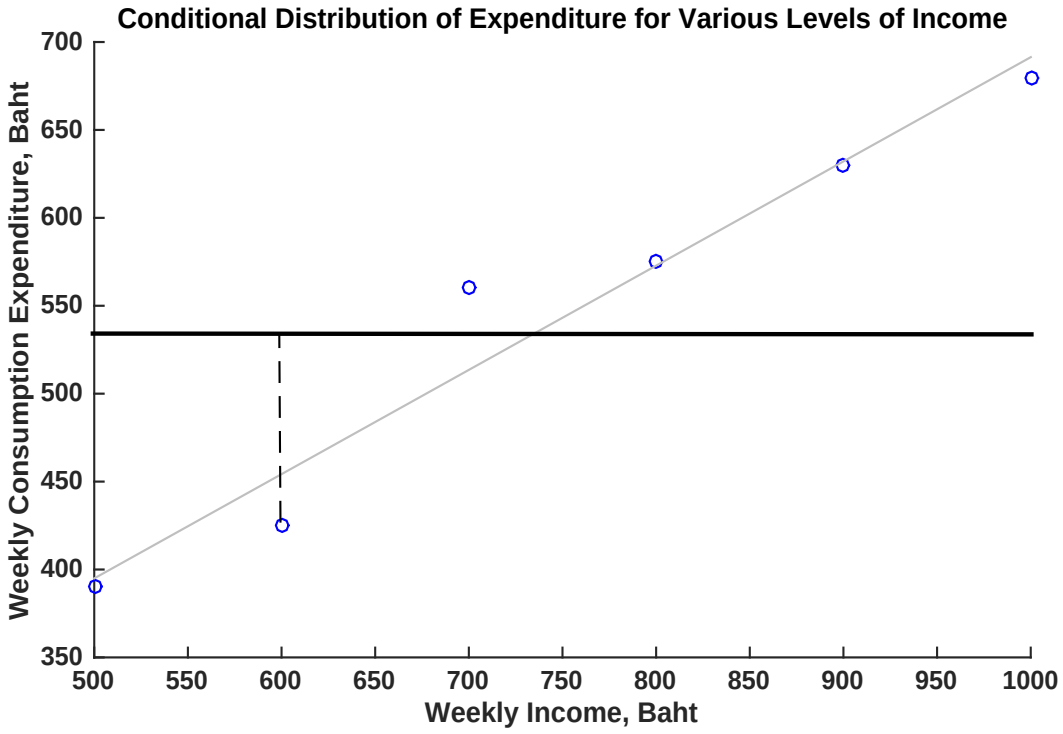


Table 3.5: Estimating the expenditure of the household with income

Family (i)	Actual Y_i	Income X_i	Regression Estimate $\hat{Y}$	Residual $Y-\hat{Y}$	Residual squared $(Y-\hat{Y})^2$
1	390	500	394.95	-4.95	24.53
2	425	600	454.24	-29.24	854.87
3	560	700	513.52	46.48	2160.04
4	575	800	572.81	2.19	4.80
5	630	900	632.10	-2.10	4.39
6	679	1000	691.38	-12.38	153.29
Sum	3259	4500	0	0	3201.90

From the table 9, we can calculate the estimation error we have committed by using the regression line as:

$$RSS = \sum (Y_i - \hat{Y}_i)^2 = \sum \hat{u}_i{}^2$$

where RSS stands for the residual sum of squares. which is the unexplained variation of the Y values about the regression line.

Total Baseline Error using the mean (SS Total) =

New or Remaining Error (SS Error or SS Residual) =

QUESTION: How much of the original estimation error have we explained away (eliminated) by using the regression model (instead of the mean)?

ANS

QUESTION: What % of estimation error have we explained (eliminated by using the regression model?

ANS

QUESTION: What does the remaining% represent?

ANS
Percent of variation (differences) in expenditures that can be accounted for by: (a) all other potential predictors not included in the model, beyond income levels, and (b) unexplainable random/chance variations.

$$r^2 = \frac{\text{ESS}}{\text{TSS}} = \frac{\sum(\hat{Y}_i - \bar{Y})^2}{\sum(Y_i - \bar{Y})^2}$$

♣ r^2 is a measure of our success regarding accuracy of our estimation effort.

♣ r^2 = % of estimation error that we have been able to explain away by using the regression model, instead of using the mean.

♣ r^2 indicates how much better we can predict Y from information about Xs, rather than from using its own mean.

♣ r^2 = % of differences (variations) in Y values that is explained by (attributable to) differences in X values.

4. Classical Normal Regression Model (CNLRM)

We know that the classical theory of statistical inference consists of:

1. Estimation

We have covered this topic since we were able to estimate the parameters β_1,β_2, and σ^2 by using the method of OLS.

We also proved that these estimators $\hat{\beta}_1$, $\hat{\beta}_2$ and $\hat{\sigma}$ satisfy several desirable statistical properties, such as unbiasedness, minimum variance, and linearity (BLUE property).

However, $\hat{\beta}_1$, $\hat{\beta}_2$ and $\hat{\sigma}$ change their values from sample to sample. The following tables show the two different sets of $\hat{\beta}_1$, $\hat{\beta}_2$ and $\hat{\sigma}$ depending on the two different sample data.

Table 4.1: Estimating the expenditure of the household with income

Family (i)	Actual Y_i	Income X_i	Regression Estimate $\hat{Y}$	Residual $Y-\hat{Y}$	Residual squared $(Y-\hat{Y})^2$
1	390	500	394.95	-4.95	24.53
2	425	600	454.24	-29.24	854.87
3	560	700	513.52	46.48	2160.04
4	575	800	572.81	2.19	4.80
5	630	900	632.10	-2.10	4.39
6	679	1000	691.38	-12.38	153.29
Sum	3259	4500	0	0	3201.90

If we use this sample data. We can estimate:

$$\hat{\beta}_1 = 98.524$$

$$\hat{\beta}_2 = 0.593$$

$$\hat{\sigma}^2 = \frac{\sum \hat{u}_i^2}{n-2} = \frac{3201.90}{6-2} = 800.476$$

Table 4.2: Estimating the expenditure of the household with income with another sample data

Family (i)	Actual Y_i	Income X_i	Regression Estimate $\hat{Y}$	Residual $Y-\hat{Y}$	Residual squared $(Y-\hat{Y})^2$
1	360	500	325.71	64.29	4132.65
2	390	600	406.43	18.57	344.90
3	440	700	487.14	72.86	5308.16
4	575	800	567.86	7.14	51.02
5	670	900	648.57	-18.57	344.90
6	730	1000	729.29	-50.29	2528.65
Sum	3165	4500	0	0	12710.29

If we use this sample data. We can estimate:

$$\hat{\beta}_1 = -77.857$$

$$\hat{\beta}_2 = 0.807$$

$$\hat{\sigma}^2 = \frac{\sum \hat{u}_i^2}{n-2} = \frac{12710.29}{6-2} = 3177.571$$

From the example, you can easily see that these estimators are **RANDOM VARIABLES**. Therefore, we need to learn another part of statistical inference which is called **Hypothesis Testing.**

2. Hypothesis Testing

The main objective is to find out how close of $\hat{\beta}_1$ and $\hat{\beta}_2$ to the true β_1 and the true β_2, respectively. Also, we would like to see how close of $\hat{\sigma}^2$ compared to the true σ^2.

To achieve this goal, we need to know the probability distributions of $\hat{\beta}_1$, $\hat{\beta}_2$,and $\hat{\sigma}^2$.
Consider the estimator of β_2:

$$\hat{\beta}_2 = \sum k_i Y_i$$

We can write the above equation as:

$$\hat{\beta}_2 = \sum k_i(\beta_1 + \beta_2 X_i + u_i)$$

From this equation, the probability distribution of $\hat{\beta}_2$ will depend on the assumption made about the probability distribution of u_i

4.1 The Normality Assumption for u_i

In the classical normal linear regression model (CNLRM), we assume that each u_i is distributed normally :

$$u_i \sim N(0, \sigma^2)$$

where
Mean:

$$E(u_i) = 0$$

Variance:

$$E[u_i - E(u_i)]^2 = E(u_i^2) = \sigma^2$$

$$cov(u_i, u_j) = E\left\{[u_i - E(u_i)][u_j - E(u_j)]\right\} = E(u_i U_j) = 0$$

Therefore,

$$u_i \sim N(0, \sigma^2)$$

Also, u_i and u_j are not only uncorrelated but also independently distributed.

we can then write the above equation as:

$$u_i \sim NID(0, \sigma^2)$$

where NID stands for normally and independently distributed.

4.2 Properties of OLS estimators under the normality assumption

1. They are unbiased.

2. They have minimum variance.

3. By 1+2 properties, they are minimum-variance unbiased, or efficient estimators.

4. $\hat{\beta}_1$ is normally distributed with:

$$\text{Mean:} E(\hat{\beta}_1) = \beta_1$$

$$\text{var}(\hat{\beta}_1) = \sigma^2_{\hat{\beta}_1} = \frac{\sum X_i^2}{n \sum x_i^2} \sigma^2$$

Therefore,

$$\hat{\beta}_1 \sim N(\beta_1, \sigma^2_{\hat{\beta}_1})$$

By the properties of the normal distribution, we can:

5. $\hat{\beta}_2$ is normally distributed with

$$\text{Mean:} E(\hat{\beta}_2) = \beta_2$$

$$\text{var}(\hat{\beta}_2) = \sigma^2_{\hat{\beta}_2} = \frac{\sigma^2}{\sum x_i^2}$$

or more compactly

$$\hat{\beta}_2 \sim N(\beta_2, \sigma^2_{\hat{\beta}_2})$$

then we can define the standard normal distribution as

6. $(n-2)(\hat{\sigma}^2/\sigma^2)$ is distributed as the χ^2 (chi-square) distribution with (n-2) df.

7. $(\hat{\beta}_1, \hat{\beta}_2)$ are distributed independently of $\hat{\sigma}^2$

8. $\hat{\beta}_1$ and $\hat{\beta}_2$ have the minimum variance in the entire class of unbiased estimators, whether linear or not.

9. we can find out the probability distribution of Y_i as following:

5. Interval Estimation and Hypothesis Testing

Interval Estimation

5.1 Confidence Intervals for Regression Coefficients β_1 and β_2

In Sum

A $100(1-\alpha)$ percent **confidence interval** for β_2 can be defined as:

$$\hat{\beta}_2 \pm t_{\alpha/2}\text{se}(\hat{\beta}_2)$$

or

$$\Pr[\hat{\beta}_2 - t_{\alpha/2}\text{se}(\hat{\beta}_2) \leq \beta_2 \leq \hat{\beta}_2 + t_{\alpha/2}\text{se}(\hat{\beta}_2)] = 1-\alpha$$

Analogously, we can define $100(1-\alpha)$ percent **confidence interval** for β_1 as:

$$\hat{\beta}_1 \pm t_{\alpha/2}\text{se}(\hat{\beta}_1)$$

or

$$\Pr[\hat{\beta}_1 - t_{\alpha/2}\text{se}(\hat{\beta}_1) \leq \beta_1 \leq \hat{\beta}_1 + t_{\alpha/2}\text{se}(\hat{\beta}_1)] = 1-\alpha$$

Example

Table 5.1: Estimating the expenditure of the household with income

Family (i)	Actual Y_i	Income X_i	$X-\bar{X}$	$Y-\bar{Y}$	$(X-\bar{X})(Y-\bar{Y})$	$(X-\bar{X})^2$
1	390	500	-250	-153.17	38291.67	62500
2	425	600	-150	-118.17	17725.00	22500
3	560	700	-50	16.83	-841.67	2500
4	575	800	50	31.83	1591.67	2500
5	630	900	150	86.83	13025.00	22500
6	679	1000	250	135.83	33958.33	62500
Sum	3259	4500	0	0	103750	175000

Table 5.2: Estimating the expenditure of the household with income

Family (i)	Actual Y_i	Income X_i	Regression Estimate $\hat{Y}$	Residual $Y-\hat{Y}$	Residual squared $(Y-\hat{Y})^2$
1	390	500	394.95	-4.95	24.53
2	425	600	454.24	-29.24	854.87
3	560	700	513.52	46.48	2160.04
4	575	800	572.81	2.19	4.80
5	630	900	632.10	-2.10	4.39
6	679	1000	691.38	-12.38	153.29
Sum	3259	4500	0	0	3201.90

Confidence Interval for β_2

Confidence Interval for β_1

5.2 Confidence Interval for σ^2

5.3 Hypothesis Testing: The Confidence-Interval Approach

Based on our sample data, the estimated marginal propensity to consume (MPC), $\hat{\beta}_2$ is 0.593. Suppose we postulate that

$$H_0 : \beta_2 = 0.6$$

$$H_1 : \beta_2 \neq 0.6$$

5.4 Hypothesis Testing: The Test of Significance Approach

5.4.1 Two-Tail Test

Based on the sample data, the estimated marginal propensity to consume (MPC), $\hat{\beta}_2$ is 0.593. Suppose we postulate that

$$H_0 : \beta_2 = 0.6$$

$$H_1 : \beta_2 \neq 0.6$$

5.4.2 One-Tail Test

Based on the sample data, the estimated marginal propensity to consume (MPC), $\hat{\beta}_2$ is 0.593. Suppose we postulate that

$$H_0 : \beta_2 \leq 0.6$$

$$H_1 : \beta_2 > 0.6$$

We can summarize the decision rules for the t test as follow:

Figure 5.1 The t test of Significance: Decision rules

Type of hypothesis	H_0: the null hypothesis	H_1: the alternative hypothesis	Decision rule: reject H_0 if
Two-tail	$\beta_2 = \beta_2^*$	$\beta_2 \neq \beta_2^*$	$\lvert t \rvert > t_{\alpha/2,\text{df}}$
Right-tail	$\beta_2 \leq \beta_2^*$	$\beta_2 > \beta_2^*$	$t > t_{\alpha,\text{df}}$
Left-tail	$\beta_2 \geq \beta_2^*$	$\beta_2 < \beta_2^*$	$t < -t_{\alpha,\text{df}}$

Notes: β_2^* is the hypothesized numerical value of β_2.
$|t|$ means the absolute value of t.
t_α or $t_{\alpha/2}$ means the critical t value at the α or $\alpha/2$ level of significance.
df: degrees of freedom, $(n-2)$ for the two-variable model, $(n-3)$ for the three-variable model, and so on.
The same procedure holds to test hypotheses about β_1.

5.4.3 Testing the significance of σ^2: The χ^2 test

Figure 5.2 The χ^2 Test : Decision rules

H_0: the null hypothesis	H_1: the alternative hypothesis	Critical region: reject H_0 if
$\sigma^2 = \sigma_0^2$	$\sigma^2 > \sigma_0^2$	$\frac{\text{df}(\hat{\sigma}^2)}{\sigma_0^2} > \chi^2_{\alpha,\text{df}}$
$\sigma^2 = \sigma_0^2$	$\sigma^2 < \sigma_0^2$	$\frac{\text{df}(\hat{\sigma}^2)}{\sigma_0^2} < \chi^2_{(1-\alpha),\text{df}}$
$\sigma^2 = \sigma_0^2$	$\sigma^2 \neq \sigma_0^2$	$\frac{\text{df}(\hat{\sigma}^2)}{\sigma_0^2} > \chi^2_{\alpha/2,\text{df}}$ or $< \chi^2_{(1-\alpha/2),\text{df}}$

Note: σ_0^2 is the value of σ^2 under the null hypothesis. The first subscript on χ^2 in the last column is the level of significance, and the second subscript is the degrees of freedom. These are critical chi-square values. Note that df is $(n-2)$ for the two-variable regression model, $(n-3)$ for the three-variable regression model, and so on.

Why do we say "we cannot reject the null hypothesis?" instead of "We accept the null hypothesis"

The Level of Significance: α

Type I error

Type II error

The Exact Level of Significance: The p Value

5.4.1 Regression Analysis and Analysis of Variance

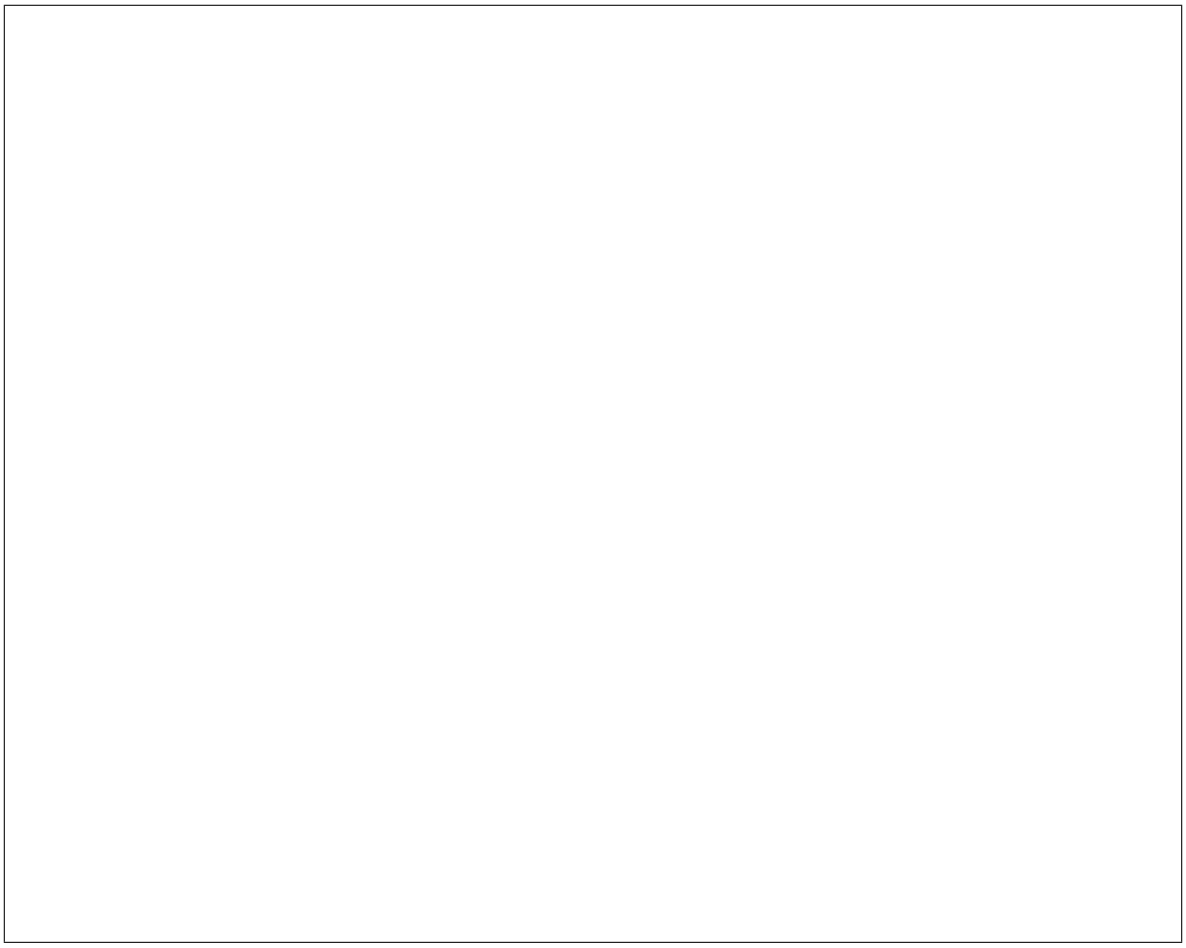


Table 5.3: ANOVA Table for the two-variable regression model

Source of variation	Sum of Square **SS**	**df**	Mean Sum of Square **MSS**
Due to regression (ESS)			
Due to residuals (RSS)			
TSS			

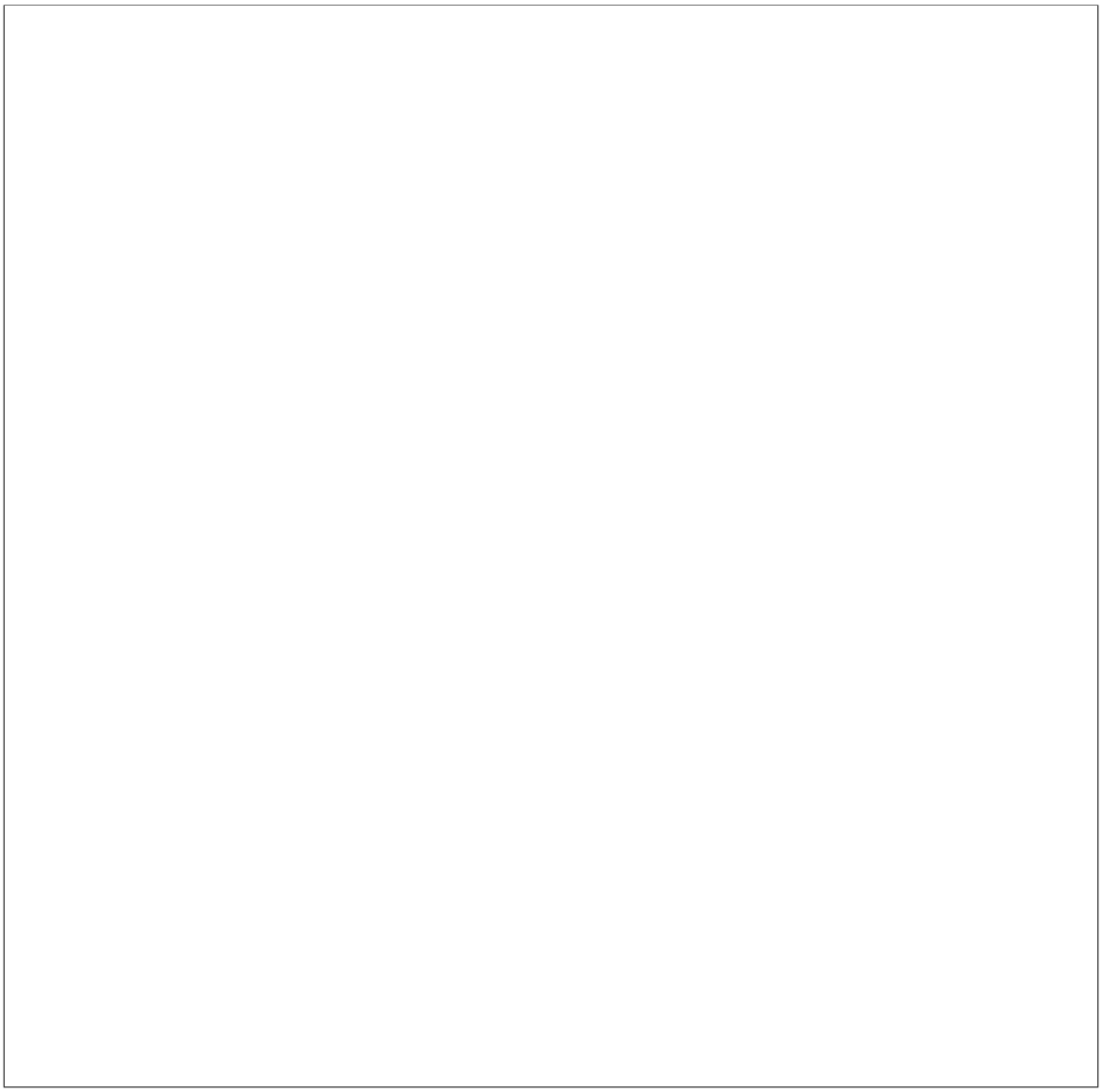


Table 5.4: Estimating the expenditure of the household

Family Number (i)	Actual Y_i	Estimate $\hat{Y}_i = \bar{Y}$	**Error in Estimation** $Y_i - \bar{Y}$	**Errors Squared** $(Y_i - \bar{Y})^2$
1	390	543	-153	23460.03
2	425	543	-118	13963.36
3	560	543	17	283.36
4	575	543	32	1013.36
5	630	543	87	7540.03
6	679	543	136	18450.69
Sum	3259	3259	0	64710.83

Table 5.5: Estimating the expenditure of the household with income

Family (i)	Actual Y_i	Income X_i	Regression Estimate $\hat{Y}$	Residual $Y - \hat{Y}$	Residual squared $(Y - \hat{Y})^2$
1	390	500	394.95	-4.95	24.53
2	425	600	454.24	-29.24	854.87
3	560	700	513.52	46.48	2160.04
4	575	800	572.81	2.19	4.80
5	630	900	632.10	-2.10	4.39
6	679	1000	691.38	-12.38	153.29
Sum	3259	4500	0	0	3201.90

Table 5.6: ANOVA Table: Estimating the expenditure of the household with income

Source of variation	Sum of Square **SS**	**df**	Mean Sum of Square **MSS**
Due to regression (ESS)			
Due to residuals (RSS)			
TSS			

(After MID-TERM)

5.5 The problem of prediction

Based on our sample data, we have the following sample regression:

We can use the above regression to "Predict" or "Forecast" the future consumption expenditure Y corresponding to some given level of income X

There are two kinds of predictions which are:

[1] **Mean prediction** We will predict the conditional mean value of Y corresponding to a chosen X (i.e X_0)

[2] **Individual Prediction** We will predict an individual Y value corresponding to (i.e X_0)

Mean Prediction

We know that

$$\hat{Y_0} \sim N(E(\hat{Y_0}), \text{var}(\hat{Y_0}))$$

where

$$E(\hat{Y_0}) = \beta_1 + \beta_2 X_0$$

and

$$\text{var}(\hat{Y_0}) = \sigma^2 \left[\frac{1}{n} + \frac{(X_0 - \bar{X})^2}{\sum x_i^2} \right]$$

If we replace the unknown σ^2 by the unbiased estimator $\hat{\sigma}^2$ we can get

$$t = \frac{\hat{Y}_0 - (\beta_1 + \beta_2 X_0)}{se(\hat{Y}_0)}$$

which is the t distribution with n-2 df.

Therefore, we can derive the confidence interval for the true $E(Y_0|X_0)$ as following:

$$Pr[\hat{\beta}_1 + \hat{\beta}_2 X_0 - t_{\frac{\alpha}{2}} se(\hat{Y}_0) \leq \beta_1 + \beta_2 X_0 \leq \hat{\beta}_1 + \hat{\beta}_2 X_0 + t_{\frac{\alpha}{2}} se(\hat{Y}_0)] = 1 - \alpha$$

Example

Individual Prediction

We can prediction an individual Y value, Y_0 corresponding to a given X value (X_0) :

but the variance in this case is:

$$\text{var}(Y_0 - \hat{Y}_0) = E[Y_0 - \hat{Y}_0]^2 = \sigma^2 \left[1 + \frac{1}{n} + \frac{(X_0 - \bar{X})^2}{\sum x_i^2}\right]$$

We can show that Y_0 follows the normal distribution:

$$Y_0 \sim N(\hat{Y}_0, \text{var}(Y_0 - \hat{Y}_0))$$

Therefore, we can construct the confidence interval for the Y_0 as well.

From our example:

6. Extensions of The Two-Variable Linear Regression Mode

6.1 Functional Form of regression Models

We will consider the following models:

[1] The log-linear model

[2] Semilog models

[3] Reciprocal models

[4] The logarithmic reciprocal model

The Log-linear Model

The Semilog Models

The Reciprocal Models

The Logarithmic Reciprocal Models

6.2 Regression Through the Origin

In this section, we consider the case that the two-variable PRF assumes the following form:

$$Y_i = \beta_2 X_i + u_i$$

This model is called **the regression through the origin** where the intercept term $\hat{\beta}_1$ is absent from the model.

Example

Since it is the linear regression model, we can apply the Ordinary Least Square (OLS) to estimate the formula for $\hat{\beta}_2$

Let us first write the sample regression function (SRF) as:

$$Y_i = \hat{\beta}_2 X_i + \hat{u}_i$$

We would like to minimize

$$\sum \hat{u}_i{}^2 = \sum (Y_i - \hat{\beta}_2 X_i)^2$$

therefore,

$$\hat{\beta}_2 = \frac{\sum X_i Y_i}{\sum X_i^2}$$

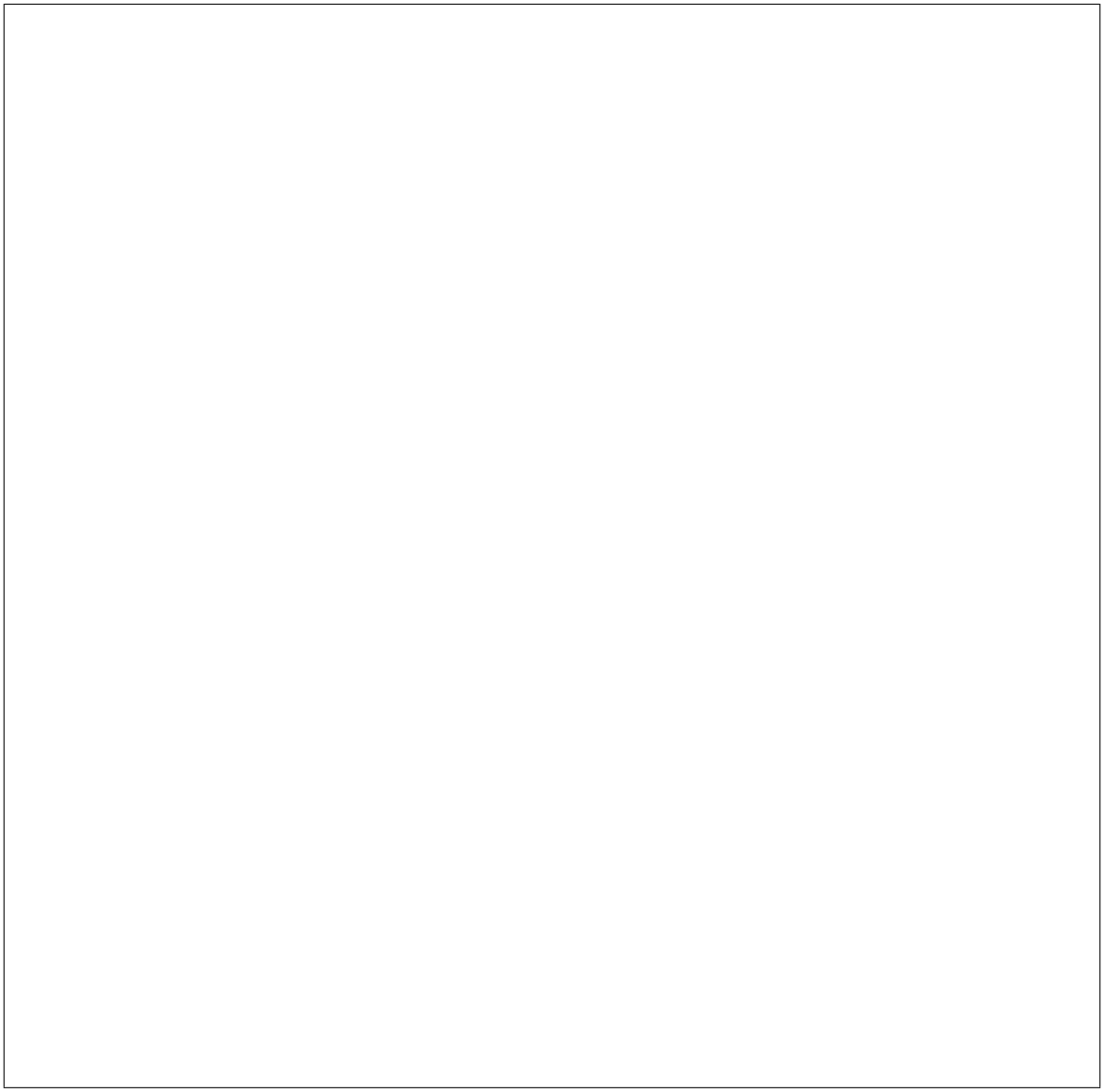
Now we can find out the variance of $\hat{\beta}_2$

$$\text{var}(\hat{\beta}_2) = \frac{\sigma^2}{\sum X_i^2}$$

$$\hat{\sigma}^2 = \frac{\sum \hat{u}_i{}^2}{n-1}$$

It should be noted that we get the condition $\sum \hat{u}_i X_i = 0$ from the normal equation. However, with the regression through the origin model, we cannot get the condition $\sum \hat{u}_i = 0$.

For the zero-intercept model, r^2 can be negative, whereas for the conventional model it cannot be negative.



Since the conventional r^2 is not appropriate for the regressions that do not contain the intercept, we therefore compute what is known as the **raw** r^2 instead:

$$\text{raw } r^2 = \frac{(\sum X_i Y_i)^2}{\sum X_i^2 \sum Y_i^2}$$

This raw r^2 has its value between 0 and 1, but we cannot directly compare its value to the conventional r^2 value. For this reason, some researchers do not report the r^2 value for zero intercept regression models.

6.2.1 Scaling and Units of Measurements

Consider our old example given in table 18 which refer to weekly family expenditure (Y) and Income (x), in baht.

Table 6.1: Weekly family Expenditure (Y), Baht and Income (X), (Unit:Baht)

X	**Y**
500	360
600	390
700	440
800	575
900	670
1000	730

By using the OLS estimation, we get the following results:

Now, we are interested in changing the units of our data. For example, we would prefer to express our sample data in the unit of 1000 baht. By using the new unit of X and Y, we can report our data in 1000 baht as in the following table.

Table 6.2: Weekly family Expenditure (Y), Baht and Income (X), (Unit: 1000 Baht)

X	**Y**
0.5	0.360
0.6	0.390
0.7	0.440
0.8	0.575
0.9	0.670
1	0.730

With the new unit, we would like to answer these two questions:
1. Do the units in which the regressand (Y) and regressor/s (X) are measured make any difference in the regression results?
2. If so, what is the sensible course to follow in choosing units of measurement for regression analysis?

To answer these questions, let:

$$Y_i = \hat{\beta}_1 + \hat{\beta}_2 X_i + \hat{u}_i$$

where Y is the weekly family expenditure and X is the income, in baht.

Now, let w_1 and w_2 are constants, called the **Scale factors**. For example, in our data, if we need to use the unit of 1000 baht instead, we can directly multiple the original data in table 18 with the scale factors equal to 0.001. In other words, $w_1 = w_2 = \frac{1}{1000} = 0.001$.

Define

$$Y_i^* = w_1 Y_i$$

$$X_i^* = w_1 X_i$$

Now consider the regression using Y_i^* and X_i^* variables:

$$Y_i^* = \hat{\beta}_1^* + \hat{\beta}_2^* X_i^* + \hat{u}_i{}^*$$

$\hat{u}_i^* = ?$

Our target is to find out the relationship between the following pairs:

1. $\hat{\beta}_1$ and $\hat{\beta}_1^*$

2. $\hat{\beta}_2$ and $\hat{\beta}_2^*$

3.var($\hat{\beta}_1$) and var($\hat{\beta}_1^*$)

4.var($\hat{\beta}_2$) and var($\hat{\beta}_2^*$)

5.$\hat{\sigma}^2$ and $\hat{\sigma}^{*2}$

6. r_{xy}^2 and $r_{x^*y^*}^2$

1. $\hat{\beta}_1$ and $\hat{\beta}_1^*$

2. $\hat{\beta}_2$ and $\hat{\beta}_2^*$

3.var($\hat{\beta}_1$) and var($\hat{\beta}_1^*$)

4.var($\hat{\beta}_2$) and var($\hat{\beta}_2^*$)

5.$\hat{\sigma}^2$ and $\hat{\sigma}^{*2}$

6. r^2_{xy} and $r^2_{x^*y^*}$

7. Multiple Regression Analysis: The Problem of Analysis

Three-Variable Model: Notation and Assumptions

Let us consider the following three-variable PRF as:

$$Y_i = \beta_1 + \beta_2 X_{2i} + \beta_3 X_{3i} + u_i$$

where

Y_i is the dependent variable (regressand)

X_{2i} and X_{3i} are the regressors or the explanatory variables

u_i is the stochastic disturbance term

Remark: the subscript i is denoted the observation i from our sample data.

In case our data are time series, the subscript t will denote the t observation.

β_1 means the average value of Y when X_2 and X_3 are set equal to zero

β_2 and β_3 are called the partial regression coefficients.

We will talk about the meaning of β_1 and β_2 shortly after knowing the assumptions of the classical linear regression model (CLRM)

Under the CLRM, we assume:

1. Zero mean value of u_i

2. No serial correlation

3. Homoscedasticity

4. Zero covariance between u_i and each X variable, or

5. No specification bias or

The model is correctly specified.

6. No exact collinearity between the X variables or

By the above assumptions, we can find out the conditional expectation of Y_i:

The meaning of partial coefficients:

β_2

β_3

7.1 OLS Estimation of the Partial Regression Coefficients

In order to find the OLS estimators, we need to write down the sample regression function (SRF) corresponding to the PRF:

$$Y_i = \hat{\beta}_1 + \hat{\beta}_2 X_{2i} + \hat{\beta}_3 X_{3i} + \hat{u}_i$$

From the FOC, we then get the normal equations:

$$\bar{Y} = \hat{\beta}_1 + \hat{\beta}_2 \bar{X}_2 + \hat{\beta}_3 \bar{X}_3$$
$$\sum Y_i X_{2i} = \hat{\beta}_1 \sum X_{2i} + \hat{\beta}_2 \sum X_{2i}^2 + \hat{\beta}_3 \sum X_{2i} X_{3i}$$
$$\sum Y_i X_{3i} = \hat{\beta}_1 \sum X_{3i} + \hat{\beta}_2 \sum X_{2i} X_{3i} + \hat{\beta}_3 \sum X_{3i}^2$$

We therefore get:

$$\hat{\beta}_1 = \bar{Y} - \hat{\beta}_2 \bar{X}_2 - \hat{\beta}_3 \bar{X}_3$$
$$\hat{\beta}_2 = \frac{(\sum y_i x_{2i})(\sum x_{3i}^2) - (\sum y_i x_{3i})(\sum x_{2i} x_{3i})}{(\sum x_{2i}^2)(\sum x_{3i}^2) - (\sum x_{2i} x_{3i})^2}$$
$$\hat{\beta}_3 = \frac{(\sum y_i x_{3i})(\sum x_{2i}^2) - (\sum y_i x_{2i})(\sum x_{2i} x_{3i})}{(\sum x_{2i}^2)(\sum x_{3i}^2) - (\sum x_{2i} x_{3i})^2}$$

Variance and Standard Errors of OLS Estimators

$$var(\hat{\beta}_1) = \left[\frac{1}{n} + \frac{\bar{X}_2^2 \sum x_{3i}^2 + \bar{X}_3{}^2 \sum x_{2i}^2 - 2\bar{X}_2 \bar{X}_3 \sum x_{2i} x_{3i}}{\sum x_{2i}^2 \sum x_{3i}^2 - (\sum x_{2i} x_{3i})^2} \right] * \sigma^2$$
$$se(\hat{\beta}_1) = +\sqrt{var(\hat{\beta}_1)}$$

$$var(\hat{\beta}_2) = \frac{\sum x_{3i}^2}{(\sum x_{2i}^2)(\sum x_{3i}^2) - (\sum x_{2i} x_{3i})^2} * \sigma^2$$
$$var(\hat{\beta}_2) = \frac{\sigma^2}{\sum x_{2i}^2 (1 - r_{23}^2)}$$
$$se(\hat{\beta}_2) = +\sqrt{var(\hat{\beta}_2)}$$

$$var(\hat{\beta}_3) = \frac{\sum x_{2i}^2}{(\sum x_{2i}^2)(\sum x_{3i}^2) - (\sum x_{2i} x_{3i})^2} * \sigma^2$$
$$var(\hat{\beta}_3) = \frac{\sigma^2}{\sum x_{3i}^2 (1 - r_{23}^2)}$$
$$se(\hat{\beta}_3) = +\sqrt{var(\hat{\beta}_2)}$$

$$cov(\hat{\beta}_2, \hat{\beta}_3) = \frac{-r_{23}\sigma^2}{(1-r_{23}^2)\sqrt{\sum x_{2i}^2}\sqrt{\sum x_{3i}^2}}$$

$$\hat{\sigma}^2 = \frac{\sum \hat{u}_i^2}{n-3}$$

7.2 Properties of OLS Estimators

Properties of OLS Estimators (Cont:)

Properties of OLS Estimators (Cont:)

The Multiple Coefficient of Determination R^2 and the Multiple Coefficient of Correlation R

In this section, we will study how to measure the proportion of the variation in Y explained by the variables X_2 and X_3 jointly. This is the same concept of r^2 that we have learned before.

The quantity that gives this information is known as the **the multiple coefficient of determination** and is denoted by R^2.

To derive R^2, we firstly write down the following equation:

$$\begin{aligned} Y_i &= \hat{\beta}_1 + \hat{\beta}_2 X_{2i} + \hat{\beta}_3 X_{3i} + \hat{u}_i \\ &= \hat{Y}_i + \hat{u}_i \end{aligned} \tag{7.1}$$

where $\hat{Y}_i$ is the estimated value of Y_i from the fitted regression line and is an estimator of true $E(Y_i|X_{2i}, X_{3i})$.

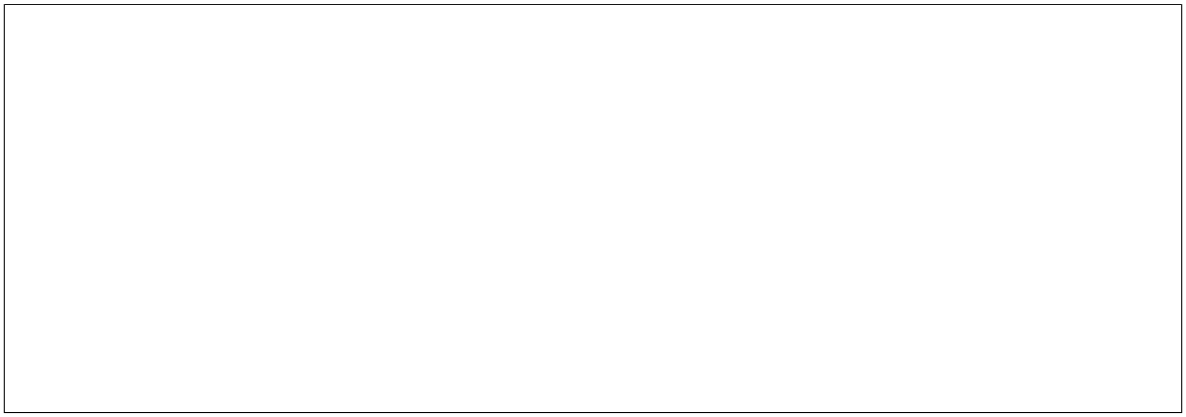
7.1 may be written as

$$\begin{aligned} y_i &= \hat{\beta}_2 x_{2i} + \hat{\beta}_3 x_{3i} + \hat{u}_i \\ &= \hat{y}_i + \hat{u}_i \end{aligned} \tag{7.2}$$

Squaring 7.2 on both sides and summing over the sample values, we obtain

$$\begin{aligned} \sum y_i^2 &= \sum \hat{y}_i{}^2 + \hat{u}_i{}^2 + 2\sum \hat{y}_i \hat{u}_i \\ &= \sum \hat{y}_i{}^2 + \sum \hat{u}_i{}^2 \end{aligned} \tag{7.3}$$

$$\begin{aligned} R^2 &= \frac{ESS}{TSS} \\ &= \frac{\hat{\beta}_2 \sum y_i x_{2i} + \hat{\beta}_3 \sum y_i x_{3i}}{\sum y_i^2} \end{aligned} \tag{7.4}$$



The three-or-more-variable analogue of r is the coefficient of multiple correlation, denoted by R, and it is a measure of the degree of association between Y and all the explanatory variables jointly. Although r can be positive or negative, R is always taken to be positive.

$$Var(\hat{\beta}_j) = \frac{\sigma^2}{\sum x_j^2} \left(\frac{1}{1 - R_j^2} \right)$$

7.2.1 R^2 and the Adjusted R^2

It should be noted that he R^2 is a nondecreasing function of the number of explanatory variables. Thus, when the number of regressors increases, R^2 almost invariably increases and never decreases. **In other words, an additional X variable will not decrease R^2!**

To explain this fact, let we write down the definition of R^2 again:

$$\begin{aligned} R^2 &= \frac{ESS}{TSS} \\ &= 1 - \frac{RSS}{TSS} \\ &= 1 - \frac{\sum \hat{u}_i^2}{\sum y_i^2} \end{aligned} \tag{7.5}$$

Therefore, in comparing two regression models **with the same dependent variable but differing number of X variables**, one should be very wary of choosing the model with the highest R^2.

In light of comparing two R^2 terms, we have to take into account the number of X variables present in the model. To achieve this goal , we can consider the alternative coefficient of determination, which is as follows:

k= the number of parameters in the model including the intercept term.
n= the number of observations in the sample data.

The above equation is known as **the adjusted** R^2, denoted by $\bar{R}^2$. The term adjusted means adjusted for the df associated with the sums of squares entering into 7.5.

We can rewrite the the adjusted R^2 as:

We can also get the equation which shows the relationship between $\bar{R}^2$ and R^2:

Besides R^2 and $\bar{R}^2$ as goodness of fit measures, other criteria are often used to judge the adequacy of a regression model. Two of these are **Akaike's Information criterion and Amemiya's Prediction criteria**, which are used to select between competing models. We will discuss these criteria in greater detail later.

8. Multiple Regression Analysis: The Problem of Inference

In this chapter, we will extend the ideas of interval estimation and hypothesis testing developed there to models involving three or more variables.

$$Y_i = \beta_1 + \beta_2 X_{2i} + \beta_3 X_{3i} + u_i$$

We have already known that if our objective is to do interval estimation and hypothesis testing, we need to assume that the u_i follow the normal distribution with zero mean and constant variance σ^2

With the normality assumption and the CLRM assumptions, we know that:

[1] The OLS estimations of partial regression coefficients are best linear unbiased estimators (BLUE).

[2] The estimators $\hat{\beta}_1$, $\hat{\beta}_2$, and $\hat{\beta}_3$ are normally distributed with means equal to true β_1,β_2, and β_3 and variances are following:

$$var(\hat{\beta}_1) = \left[\frac{1}{n} + \frac{\bar{X}_2^2 \sum x_{3i}^2 + \bar{X}_3{}^2 \sum x_{2i}^2 - 2\bar{X}_2\bar{X}_3 \sum x_{2i}x_{3i}}{\sum x_{2i}^2 \sum x_{3i}^2 - (\sum x_{2i}x_{3i})^2}\right] * \sigma^2$$

$$se(\hat{\beta}_1) = +\sqrt{var(\hat{\beta}_1)}$$

$$var(\hat{\beta}_2) = \frac{\sum x_{3i}^2}{(\sum x_{2i}^2)(\sum x_{3i}^2) - (\sum x_{2i}x_{3i})^2} * \sigma^2$$

$$var(\hat{\beta}_2) = \frac{\sigma^2}{\sum x_{2i}^2(1 - r_{23}^2)}$$

$$se(\hat{\beta}_2) = +\sqrt{var(\hat{\beta}_2)}$$

$$var(\hat{\beta}_3) = \frac{\sum x_{2i}^2}{(\sum x_{2i}^2)(\sum x_{3i}^2) - (\sum x_{2i}x_{3i})^2} * \sigma^2$$

$$var(\hat{\beta}_3) = \frac{\sigma^2}{\sum x_{3i}^2(1 - r_{23}^2)}$$

$$se(\hat{\beta}_3) = +\sqrt{var(\hat{\beta}_3)}$$

Moreover, $\frac{(n-3)\hat{\sigma^2}}{\sigma^2}$ follows the χ^2 distribution with n-3 df. We can also show that, if we replace the true σ^2 by its unbiased estimator $\hat{\sigma}^2$ in the computation of the standard errors, we then get

$$t = \frac{\hat{\beta}_1 - \beta_1}{se(\hat{\beta}_1)}$$

$$t = \frac{\hat{\beta}_2 - \beta_2}{se(\hat{\beta}_2)}$$

$$t = \frac{\hat{\beta}_3 - \beta_3}{se(\hat{\beta}_3)}$$

follows the t distribution with n-3 df.

Example Consider the following regression:

$$\begin{aligned} \widehat{\log(\text{salary})} &= 4.32 + 0.280\ \log(\text{sales}) + 0.0174\ \text{ROE} + 0.00024\ \text{ROS} \\ se &= (0.32)\ (0.035)\qquad\quad (0.0041)\qquad\ (0.00054) \end{aligned} \tag{8.1}$$

$$R^2 = 0.283$$

where
salary =salary of CEO
sales=annual firm sales
ROE=return on equity in percent
ROS =return on firm's stock

Interpreter the partial regression coefficients

Questions What about the statistical significance of the observed results?

For the coefficient of log(sales) of 0.280, Is this coefficient statistically significant different from zero?

For the coefficient of ROE of 0.0174, Is this coefficient statistically significant different from zero?

For the coefficient of ROS of 0.00024, Is this coefficient statistically significant different from zero?

Are these three coefficients statistically significant?
To answer these questions, we have to learn the kinds of hypothesis testing.

8.1 Hypothesis Testing About Individual Regression Coefficients

We can use the t-test to test a hypothesis about any individual partial regression coefficient.

8.1.1 Two-tail test:

Let us postulate that

$H_0: \quad \beta_2 = 0$

$H_1: \quad \beta_2 \neq 0$

8.1.2 One-tail test:

Let us postulate that

$H_0: \ \beta_2 \leq 0$

$H_1: \ \beta_2 > 0$

8.2 Testing The Overall Significance of the Sample Regression

In the previous section, we test the significance of the estimated partial regression coefficients individually, that is under the separate hypothesis that each true population partial regression coefficient was zero. But now we are interested in testing β_2 , β_3 and β_4 are jointly or simultaneously equal to zero. In other words, we would like to test the following hypothesis:

$$H_0 \;\; \beta_2 = \beta_3 = \beta_4 = 0$$

In order to reach this goal, we have to learn the following test.

The Analysis of Variance Approach to Testing the Overall Significance of an Observed Multiple Regression: The F-Test

The joint hypothesis can be tested by the **Analysis of Variance (ANOVA)** which can be demonstrated as follows:

Table 8.1: ANOVA Table for the three-variable regression model

Source of variation	Sum of Square **SS**	**df**	Mean Sum of Square **MSS**
Due to regression (ESS)			
Due to residuals (RSS)			
TSS			

Decision Rule Given the k- variable regression model:

$$Y_i = \beta_1 + \beta_2 X_{2i} + \beta_3 X_{3i} + ... + \beta_k X_{ki} + u_i$$

To test the hypothesis

$$H_0 : \quad \beta_2 = \beta_3 = ... = \beta_k = 0$$

(i.e ., all slope coefficients are simultaneously zero) versus

H_1 Not all slope coefficients are simultaneously zero

If $F > F_\alpha(k-1, n-k)$, we reject H_o ; otherwise we cannot reject it, where $F_\alpha(k-1, n-k)$ is the critical F value at the α level of significance and (k-1) numerator df and (n-k) denominator df.

An important Relationship between R^2 and F

Table 8.2: ANOVA Table in Terms of R^2

Source of variation	Sum of Square **SS**	**df**	Mean Sum of Square **MSS**
Due to regression (ESS)			
Due to residuals (RSS)			
TSS			

Decision Rule Testing the overall significance of a regression in terms of R^2

Given the k- variable regression model:

$$Y_i = \beta_1 + \beta_2 X_{2i} + \beta_3 X_{3i} + ... + \beta_k X_{ki} + u_i$$

To test the hypothesis

$$H_0 : \quad \beta_2 = \beta_3 = ... = \beta_k = 0$$

(i.e ., all slope coefficients are simultaneously zero) versus

H_1 Not all slope coefficients are simultaneously zero

Compute

$$F = \frac{R^2/(k-1)}{(1-R^2)/(n-k)}$$

If $F > F_\alpha(k-1, n-k)$, we reject H_o ; otherwise we cannot reject it, where $F_\alpha(k-1, n-k)$ is the critical F value at the α level of significance and (k-1) numerator df and (n-k) denominator df.

8.3 The "Incremental" or "Marginal" Contribution of an Explanatory Variable

Let consider the following regression:

$$Y_i = \alpha_1 + \alpha_2 X_{2i} + u_i$$

Having run the above regression, let us suppose we decide to add the additional variable, X_{3i}, to the model and obtain the multiple regression as follow:

$$Y_i = \beta_1 + \beta_2 X_{2i} + \beta_3 X_{3i} + u_i$$

Comparing between these two regressions, we might need to answer the below questions:

[1]. What are the marginal, or incremental, contribution of X_{3i}, knowing that X_{2i} is already in the model and that it is significantly related to Y_i.

[2]. Is the incremental contribution of X_{3i} statistically significant?

[3]. What is the criterion for adding variables to the model?

By contribution we mean whether the additional of the variable, X_{3i}, to the model increases ESS (and hence R^2) "significantly" in relation to the RSS. This contribution is called **the incremental, or marginal** contribution of an additional variable.

To assess the incremental contribution of X_3 after allowing for the contribution of X_2, we form

$$\begin{aligned} F &= \frac{Q_2/\text{df}}{Q_4/\text{df}} \\ &= \frac{(ESS_{new} - ESS_{old})/\text{number of new regressors}}{RSS_{new}/df(\text{=n-number of parameters in the new model})} \end{aligned} \tag{8.2}$$

Under the normality assumption of u_i and CLRM assumptions, this F value follows the F distribution with 1 and n-number of parameters in the new model.

Table 8.3: ANOVA Table To Assess Incremental Contribution of A Variable(s)

Source of variation	Sum of Square **SS**	**df**	Mean Sum of Square **MSS**
ESS due to X_2 alone	$Q_1 = \hat{\alpha_2}^2 \sum x_2^2$	1	$\frac{Q_1}{1}$
ESS due to the addition of X_3	$Q_2 = Q_3 - Q_1$	1	$\frac{Q_2}{1}$
ESS due to both X_2,X_3	$Q_3 = \hat{\beta}_2 \sum x_{2i} y_i + \hat{\beta}_3 \sum x_{3i} y_i$	2	$\frac{Q_3}{2}$
RSS	$Q_4 = Q_5 - Q_3$	n-3	$\frac{Q_4}{n-3}$
TSS	$Q_5 = \sum y_i^2$	n-1	

As usual method, we can re write 8.2 in term of R^2 only. Thus the F ratio of 8.2 is equivalent to the following F ratio:

$$\begin{aligned} F &= \frac{R^2_{new} - R^2_{old}/\text{df}}{(1 - R^2_{new})/\text{df}} \\ &= \frac{(R^2_{new} - R^2_{old})/\text{number of new regressors}}{1 - R^2_{new}/df(\text{=n-number of parameters in the new model})} \end{aligned} \tag{8.3}$$

This F ratio follows the F distribution with 1 and n-number of parameters in the new model.

Example

Consider the child mortality example. We considered the behavior of child mortality (CM) in relation to per capita GNP (PGNP). There we found that PGNP has a negative impact on CM, as one would expect. Now let us bring in female literacy as measured by the female literacy rate (FLR). A priori, we expect that FLR too will have a negative impact on CM. Our sample consists of 64 countries.

In model 1, we regressed child mortality (CM) on per capita GNP (PGNP) and female literacy rate (FLR).

Model 1:

$$\begin{aligned} \widehat{CM_i} &= 263.6416 - 0.0056PGNP_i - 2.2316FLR_i \\ se &= (11.5932) \quad (0.0019) \qquad (0.2099) \quad R^2 = 0.7077 \end{aligned} \tag{8.4}$$

Now we extend this model to model 2 by including total fertility rate (TFR):

Model 2:

$$\begin{aligned} \widehat{CM_i} &= 168.3067 - 0.00555GNP_i - 1.7680FLR_i + 12.8686TFR_i \\ se &= (32.8916) \quad (0.0018) \qquad (0.2480) \qquad (?) \quad R^2 = 0.7474 \end{aligned} \tag{8.5}$$

Questions

1.How would you choose between models 1 and 2? Which statistical test would you use to answer this question? Show the necessary calculations.
2. We have not given the standard error of the coefficient of TFR. Can you find it out? (Hint: Recall the relationship between the t and F distributions.

8.4 Testing the Equality of Two Regression Coefficients

Suppose we have the following model:

$$Y_i = \beta_1 + \beta_2 X_{2i} + \beta_3 X_{3i} + \beta_4 X_{4i} + ... + \beta_k X_{ki} + u_i$$

We would like to test the hypotheses:

$$H_0: \ \beta_3 = \beta_4 \ \text{ or } \ (\beta_3 - \beta_4) = 0$$
$$H_1: \ \beta_3 \neq \beta_4 \ \text{ or } \ (\beta_3 - \beta_4) \neq 0$$

Under the classical assumptions, it can be shown that:

$$t = \frac{(\hat{\beta}_3 - \hat{\beta}_4) - (\beta_3 - \beta_4)}{se(\hat{\beta}_3 - \hat{\beta}_4)}$$

where the t follows the t distribution with (n-k) df because the above equation is a k-variable model, where k is the total number of parameters estimated, including the constant term.

The $se(\hat{\beta}_3 - \hat{\beta}_4)$ is calculated from the following formula:

$$se(\hat{\beta}_3 - \hat{\beta}_4) = \sqrt{var(\hat{\beta}_3) + var\hat{\beta}_4 - 2cov(\hat{\beta}_3, \hat{\beta}_4)}$$

Example

among other things, you were asked to consider the following demand function for chicken:

$$\begin{aligned} \widehat{\ln Y_t} &= 2.0328 + 0.4515 \ln X_{2t} - 0.3772 \ln X_{3t} \\ se &= (0.1162) \quad (0.0247) \qquad (0.0635) \quad R^2 = 0.9801 \end{aligned} \tag{8.6}$$

where Y = per capita consumption of chicken, lb, X_2 = real disposable per capita income, \$, X_3 = real retail price of chicken per lb.

Question

For the above demand function, how would you test the hypothesis that the income elasticity is equal in value but opposite in sign to the price elasticity of demand? Show the necessary calculations. [Note: $\text{cov}(\hat{\beta}_2, \hat{\beta}_3) = -0.00142$. and the sample data = 23 observations]

8.5 Restricted Least Squares: Testing Linear Equality Restriction

In economic theories, the coefficients in a regression model need to satisfy some linear equality restrictions, For example, in microeconomics, consider the Cobb-Douglas production function:

$$Y_i = \beta_1 X_{2i}^{\beta_2} X_{3i}^{\beta_3} e^{u_i}$$

where Y=output, X_2 = labor input, and X_3=capital input. We can transform the above equation to be the log form as:

$$\ln Y_i = \beta_0 + \beta_2 \ln X_{2i} + \beta_3 \ln X_{3i} + u_i$$

where $\beta_0 = \ln \beta_1$

Now, if there are the constant returns to scale, economic theory would suggest that

$$\beta_2 + \beta_3 = 1$$

which is an example of a linear equality restriction.
In order to test the above linear equality restriction, we can follow two approaches which are:

[1]. The t-test approach

[2]. The F-test approach: Restricted Least Squares.

First Approach: The t-Test
A test of the hypothesis or restriction can be conducted by the t-test:

$$t = \frac{(\hat{\beta}_2 + \hat{\beta}_3) - (\beta_2 + \beta_3)}{se(\hat{\beta}_2 + \hat{\beta}_3)}$$

where the t follows the t distribution with (n-k) df for a k-variable model, where k is the total number of parameters estimated, including the constant term. In this case, df=n-3.

The $se(\hat{\beta}_2 + \hat{\beta}_3)$ is calculated from the following formula:

$$se(\hat{\beta}_2 + \hat{\beta}_3) = \sqrt{var(\hat{\beta}_2) + var\hat{\beta}_3 + 2cov(\hat{\beta}_2, \hat{\beta}_3)}$$

Example
Consider the Cobb-Douglas production function to the Mexican economy (1955-1974: n=20):

$$\begin{aligned} \widehat{\ln GDP_t} &= -1.6524 + 0.3397 \ln Labor_t + 0.8460 \ln Capital_t \\ t &= (-2.7259) \quad (1.8295) \qquad (9.0625) \quad R^2 = 0.9951 \;\; RSS_{UR} = 0.0136 \end{aligned} \tag{8.7}$$

where GDP = Real GDP, Millions of 1960 pesos, *Labor* = Employment, Thousands of People, *Capital* = Fixed Capital, Millions of 1960 pesos.

Question
As you can see, the output/labor elasticity is about 0.34 and the output/capital elasticity is about 0.85. If we add these coefficients, we obtain 1.19, suggesting that perhaps the Mexican economy during the stated time period was experiencing increasing returns to scale. However, we do not know if 1.19 is statistically different from 1.

Therefore, we have to test this linear equality restriction.

8.6 The F-Test Approach: Restricted Least Squares

From the Cobb-Douglas production function:

$$\ln Y_i = \beta_0 + \beta_2 \ln X_{2i} + \beta_3 \ln X_{3i} + u_i \tag{8.8}$$

if there are the constant returns to scale, economic theory would suggest that

$$\beta_2 + \beta_3 = 1$$

We can rewrite it as:

$$\beta_2 = 1 - \beta_3$$

or

$$\beta_3 = 1 - \beta_2$$

Using either of these equalities, we can eliminate one of the β coefficients. Therefore, we can rewrite the Cobb-Douglas production function as:

$$\ln (Y_i/X_{2i}) = \beta_0 + \beta_3 \ln (X_{3i}/X_{2i}) + u_i \tag{8.9}$$

where $\frac{Y_i}{X_{2i}}$=output/labor ratio
$\frac{X_{3i}}{X_{2i}}$= capital labor ratio.

It should be noted that:
8.8 is known as **unrestricted Least Squares (URLS)**
8.9 is known as **restricted Least Squares (RLS)**

We can compare the unrestricted and restricted least-squares regressions by applying the F-test as follows:

$\sum \hat{U}_{UR}^2$=RSS of the unrestricted regression 8.8

$\sum \hat{U}_R^2$ = RSS of the restricted regression 8.9

m=number of linear restrictions (in this example, we have 1 restriction)

k= number of parameters in the unrestricted regression

n= number of observations

Then, we have

$$\begin{aligned} F &= \frac{(RSS_R - RSS_{UR})/m}{RSS_{UR}/(n-k)} \\ &= \frac{(\sum \hat{U}_R^2 - \sum \hat{U}_{UR}^2)/m}{\sum \hat{U}_{UR}^2/(n-k)} \end{aligned} \tag{8.10}$$

follows the F-distribution with m, (n-k) df.

We can also rewrite the F-test in terms of R^2 as follows:

$$F = \frac{R_{UR}^2 - R_R^2/\text{m}}{(1-R_{UR}^2)/\text{n-k}} \tag{8.11}$$

Example
Consider the Cobb-Douglas production function to the Mexican economy(1955-1974: n=20):

$$\begin{aligned} \widehat{\ln GDP_t} &= -1.6524 + 0.3397 \ln Labor_t + 0.8460 \ln Capital_t \\ t &= (-2.7259) \quad (1.8295) \qquad (9.0625) \quad R^2 = 0.9951 \;\; RSS_{UR} = 0.0136 \end{aligned} \tag{8.12}$$

where GDP = Real GDP, Millions of 1960 pesos, *Labor* = Employment, Thousands of People, *Capital* = Fixed Capital, Millions of 1960 pesos.

The restriction of constant return to scale, which gives the following regression:

$$\begin{aligned} \widehat{\ln (GDP/Labor)}_t &= -0.4947 + 1.0153 \ln (Capital/Labor)_t \\ t &= (-4.0612) \quad (28.1056) \quad R_R^2 = 0.9777 \;\; RSS_R = 0.0166 \end{aligned} \tag{8.13}$$

8.7 Testing for Structural or Parameter Stability of Regression Models: The Chow Test

Sometime when we estimate the regression model, it may happen that there is a **Structural Change** in the relationship between the regressand Y and the regressors X's, especially the model involving time series data. The structural change may be due to the external forces (i.e the financial crisis of 2007-2008) or due to policy changes (such as the switch from a fixed exchange rate system to a flexible exchange rate system in 1997).

The question is **"How do we figure out that there is a structural change in our sample data?"**

To answer this question, consider the following example.

Based on the sample data, we found out that in 1982 the United State suffers its worst peacetime regression. This event might disturb the relationship between savings and DPI.

To see this effect, we can divide our sample data into two time periods: 1970-1981 (Pre-1982 crisis) and 1982-1995 (Post-1982 crisis).

Therefore we have three possible regressions:

Time period 1970-1981: $Y_t = \beta_1 + \beta_2 X_t + u_1 t$ where $n_1 = 12$

Time period 1982-1995: $Y_t = \gamma_1 + \gamma_2 X_t + u_2 t$ where $n_2 = 14$

Time period 1970-1995: $Y_t = \alpha_1 + \alpha_2 X_t + u_t$ where $n = n_1 + n_2 = 26$

For our sample data, we can get the following results:

Time period 1970-1981:

$$\begin{aligned} \hat{Y}_t &= 1.0161 + 0.0803 X_t \\ t &= (0.00873) \quad (9.6015) \end{aligned} \tag{8.14}$$

$R^2 = 0.9021 \quad RSS_1 = 1785.032 \quad df = 10$

Time period 1982-1995:

$$\begin{aligned} \hat{Y}_t &= 153.4947 + 0.0148 X_t \\ t &= (4.6922) \quad (1.7707) \end{aligned} \tag{8.15}$$

$R^2 = 0.2971 \quad RSS_2 = 10,005.22 \quad df = 12$

Time period 1970-1995:

$$\begin{aligned} \hat{Y}_t &= 62.4226 + 0.0376 X_t \\ t &= (4.8917) \quad (8.8937) \end{aligned} \tag{8.16}$$

$R^2 = 0.7672 \quad RSS_3 = 23,248.30 \quad df = 24$

We can apply **the Chow test** to investigate the structural changes that may be caused by differences in the intercept or the slope coefficient or both.

The chow test assumes that:

[1] $u_{1t} \sim N(0, \sigma^2)$ and $u_{2t} \sim N(0, \sigma^2)$

[2] The two error terms u_{1t} and u_{2t} are independently distributed.

Chow Test

H_0 : There is no structural change in the model

H_1 : There is structural change in the model

Then, we need to construct the F-ratio:

$$F = \frac{(RSS_R - RSS_{UR})/k}{RSS_{UR}/(n_1 + n_2 - 2k)} \tag{8.17}$$

where the F ratio follows the F distribution with k and $(n_1 + n_2 - 2k)$ df in the numerator and denominator, respectively.

We do not reject the null hypothesis of parameter stability (i.e no structural change) if the computed F value does not exceed the critical value F value obtained from the F table.

9. Dummy Variable Regression Models

In the previous chapter, the dependent and independent variables in our multiple regression models have had **quantitative** meaning. For example, the salary of CEO,annual firm sales,return on equity in percent, and return on firm's stock. In each case the magnitude of the variable conveys useful information.

However, in the empirical work, we must also incorporate **qualitative factors** into regression models. The gender or race of an individual, the industry of a firm (manufacturing, retail, and so on), and the region in Thailand where a city is located (north, south, west, and so on) are all considered as the qualitative factors.

9.1 Describing Qualitative Information

Normally, qualitative factors often come in the form of binary information:

Example:

[1] A person is female or male or female.

[2] A firm offers a certain kind of employee pension plan or it does not.

[3] A farm is located nearby the dam or not.

All of these examples, the relevant information can be captured by defining a **binary variable** or a zero-one variable.

In econometrics, binary variables are most commonly called **dummy variables**, although this name is not especially descriptive.

In defining a dummy variable, we must decide which event is assigned the value one and which is assigned the value zero.

Question: Why do we use the the values zero and one to describe qualitative information?

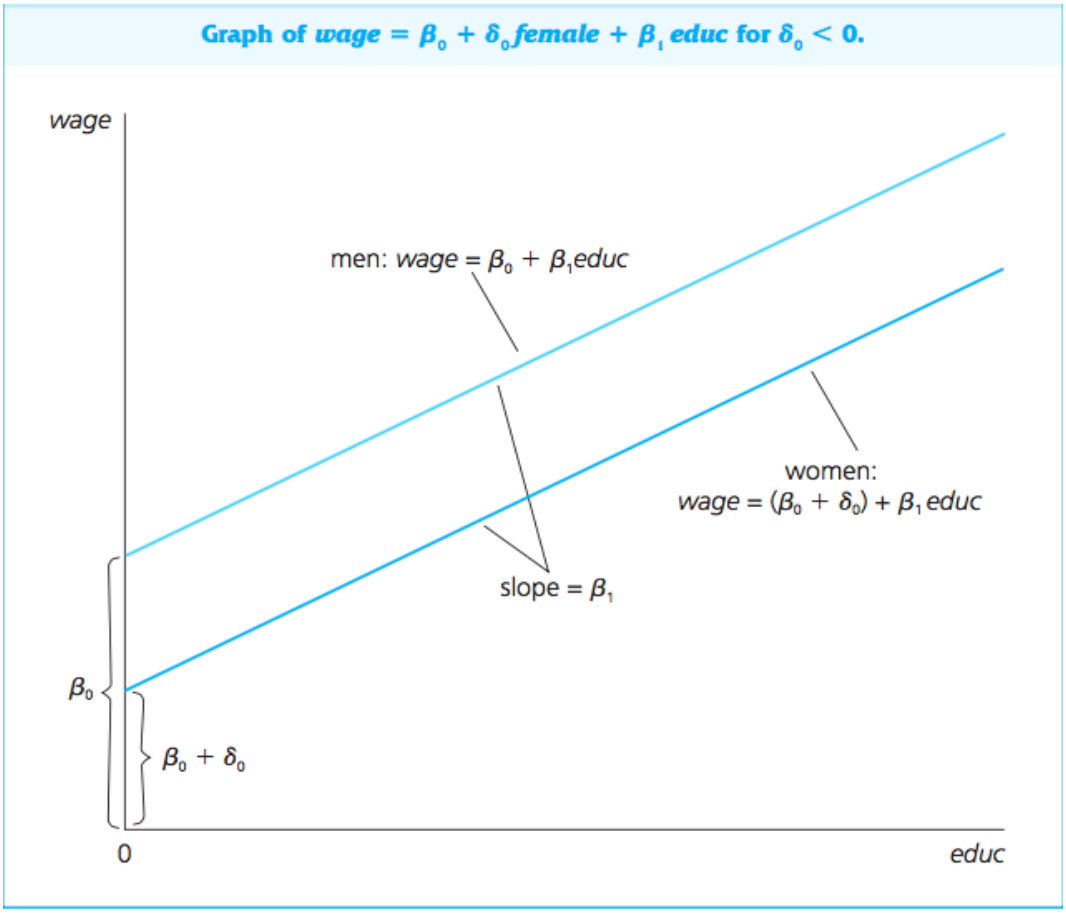
Answer: These values are arbitrary: any two different values would do. The real benefit of capturing qualitative information using zero-one variable is that it leads to regression models where the parameters have very natural interpretations.

9.1.1 A Single Dummy Independent Variable

Suppose we would like to estimate the following simple model of hourly wage determination:

$$wage_i = \beta_0 + \delta_0 \text{female} + \beta_1 \text{edu} + u_i$$

Figure 9.1: Graph of Wage



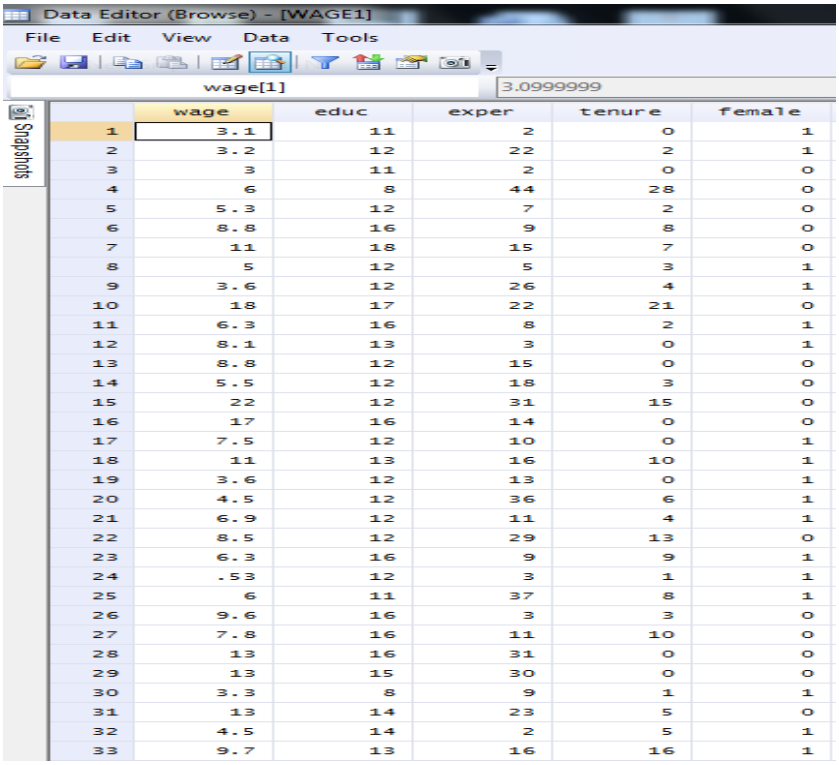
Now, we added more variables into the wage model. Taking males as the base group, a model that controls for experience and tenure in addition to education is

$$wage_i = \beta_0 + \delta_0 \text{female} + \beta_1 \text{edu} + \beta_2 \text{exper} + \beta_3 \text{tenure} + u_i$$

If edu, expert, and tenure are all relevant productivity characteristics, the null hypothesis of no difference between men and women (No wage discrimination) is:

In table 9.1, it represents the partial listing of the sample data of wage model. We see that Person 1 is female, Person 2 is female , Person 3 is male, and so on.

Table 9.1: A Partial Listing of the Wage Data.



	wage	educ	exper	tenure	female
1	3.1	11	2	0	1
2	3.2	12	22	2	1
3	3	11	2	0	0
4	6	8	44	28	0
5	5.3	12	7	2	0
6	8.8	16	9	8	0
7	11	18	15	7	0
8	5	12	5	3	1
9	3.6	12	26	4	1
10	18	17	22	21	0
11	6.3	16	8	2	1
12	8.1	13	3	0	1
13	8.8	12	15	0	0
14	5.5	12	18	3	0
15	22	12	31	15	0
16	17	16	14	0	0
17	7.5	12	10	0	1
18	11	13	16	10	1
19	3.6	12	13	0	1
20	4.5	12	36	6	1
21	6.9	12	11	4	1
22	8.5	12	29	13	0
23	6.3	16	9	9	1
24	.53	12	3	1	1
25	6	11	37	8	1
26	9.6	16	3	3	0
27	7.8	16	11	10	0
28	13	16	31	0	0
29	13	15	30	0	0
30	3.3	8	9	1	1
31	13	14	23	5	0
32	4.5	14	2	5	1
33	9.7	13	16	16	1

Table 9.2: The command function to estimate the wage model in STATA program

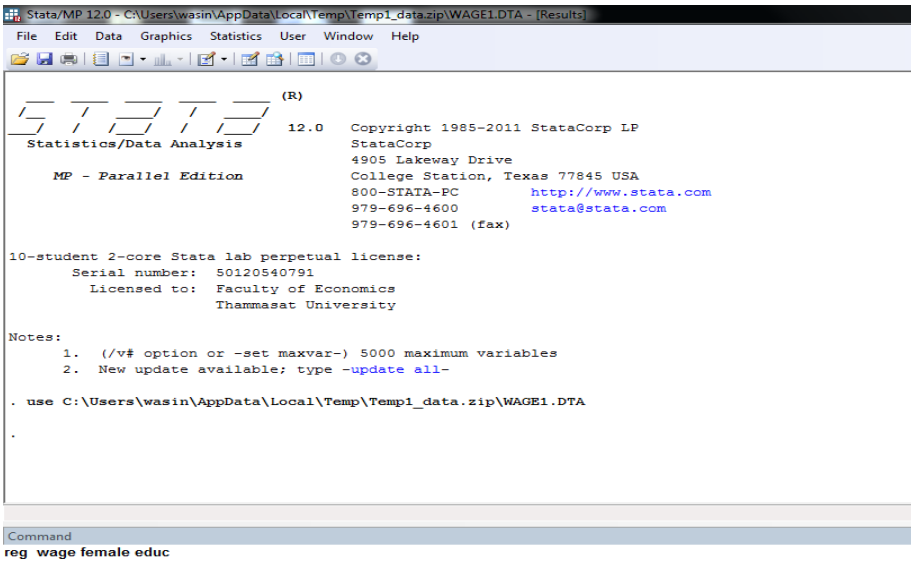


Table 9.3: $wage_i = \beta_0 + \delta_0 \textbf{female} + \beta_1 \textbf{edu} + \beta_2 \textbf{exper} + \beta_3 \textbf{tenure} + u_i$

```
. reg wage female educ exper tenure
```

Source	SS	df	MS		
Model	2603.10658	4	650.776644	Number of obs	= 526
Residual	4557.30771	521	8.7472317	F(4, 521)	= 74.40
				Prob > F	= 0.0000
Total	7160.41429	525	13.6388844	R-squared	= 0.3635
				Adj R-squared	= 0.3587
				Root MSE	= 2.9576

wage	Coef.	Std. Err.	t	P>|t|	[95% Conf.	Interval]
female	-1.810852	.2648252	-6.84	0.000	-2.331109	-1.290596
educ	.5715048	.0493373	11.58	0.000	.4745802	.6684293
exper	.0253959	.0115694	2.20	0.029	.0026674	.0481243
tenure	.1410051	.0211617	6.66	0.000	.0994323	.1825778
_cons	-1.567939	.7245511	-2.16	0.031	-2.991339	-.144538

Example: the Hourly Wage Equation:

$$\begin{aligned} \widehat{\text{wage}} &= -1.5679 - 1.8109 \ \text{female} + 0.5715 \ \text{edu} + 0.025 \ \text{exper} + 0.141 \ \text{tenure} \\ &= (0.7246) \quad (0.2648) \quad (0.0493) \quad (0.0116) \quad (0.0212) \end{aligned} \tag{9.1}$$

$R^2 = 0.3635 \quad n = 526$

Interpret the model:

The intercept:

Table 9.4: $wage_i = \beta_0 + \delta_0 \textbf{female} + u_i$

```
reg wage female
```

Source	SS	df	MS		
Model	828.220467	1	828.220467	Number of obs =	526
Residual	6332.19382	524	12.0843394	F(1, 524) =	68.54
				Prob > F =	0.0000
				R-squared =	0.1157
				Adj R-squared =	0.1140
Total	7160.41429	525	13.6388844	Root MSE =	3.4763

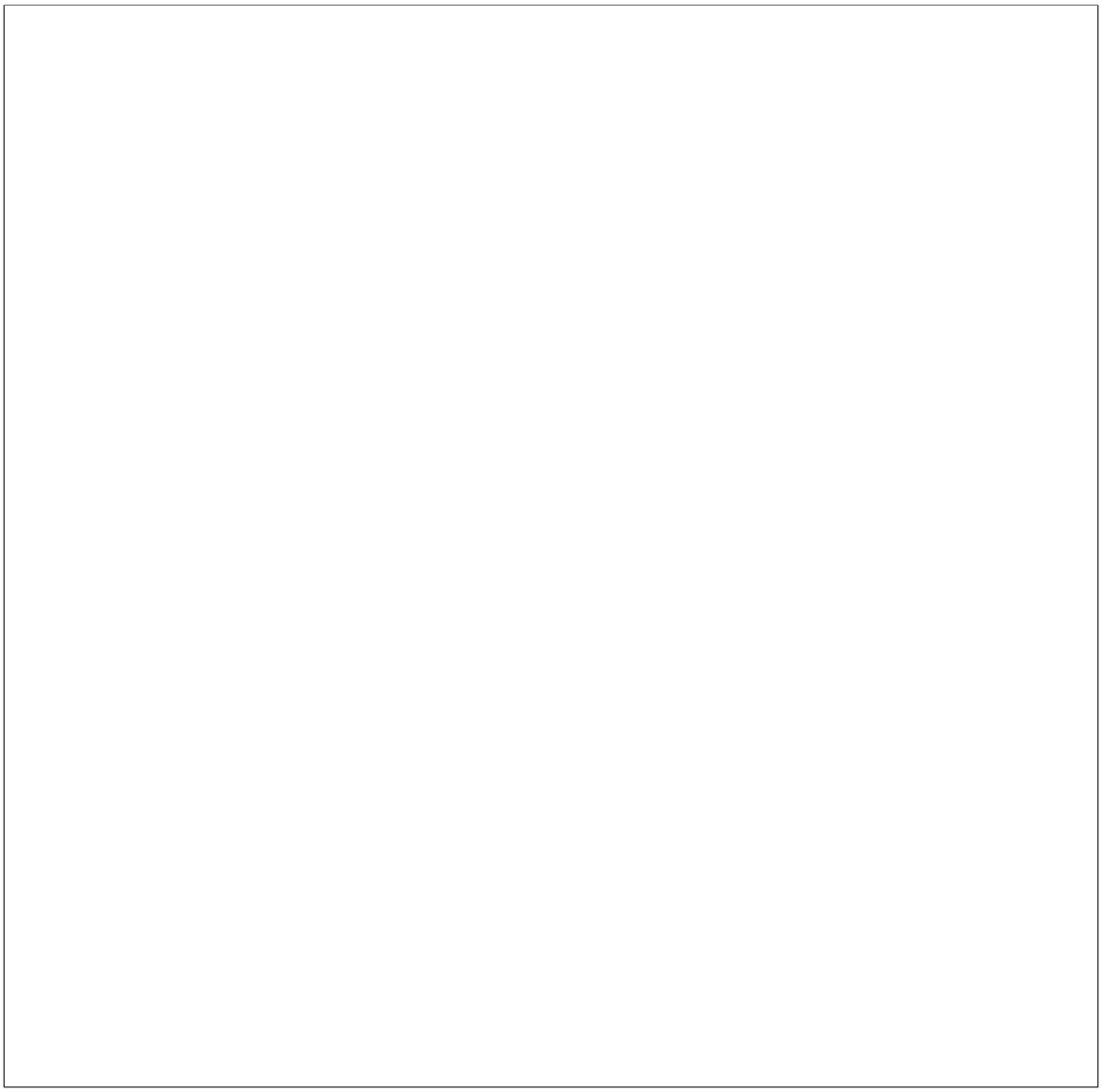
wage	Coef.	Std. Err.	t	P>|t|	[95% Conf.	Interval]
female	-2.51183	.3034092	-8.28	0.000	-3.107878	-1.915782
_cons	7.099489	.2100082	33.81	0.000	6.686928	7.51205

The coefficient on female

It is informative to compare the coefficient on female in the above equation to the estimate we get when all other explanatory variables are dropped from the equation:

$$\begin{aligned} \widehat{\text{wage}} &= 7.0995 - 2.5118 \ \text{female} \\ se &= (0.2100) \qquad (0.3034) \end{aligned} \tag{9.2}$$

$R^2 = 0.1157 \quad n = 526$



9.2 Interpreting Coefficients on Dummy Explanatory Variables When the Dependent Variable is log(y)

In this section, we will study a model that has the dependent variable appearing in logarithmic form, with one or more dummy variables approving as independent variables.
Question: How do we interpret the dummy variable coefficients in this case?

Answer: Not surprisingly, the coefficients have a percentage interpretation.

Let us reestimate the wage equation , using log (wage) as the dependent variable and adding quadratics in expert and tenure:

$$log(wage_i) = \beta_0 + \delta_0 female + \beta_1 edu + \beta_2 exper + \beta_3 exper^2 + \beta_4 tenure + \beta_5 tenure^2 + u_i$$

The Stata result is shown in table 9.5.

Table 9.5:

```
reg lwage female educ exper expersq tenure tenursq
```

Source	SS	df	MS		
Model	65.3791009	6	10.8965168	Number of obs	= 526
Residual	82.9506505	519	.159827843	F(6, 519)	= 68.18
				Prob > F	= 0.0000
Total	148.329751	525	.28253286	R-squared	= 0.4408
				Adj R-squared	= 0.4343
				Root MSE	= .39978

lwage	Coef.	Std. Err.	t	P>|t|	[95% Conf.	Interval]
female	-.296511	.0358055	-8.28	0.000	-.3668524	-.2261696
educ	.0801967	.0067573	11.87	0.000	.0669217	.0934716
exper	.0294324	.0049752	5.92	0.000	.0196585	.0392063
expersq	-.0005827	.0001073	-5.43	0.000	-.0007935	-.0003719
tenure	.0317139	.0068452	4.63	0.000	.0182663	.0451616
tenursq	-.0005852	.0002347	-2.49	0.013	-.0010463	-.0001241
_cons	.416691	.0989279	4.21	0.000	.2223425	.6110394

Example: Log Hourly Wage Equation:

$$\begin{aligned}\widehat{\log(\text{wage})} &= 0.4167 - 0.2965\ \text{female} + 0.0802\ \text{edu} + 0.0294\ exper - 0.0006\ exper^2 \\ &\quad (0.0989) \qquad (0.0358) \qquad (0.0068) \qquad (0.0050) \qquad (0.0001) \\ &\quad + 0.0317\ tenure - 0.0006\ tenure^2 \\ &\quad (0.0068) \qquad (0.0002)\end{aligned} \tag{9.3}$$

$R^2 = 0.4408 \quad n = 526$

Interpret the model:

The coefficient on female

9.3 Using Dummy Variables for Multiple Categories

We can use several dummy independent variables in the same equation. For example, we could add the dummy variable **married** to the wage model.

The previous model:

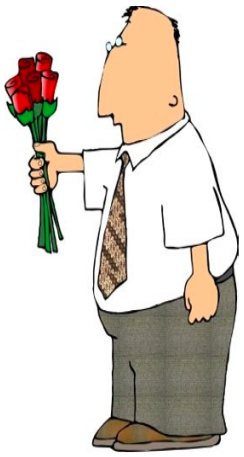
$$log(wage_i) = \beta_0 + \delta_0 female + \beta_1 edu + \beta_2 exper + \beta_3 exper^2 + \beta_4 tenure + \beta_5 tenure^2 + u_i$$

Now, Let us estimate a model that allows for wage differences among four groups:

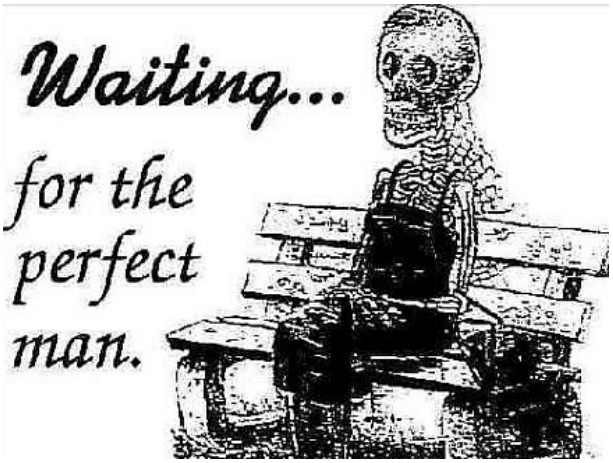
[1.] Married Men

[2] Married Women

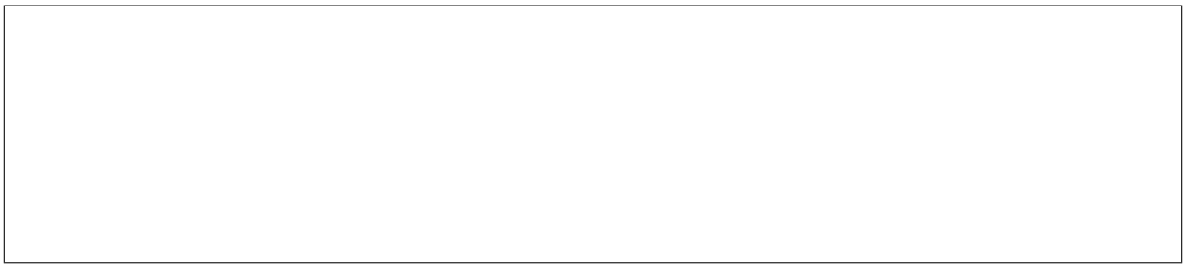
[3] Single Men



[4] Single Women



To do this, we must select a base group:



Now, we need to define dummy variables for each of the remaining groups.

Therefore, our model is:

$$log(wage_i) = \beta_0 + \delta_0 marrmale + \delta_1 marrfem + + \delta_2 singfem +$$

$$\beta_1 edu + \beta_2 exper + \beta_3 exper^2 + \beta_4 tenure + \beta_5 tenure^2 + u_i$$

We of course drop the dummy variable (female). (Why?)

Table 9.5:

```
reg lwage marrmale marrfem singfem educ exper expersq tenure tenursq
```

Source	SS	df	MS		
Model	68.3617623	8	8.54522029	Number of obs	= 526
Residual	79.9679891	517	.154676961	F(8, 517)	= 55.25
				Prob > F	= 0.0000
Total	148.329751	525	.28253286	R-squared	= 0.4609
				Adj R-squared	= 0.4525
				Root MSE	= .39329

lwage	Coef.	Std. Err.	t	P>|t|	[95% Conf.	Interval]
marrmale	.2126757	.0553572	3.84	0.000	.103923	.3214284
marrfem	-.1982676	.0578355	-3.43	0.001	-.311889	-.0846462
singfem	-.1103502	.0557421	-1.98	0.048	-.219859	-.0008414
educ	.0789103	.0066945	11.79	0.000	.0657585	.092062
exper	.0268006	.0052428	5.11	0.000	.0165007	.0371005
expersq	-.0005352	.0001104	-4.85	0.000	-.0007522	-.0003183
tenure	.0290875	.006762	4.30	0.000	.0158031	.0423719
tenursq	-.0005331	.0002312	-2.31	0.022	-.0009874	-.0000789
_cons	.3213781	.100009	3.21	0.001	.1249041	.5178521

$$\begin{aligned}\widehat{\log(\text{wage})} &= 0.3214 + 0.2127\ \text{marrmale} - 0.1983\ \text{marrfem} - 0.1104\ sing fem \\ &\quad (0.1000)\qquad (0.0554)\qquad (0.0578)\qquad (0.0557) \\ &\quad +0.0789\ edu + 0.0268\ exper - 0.0005\ exper^2 + 0.0291\ tenure - 0.0005\ tenure^2 \\ &\quad (0.0067)\qquad (0.0268)\qquad (0.0001)\qquad (0.0068) + \qquad (0.0002)\end{aligned} \tag{9.4}$$

$R^2 = 0.4609 \quad n = 526$

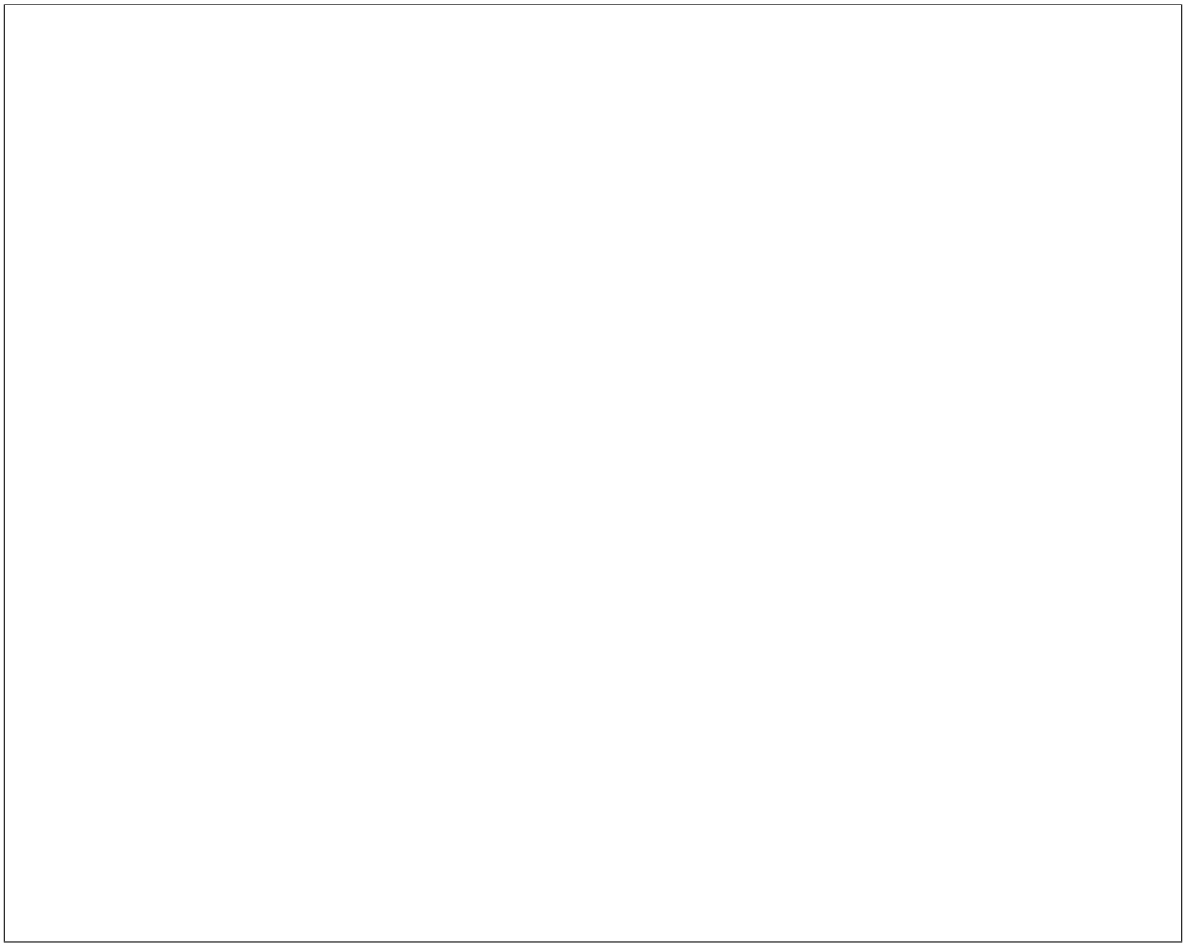
Interpret the model:

9.3.1 Interactions Involving Dummy Variables

8.5.1 The Interactions Among Dummy Variables:

We can recast the model by adding an **interaction term** between female and married to the model where female and married appear separately. This allows the marriage premium to depend on gender. The estimated model with the female-married interaction term is :

$$\begin{aligned}\widehat{\log(\text{wage})} &= 0.321 - 0.110\ \text{female} + 0.213\ \text{married} \\ &\quad (0.100) \qquad (0.056) \qquad (0.055) \\ &\quad +0.301\ \ female \cdot married + ..., \\ &\quad (0.072)\end{aligned} \tag{9.5}$$



8.5.2 The interaction between Dummy Variable/s and Explanatory Variable/s: the Allowing for the Different Slopes

There are also occasions for interacting dummy variables with explanatory variables that are not dummy variables to allow for a **difference in slope.**

To see the interaction between female and edu, we can rewrite the model as follow:

$$wage_i = \beta_0 + \delta_0 \text{female} + \beta_1 \text{edu} + \delta_1 female \cdot edu + u_i$$

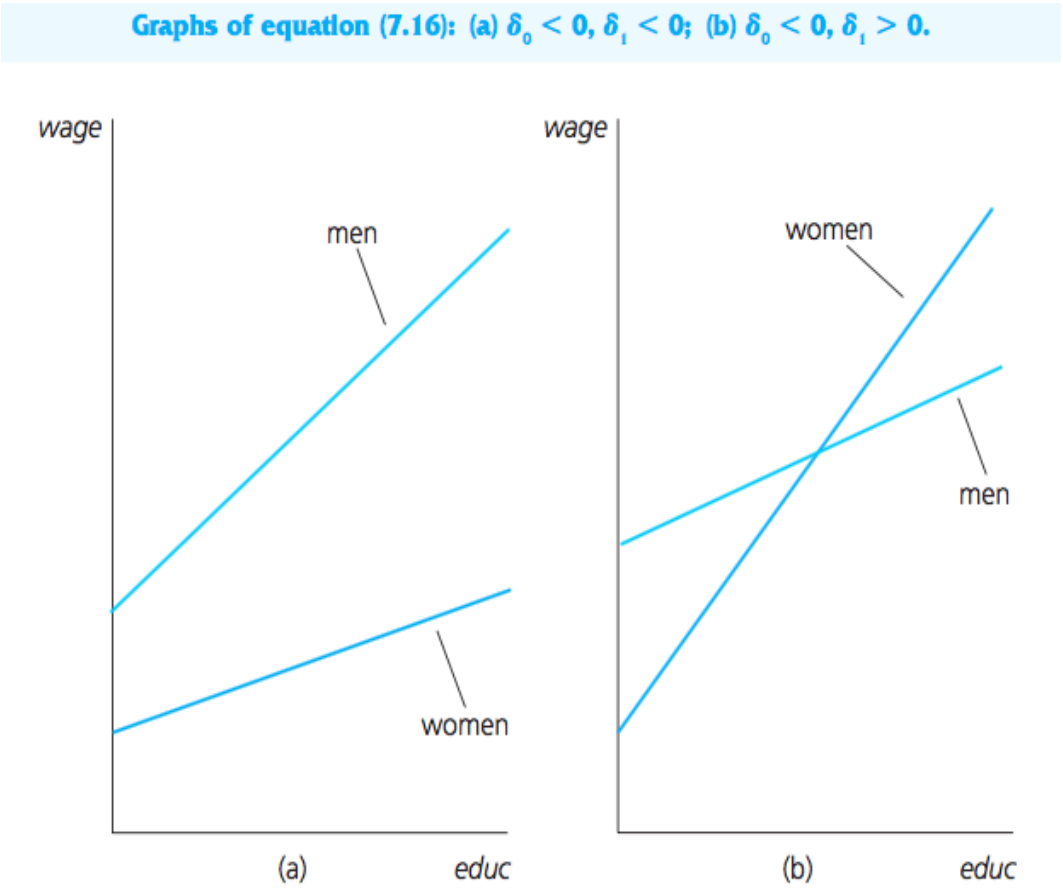
Men Group we plug female =0

Therefore:

Women Group we plug female =1

Therefore:

Figure 9.2: Graph of the Wage Model with an Interaction between female and education



Example

Table 9.5:

```
gen femed = female*educ
```

```
reg lwage female educ femed exper expersq tenure tenursq
```

Source	SS	df	MS		
Model	65.4081534	7	9.34402192	Number of obs =	526
Residual	82.921598	518	.160080305	F(7, 518) =	58.37
				Prob > F =	0.0000
Total	148.329751	525	.28253286	R-squared =	0.4410
				Adj R-squared =	0.4334
				Root MSE =	.4001

lwage	Coef.	Std. Err.	t	P>|t|	[95% Conf.	Interval]
female	-.2267886	.1675394	-1.35	0.176	-.5559289	.1023517
educ	.0823692	.0084699	9.72	0.000	.0657296	.0990088
femed	-.0055645	.0130618	-0.43	0.670	-.0312252	.0200962
exper	.0293366	.0049842	5.89	0.000	.019545	.0391283
expersq	-.0005804	.0001075	-5.40	0.000	-.0007916	-.0003691
tenure	.0318967	.006864	4.65	0.000	.018412	.0453814
tenursq	-.00059	.0002352	-2.51	0.012	-.001052	-.000128
_cons	.388806	.1186871	3.28	0.001	.1556388	.6219732

$$\begin{aligned}\widehat{\log(\text{wage})} &= 0.3889 - 0.2268\ \text{female} + 0.082\ \text{edu} - 0.0056\ female \cdot edu \\ &\quad (0.1187)\qquad (0.1675)\qquad (0.0085)\qquad (0.0131) \\ &\quad +0.0293\ exper - 0.0006\ exper^2 + 0.0319\ tenure - 0.00059\ tenure^2 \\ &\quad (0.0050)\qquad (0.0001)\qquad (0.0069) + \qquad (0.0002)\end{aligned} \tag{9.6}$$

$R^2 = 0.4410 \quad n = 526$

10. Multicollinearity

10.1 Nature of Multicollinearity

One of the required assumptions of the classical linear regression model (CLRM) in Chapter 7 is that there is no *perfect multicollinearity*. In this chapter, we take a critical look at this assumption. To clarify, we firstly begin with the multiple regression model as in Equation 10.1, in general, where $X_1 = 1$ for all observations to enable the intercept term to enter the model.

$$Y_i = \beta_1 X_1 + \beta_2 X_{2i} + \beta_3 X_{3i} + \cdots + \beta_k X_{ki} + u_i \tag{10.1}$$

If the independent variables above can be algebraically formed as Equation 10.2:

$$\lambda_1 X_{1i} + \lambda_2 X_{2i} + \lambda_3 X_{3i} + \cdots + \lambda_k X_{ki} = 0 \tag{10.2}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_k$ are constant such that not all of them are equal to zero simultaneously.

They are said to have exact linear relationship or **perfect multicollinearity**. That is, we can acquire the value of any independent variable in the model through the linear combination of other independent variables. For instance, if we want to find the value of X_2, we can apply the addition, subtraction, multiplication and division among other independent variables.

On the other hand, if the formation of independent variables follows Equation 10.3 , rather than Equation 10.2, they are said to have **imperfect multicollinearity**. Specifically, we cannot obtain any independent variable in the model from the linear combination of other independent variables.

$$\lambda_1 X_{1i} + \lambda_2 X_{2i} + \lambda_3 X_{3i} + \cdots + \lambda_k X_{ki} + \nu_i = 0 \tag{10.3}$$

where ν_1 is the stochastic disturbance term.

To see the difference between perfect and less than perfect multicollinearity, we can rewrite Equation 10.2 and Equation 10.3 as:

According to Equation 10.2 and 10.3, consider Table 10.1 which illustrates the collection of 5 observations for each independent variable (X_2 and X_3). It is obvious that we can multiply X_2 by the constant term to transform it into X_3. In this case, we can establish the relationship, as in Equation 10.2, between these two independent variables by letting $\lambda_1 = -3$ and $\lambda_2 = 1$.

$$-3X_{2i} + X_{3i} = 0$$

Table 10.1: Perfect multicollinearity in explanatory variables

Observation	X_{2i}	X_{3i}	$3X_{2i}$	$\nu_i = -3X_{2i} + X_{3i}$
1	6	18	18	0
2	12	36	36	0
3	7	21	21	0
4	-5	-15	-15	0
5	2	6	6	0

For the case of imperfect multicollinearity, consider Table 10-2. It can be found that we cannot form the relationship as Equation 10.2 due to the difference between independent variables (X_2 and X_3). Even we multiply X_2 by -3, the random disturbance term (ν_i) still exists. In Table 10-2, after the forth observation of X_2 is multiplied by 3, it is still different from -12 by 3. Hence, the relationship between these two independent variables can be written as

$$-3X_{2i} + X_{3i} + \nu_i = 0$$

Table 10-2: Imperfect multicollinearity in explanatory variables

Observation	X_{2i}	X_{3i}	$3X_{2i}$	$\nu_i = -3X_{2i} + X_{3i}$
1	6	16	18	-2
2	12	45	36	9
3	7	18	21	-3
4	-5	-12	-15	3
5	2	7	6	1

The relationship discussed in this chapter involves only the linear one. Although the independent variable is squared or cubed, as in Equation 10.4, it does not always mean that the model constructed from these variables will suffer the perfect or imperfect multicollinearity. The important factor to consider is whether X_i and X_i^3 can be written in the form of Equation 10.3 or 10.4.

$$Y_i = \beta_1 + \beta_2 X_i + \beta_3 X_i^3 + u_i \tag{10.4}$$

In the following sections of this chapter, the consequence of the perfect or imperfect linear relationship of independent variables will be discussed. However, to completely understand the characteristics of multicollinearity, it is essential to know the sources of the problem. In principle, multicollinearity is originated from:

1. *Method of data collection used in the regression model*: sometimes researchers collect the data in the limited amount, causing the sample to concentrate in some group of population rather than to represent the population as a whole.

2. *Restriction imposed in the model*: in the study of relationship between a single dependent variable and multiple independent variables, possibly the linear relationship exists among those independent variables. To illustrate, suppose we study the dependence of the sale of goods Y on the prices of goods X and Y, where X is used to produce Y. With this relationship, when the price of goods X increases, it almost certainly raises the price of goods Y. Hence, there seems to be highly linear relationship between these two variables.

3. *Application of polynomial to the model*: such as Equation 10.4, the X_i^3 is included as another variable. If the data used in the study is restricted within the narrow range, the value of two variable, namely X_i and X_i^3, might not be notably different, resulting in the linear relationship between them.

4. *Over-determination of the model*: some models have higher amount of parameters than the amount of observation collected. The evident example of these models is in the medical or human behavior field in which the amount of patients or volunteers is less than the independent variables. Usually, researchers have to discard some variables to make the study possible.

5. *Common trend of independent variables*: the time series data of revenue, expenditure and population seems to move together because, as the time passes, they tend to increase collectively. Thus, the linear relationship among them might occur.

10.2 Estimation in the Presence of Perfect Multicollinearity

According to Chapter 7, if we have the regression model as shown in Equation 10.1 , we can employ the ordinary least square to estimate β_2 and β_3. By the method of calculus, minimizing the sum of disturbance term squared, we obtain the estimators ($\hat{\beta}_2$ and $\hat{\beta}_3$) as follow:

$$Y_i = \hat{\beta}_1 + \hat{\beta}_2 X_{2i} + \hat{\beta}_3 X_{3i} + \hat{u}_i \tag{10.5}$$

$$\hat{\beta}_2 = \frac{(\sum y_i x_{2i})(\sum x_{3i}^2) - (\sum y_i x_{3i})(\sum x_{2i} x_{3i})}{(\sum x_{2i}^2)(\sum x_{3i}^2) - (\sum x_{2i} x_{3i})^2} \tag{10.6}$$

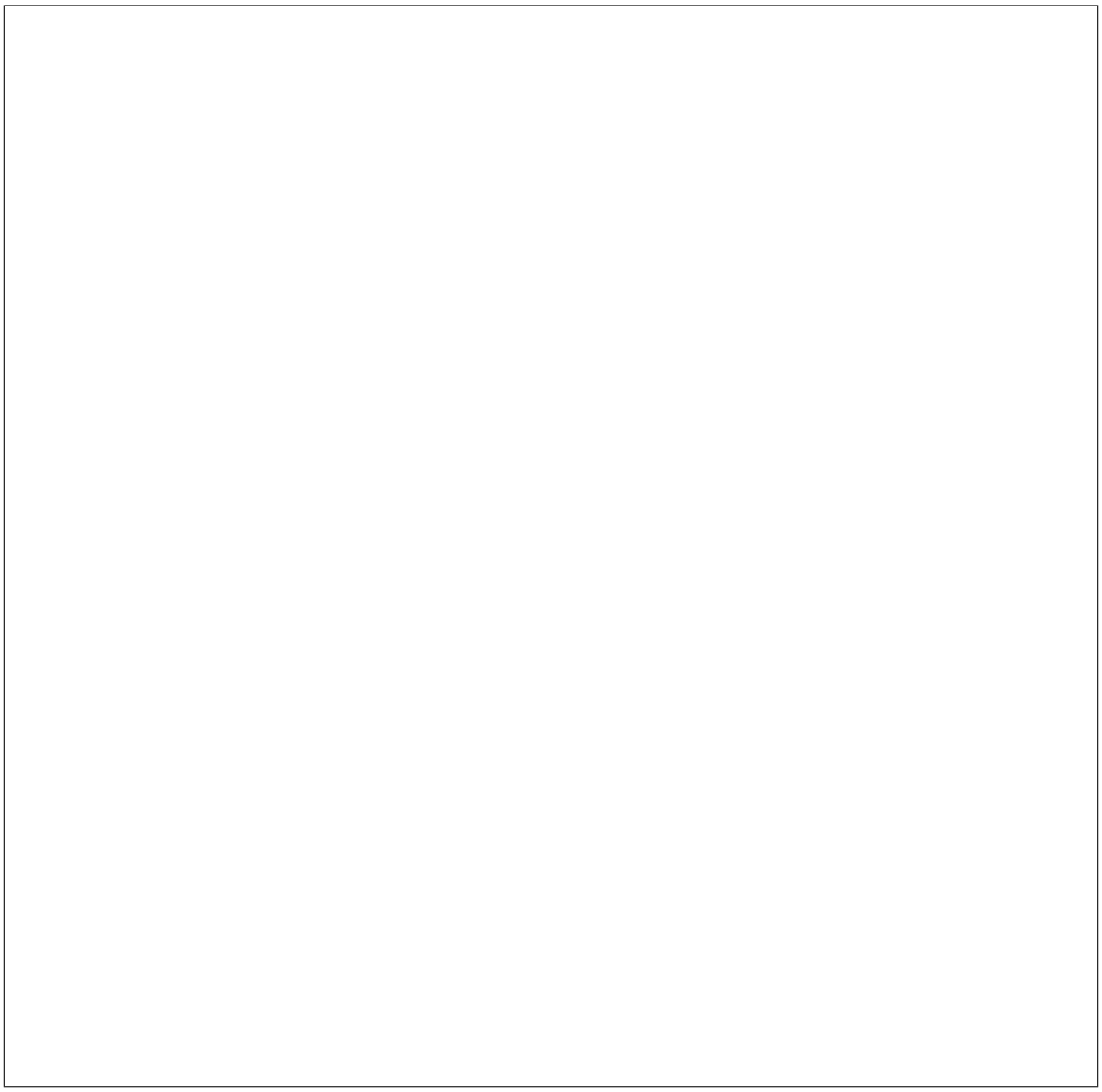
$$\hat{\beta}_3 = \frac{(\sum y_i x_{3i})(\sum x_{2i}^2) - (\sum y_i x_{2i})(\sum x_{2i} x_{3i})}{(\sum x_{2i}^2)(\sum x_{3i}^2) - (\sum x_{2i} x_{3i})^2} \tag{10.7}$$

where
$y_i = Y_i - \bar{Y}$,
$x_{2i} = X_{2i} - \bar{X}_{2i}$,
$x_{3i} = X_{3i} - \bar{X}_{3i}$

Nonetheless, if the explanatory variables, X_2 and X_3, suffer perfect multicollinearity, namely $X_{3i} = \lambda X_{2i}$ where λ is the constant greater than zero, from the relationship stated, we can substitute $x_{3i} = \lambda x_{2i}$ in equation 10.6 and 10.7 and get

We can get the same result for $\hat{\beta}_3$

$$\hat{\beta}_3 = \frac{\lambda^2[(\sum y_i x_{2i})(\sum x_{3i}^2) - (\sum y_i x_{2i})(\sum x_{3i} x_{3i})]}{\lambda^2[(\sum x_{2i}^2)(\sum x_{2i}^2) - (\sum x_{2i})^2]} = \frac{0}{0} \tag{10.8}$$



On the other hand, consider the case where there is imperfect multicollinearity among regressors. For the model with two regressors, let $X_{3i} = \lambda X_{2i} + v_i$ and substitute $x_{3i} = \lambda x_{2i} + v_i$ into equation 10.6, the estimator of β_2 can be obtained by equation 10.9 which is different from the case with perfect multicollinearity problem. The same is true for both estimators of β_1 and β_3.

$$\hat{\beta}_2 = \frac{(\sum y_i x_{2i})(\lambda^2 \sum x_{2i}^2 + \sum v_i^2) - (\lambda \sum y_i x_{2i} + \sum y_i v_i)(\lambda \sum x_{2i}^2)}{(\sum x_{2i}^2)(\lambda^2 \sum x_{2i}^2 + \sum v_i^2) - (\lambda \sum x_{2i}^2)^2} \neq \frac{0}{0} \tag{10.9}$$

where
$X_{3i} = \lambda X_{2i} + v_i$
$\bar{X}_{3i} = \lambda \bar{X}_{2i}$
$X_{3i} - \bar{X}_{3i} = \lambda (X_{2i} - \bar{X}_{2i}) + v_i$
$x_{3i} = \lambda x_{2i} + v_i$
where $\lambda \neq 0$ and $\sum x_i v_i = 0$

10.3 Practical Consequences of Multicollinearity

In case of near or high multicollinearity, we might encounter the following consequences:

1. Although BLUE, the OLS estimators have large variances and covariances, making precise estimation difficult.

2. Because of consequence 1, the confidence intervals tend to be much wider, leading to the acceptance of the **zero null hypothesis** (i.e., the true population coefficient is zero) more readily.

3. Also because of consequence 1, the t ratio of one or more coefficients tends to be statistically insignificant.

4. Although the t ratio of one or more coefficients is statistically insignificant, R^2 , the overall measure of goodness of fit, can be very high.

5. The OLS estimators and their standard errors can be sensitive to small changes in the data.

The preceding consequences can be demonstrated as follows.

Large Variance and Covariances of OLS Estimateors

$$Var(\hat{\beta}_2) = \frac{\sigma^2}{\sum x_{2i}^2 (1 - r_{23}^2)} \tag{10.10}$$

$$Var(\hat{\beta}_3) = \frac{\sigma^2}{\sum x_{3i}^2 (1 - r_{23}^2)} \tag{10.11}$$

$$cov(\hat{\beta}_2, \hat{\beta}_3) = \frac{-r_{23}\sigma^2}{(1 - r_{23}^2)\sqrt{\sum x_{2i}^2 x_{3i}^2}} \tag{10.12}$$

where r_{23} is the correlation coefficient between regressors X_2 and X_3 and can be computed by equation 10.13 and the value ranges from -1 to 1.

$$r_{23} = \frac{(\sum x_{2i} x_{3i})^2}{\sum x_{2i}^2 \sum x_{3i}^2} \tag{10.13}$$

According to equation 10.10 and 10.11, it can be seen that the higher the correlation coefficient, the higher the variance of estimators. To ease the analysis, redefine equation 10.10 and 48 by **Variance Inflation Factor (VIF)** by letting,

$$\text{VIF} = \frac{1}{1 - r_{23}^2} \tag{10.14}$$

We get

$$Var(\hat{\beta}_2) = \frac{\sigma^2}{\sum x_{2i}^2}\text{VIF} \tag{10.15}$$

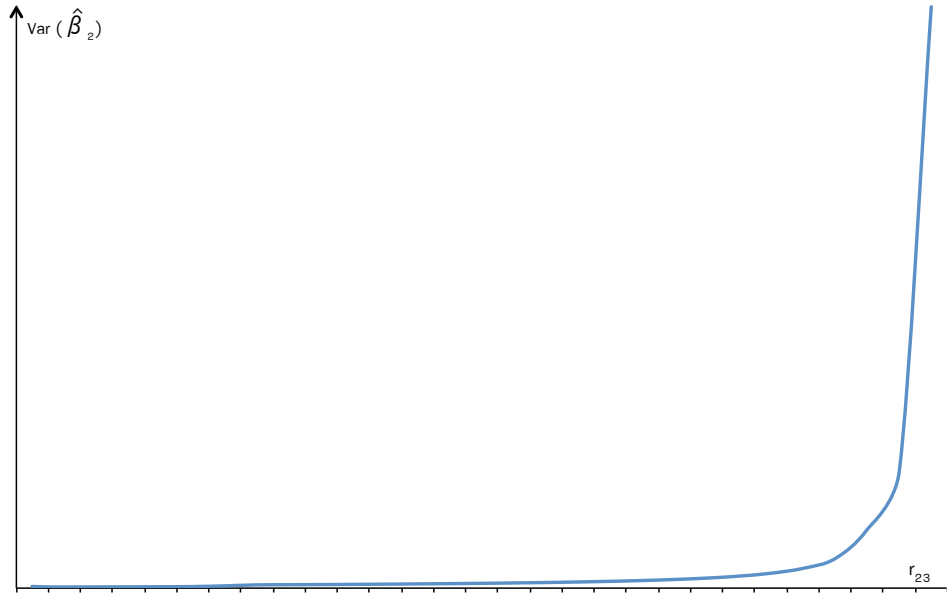
$$Var(\hat{\beta}_3) = \frac{\sigma^2}{\sum x_{3i}^2}\text{VIF} \tag{10.16}$$

As $r_{23} \to 1$, or the correlation approaches one, VIF $\to \infty$ and the variance will be higher and approaches infinity.
On the contrary, as $r_{23} \to 0$, or the correlation coefficient approaches zero (namely, no linear relationship), VIF $\to 1$ and the variance will be lower. Consider Table 10.3 and Figure 10.1, it can be seen that the higher the correlation , the higher the VIF and the higher the variance of estimators.

Table 10.3: The consequence of an increase in the correlation coefficient on the variance of estimators

r_{23}	VIF	$Var(\hat{\beta}_2)$	$Var(\hat{\beta}_3)$
		Let	
0.00	1	$\frac{\sigma^2}{\sum x_{2i}^2}$VIF = B	$\frac{\sigma^2}{\sum x_{2i}^2}$VIF = C
0.50	1.33	1.33B	1.33C
0.70	1.96	1.96B	1.96C
0.80	2.78	2.78B	2.78C
0.90	5.76	5.76B	5.76C
0.97	16.92	16.92B	16.92C
0.99	50.25	50.25B	50.25C

Figure 10.1: The consequence of an increase in the correlation coefficient on the variance estimators



When the variance rises due to the level of multicollinearity among independent variables, the standard deviation will certainly rise and at least two negative effects will result. First, the interval estimation will be impaired because the confidence interval will be widen and Table 11.4 (for 95 percent confidence interval and large number of observations). The other negative effect is on hypothesis test since the t-statistic, as in equation 10.17, will be lower and might result in misleading conclusion from hypothesis test.

Table 10.4: The consequence of an increase in the correlation coefficient on 95 percent confidence interval

r_{23}	95% Confidence interval of β_2
0.00	$\hat{\beta}_2 \pm 1.96\sqrt{\frac{\sigma^2}{\sum x_{2i}^2}}$
0.50	$\hat{\beta}_2 \pm 1.96\sqrt{\frac{\sigma^2}{\sum x_{2i}^2}1.33}$
0.70	$\hat{\beta}_2 \pm 1.96\sqrt{\frac{\sigma^2}{\sum x_{2i}^2}1.96}$
0.80	$\hat{\beta}_2 \pm 1.96\sqrt{\frac{\sigma^2}{\sum x_{2i}^2}2.78}$
0.90	$\hat{\beta}_2 \pm 1.96\sqrt{\frac{\sigma^2}{\sum x_{2i}^2}5.76}$
0.97	$\hat{\beta}_2 \pm 1.96\sqrt{\frac{\sigma^2}{\sum x_{2i}^2}16.92}$
0.99	$\hat{\beta}_2 \pm 1.96\sqrt{\frac{\sigma^2}{\sum x_{2i}^2}50.25}$

$$\hat{t} = \left(\frac{\hat{\beta}_2 - \beta_2}{se(\hat{\beta}_2)\uparrow}\right)\downarrow \tag{10.17}$$

In any models, we might have the high value of coefficient of determination (R^2) and statistically overall significant of the model from F-test. The implication is that the model possesses the explanatory power over the dependent variable. However, it is possible that we might not get the statistically significant result from the test of individual coefficients. That is, some coefficient is not significantly different from zero which means the variable associated with that coefficient lacks explanatory variable because the t-statistic is lower due to multicollinearity problem. This situation is called **conflicting test**, namely the result from t-test contradicts with the one from F-test.

To conclude, if there is perfect multicollinearity among independent variables, we are unable to estimate the parameters in the model. Also, the variance of estimators will approach infinity. Furthermore, if there is imperfect multicollinearity, OLS is still applicable to estimate the parameters. Yet, it has to be aware that the variance of estimators might be so high that some aspects of regression analysis, such as interval estimation and hypothesis test, are negatively influenced.

10.4 Detection of Multicollinearity

We have already discussed the consequence of both perfect and imperfect multicollinearity among regressors. For the regression analysis, the harmful problem is the situation when there is perfect multicollinearity which will invalidate the estimation of the model in order to explain the true relationship in the population. The case of perfect multicollinearity, thus, can be easily detected.

For the imperfect multicollinearity, if the degree of multicollinearity is not immense, the estimators are still BLUE. Yet, if the degree is huge, the problem will become damaging. Statistically, the extent of multicollinearity can be tested through various approaches. Some of them are discussed here.

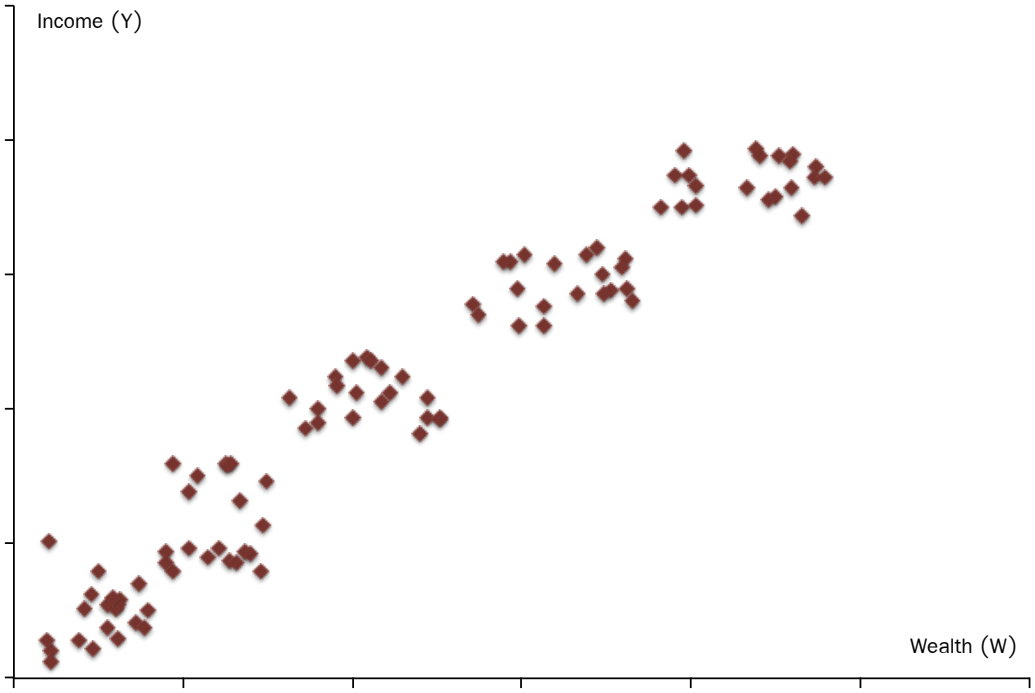
1. *There is conflicting test between t- and F-test*: if we find that the conclusion derived from the two tests are inconsistent, specifically R^2 is high and F-test results in statistical overall significance; whereas, at least, one null hypothesis of some t-tests cannot be rejected, it is reasonable to suspect the multicollinearity problem.

2. *Correlation of regressors is greater than 0.8*: the higher the correlation, the higher the variance of estimators.

3. *Variance inflation factor is greater than 10*: when the regressors face the multicollinearity problem, the value of VIF might be so high that the resulting high variance of estimators adversely affects the regression analysis.

4. *Scatter plot of two regressors is relatively linear*: when we plot the value of on regressor against another and we find that both of them tend to change in the same way, this fact might suggest the existence of multicollinearity. Figure 11.2 depicts the case where income and wealth, which is usually perceived to explain consumption expenditure, are prone to move together.

Figure 10.2: Scatter plot between two regressors, namely income and wealth, showing the linear relationship between both of them



10.5 Remedial Measure for Multicollinearity

In principle, the problem of multicollinearity among explanatory variables is not actually serious as we still have BLUE estimators. Notwithstanding, the problem become more severe as the degree of multicollinearity rises and can be alleviated through:

1. *Do nothing*: if the degree of multicollinearity is low, the model is still valid as the BLUE property of estimators is attained.

2. *Apply prior relationship among explanatory variables*: consider the model

$$Y_i = \beta_1 + \beta_2 X_{2i} + \beta_3 X_{3i} + u_i$$

If we know before that the linear relationship between explanatory variables X_2 and X_3 can be written as $\beta_3 = 0.7\beta_2$, we can use this fact to eliminate the problem by

$$\begin{aligned} Y_i &= \beta_1 + \beta_2 X_{2i} + 0.7\beta_2 X_{3i} + u_i \\ &= \beta_1 + \beta_2 (X_{2i} + 0.7X_{3i}) + u_i \\ &= \beta_1 + \beta_2 X_i^* + u_i \end{aligned}$$

where $X_i^* = X_{2i} + 0.7X_{3i}$

3. *Discard some explanatory variables*: the removal of the variables could mitigate the problem; but, another problem, namely **specification bias** problem, might occur instead. For example, suppose we want to construct the model where the production is the explained variables; and labor and capital are the explanatory ones. If there is linear relationship between labor and capital, the elimination of one variable might assuage the multicollinearity problem, but might be contrary to economic reasoning. Hence, the decision of which variables will be disposed of should be based on economic theory.

4. *Collect more observation*: this practice will increase $\sum x_i^2$ which is the component of the variances [1]. Accordingly, the variances will be lower despite high correlation among explanatory variables.

$$Var(\hat{\beta}_2) = \frac{\sigma^2}{\sum x_{2i}^2 (1 - r_{23}^2)}$$

5. *Transform the variables*: although there is linear relationship among explanatory variables, it is not necessary that the *first difference* or *ratio transformation* of the variables will have that relationship.

[1] As the data set gets larger, the sample statistic will approach the population parameter. Consequently, we can reasonably state that mean of X is almost stable under the larger data set. In this case, the increase in the size of data set is likely to increase the sum of the square of deviation from the mean

For the first difference of variables, consider the model in period t and t-1

$$\begin{aligned} Y_t &= \beta_1 + \beta_2 X_{2t} + \beta_3 X_{3t} + u_t \\ Y_{t-1} &= \beta_1 + \beta_2 X_{2,t-1} + \beta_3 X_{3,t-1} + u_{t-1} \end{aligned}$$

$$\begin{aligned} Y_t - Y_{t-1} &= \beta_2 (X_{2t} - X_{2,t-1}) + \beta_3 (X_{3t} - X_{3,t-1}) + \nu_t \\ \Delta Y_t &= \beta_2 \Delta X_2 + \beta_3 \Delta X_3 + \nu_t \\ \text{where} & \\ \nu_t &= u_t - u_{t-1} \\ \Delta Y_t &= Y_t - Y_{t-1} \\ \Delta X_2 &= X_{2t} - X_{2,t-1} \\ \Delta X_3 &= X_{3t} - X_{3,t-1} \end{aligned}$$

This transformation perhaps results in no linear relationship among new regressors. Unfortunately, another serious econometric problem might take place which is the **autocorrelation** problem which will be discussed in Chapter 12.

For the ratio transformation of variables, consider the model

$$\begin{aligned} Y_t &= \beta_1 + \beta_2 X_{2t} + \beta_3 X_{3t} + u_t \end{aligned}$$

$$\begin{aligned} \frac{Y_t}{X_{3t}} &= \beta_1 \frac{1}{X_{3t}} + \beta_2 \frac{X_{2t}}{X_{3t}} + \beta_3 \frac{X_{3t}}{X_{3t}} + \frac{u_t}{X_{3t}} \\ \frac{Y_t}{X_{3t}} &= \beta_1 \frac{1}{X_{3t}} + \beta_3 + \beta_2 \frac{X_{2t}}{X_{3t}} + \frac{u_t}{X_{3t}} \\ Y_t^* &= \beta_1^* + \beta_2 X_{2t}^* + u_t^* \\ \text{where} & \\ Y_t^* &= \frac{Y_t}{X_{3t}} \\ \beta_1^* &= \beta_1 \frac{1}{X_{3t}} + \beta_3 \\ X_{2t}^* &= \frac{X_{2t}}{X_{3t}} \\ u_t^* &= \frac{u_t}{X_{3t}} \end{aligned}$$

With this remedial measure, we can reduce the degree of multicollinearity since there is one explanatory variable left in the model. However, when we consider the random disturbance term in this new model, it is possible that the variance of the disturbance term might not be constant, namely **heteroscedasticity**, which will be discussed in the next chapter.

11. Heteroskedasticity

One of the assumptions of the classical linear regression model (Assumption 3 on p. 116) is that the disturbances u_i appearing in the population regression function are homoscedastic. In other words, they all have the same variance. In this chapter, students will examine the validity of this assumption and figure out the consequences if this assumption is not fulfilled.

11.0.1 Nature of Heteroscedasticity

As mentioned in CHAPTER 7: Multiple Regression Analysis: The Problem of Analysis. We assume that the variance of each disturbance term u_i, conditional on the chosen values of the explanatory variables, is constant. We called this as **homoscedasticity** or equal (homo) spread (scedasticity), that is equal variance.:

$$E(u_i^2) = \sigma^2 \quad i = 1, 2, ..., n \tag{11.1}$$

Depicted by Figure 11.1, the variances of disturbance terms given any independent variable are constant and equal. Specifically, conditional on X_1, the variance is equal to σ^2 which is the same as the variance of disturbance term conditional on X_2 and on the other values of independent variable.

On the other hand, if this assumption is violated, the problem occurring is called **heteroscedasticity**. That is, conditional on X_1, the variance of disturbance term is σ_1^2; whereas, conditional on X_2, the variance of disturbance term is σ_2^2. In brief, the conditional variance of disturbance term would vary across the values of independent variable, as illustrated, mathematically, by Equation 11.2 and, graphically, by Figure 11.2.

$$E(u_i^2) = \sigma_i^2 \tag{11.2}$$

Figure 11.1: Homoscedasticity

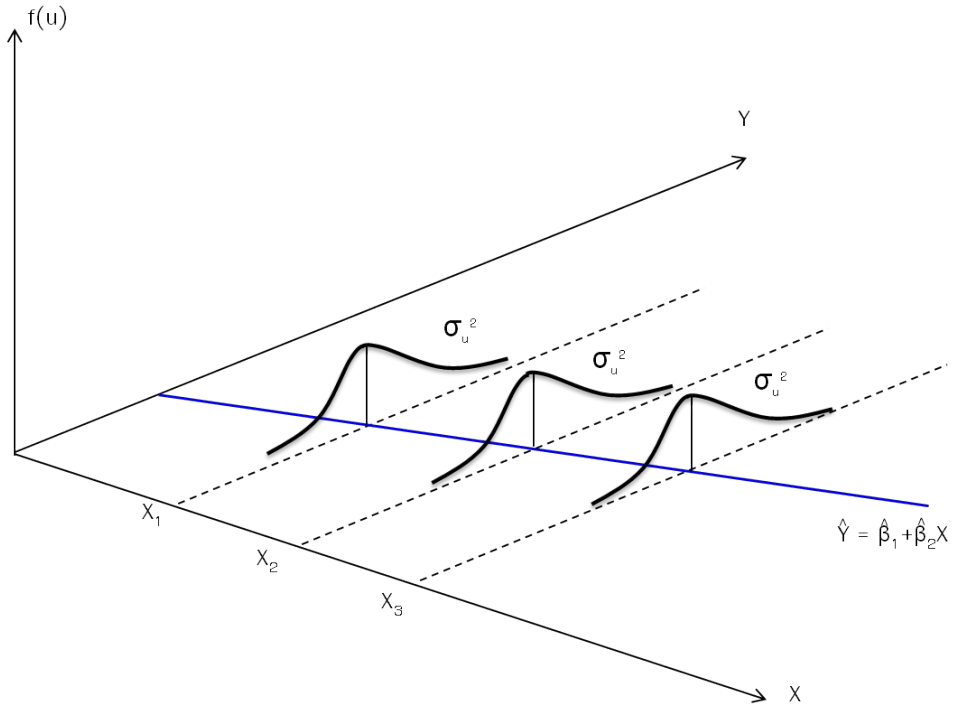
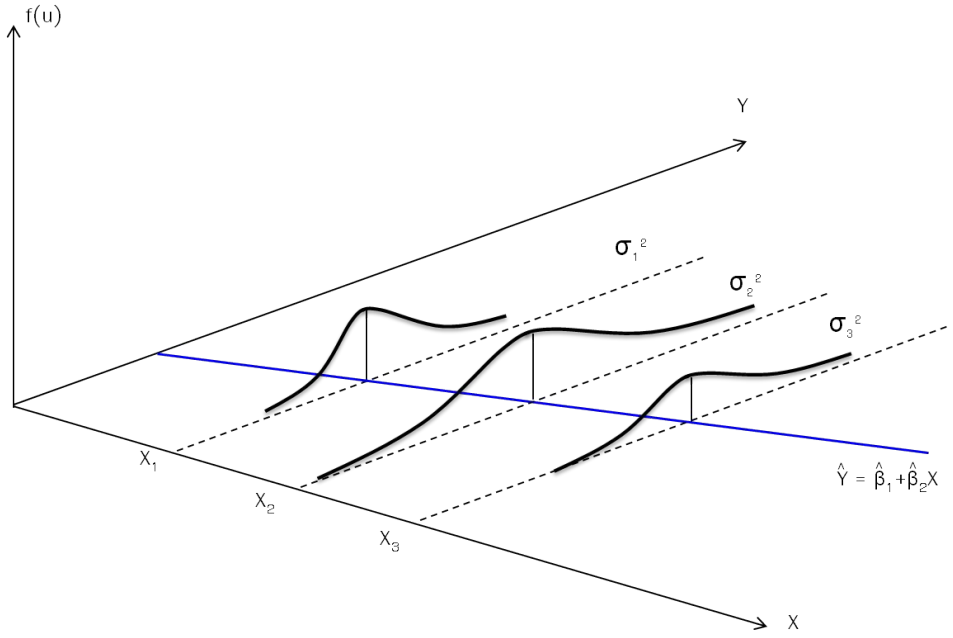


Figure 11.2: Heteroscedasticity



Generally, there is a variety of causes for the existence heteroscedasticity problem in the regression model studied by most economists. Yet, only 6 main sources are discussed here.

1. Normally, error learning is the nature of human. At the initial stage of their work, people probably commit a large number of mistakes. As they carry on working and become more specialized, the amount of errors would be reduced. In this case, it seems that the variance at the initial stage will be high but would decrease as people learn from their errors.

2. Considering the relationship between independent variable X and dependent variable Y, there seems to be possible that as the value of independent variable increase, the variance of the value of dependent one will increases. The feasible reason is the characteristics of those variables such as the relationship between the profit of the company (independent variable) and dividend (dependent variable). As the profit rises, the board of director may have a variety of dividend policies. Some companies may pay a small amount of dividend in order to keep the profit for further development. Some may pay a large amount of dividend to satisfy the shareholders. On the contrary, if the profit is low, the dividend policy will not diverge across the companies since the companies seem to be at the growth stage and decide to keep their profit as retained earnings.

3. The collection of data is another source. As the collection technique employed by the researcher is improved, the collection error would be lower. Contrarily, with the poor collection technique, the data obtained to construct the regression model would probably incur more and more errors, causing the condition variance of disturbance term to vary.

4. The existence of outliers in the independent and dependent variables may make the conditional variance of disturbance term on independent variables volatile. Mostly, if the researchers collect too small amount of data, that set of data tends to include the outliers and undermine the regression analysis. As the amount of data increases, those outliers may become normal relative to other observations.

5. The misspecification of the model could result in the heteroscedasticity problem as well. In some cases, econometricians drop some important and necessary independent variable from the regression model. The disturbance term, then, will incorporate the characteristics of the missing variables, resulting in heteroscedasticity problem. For example, suppose the researchers want to establish the model to explain the relationship of price and quantity of good X, as suggested by the theory of demand. However, if it turns out that good Y is the substitute for good X. This mistake of failing to include price of good Y, which has the explanatory power over the demand for good X, will result in misspecification error. The disturbance term will have the characteristics of good Y, which, in turn, leads to heteroscedasticity.

6. Heteroscedasticity problem usually happens in the model that applies the **cross-sectional data** which tends to be highly diverse because the data is collected in the same time period. To illustrate, the census acquired from numerous provincial areas may cover the wide range of value and results in the stated problem. On the other hand, for the **time series data**, it is prone to be the collection of the same sample for different period of time. With the same sample, the range of the value covered seems to be narrow. Hence, the stated problem, generally, does not occur with this kind of data.

11.0.2 OLS Estimation in the presence of Heteroscedasticity

For the estimation of regression model through OLS methods, given the assumption of heteroscedasticity by letting $E(u_i^2) = \sigma_i^2$, the model can be written as

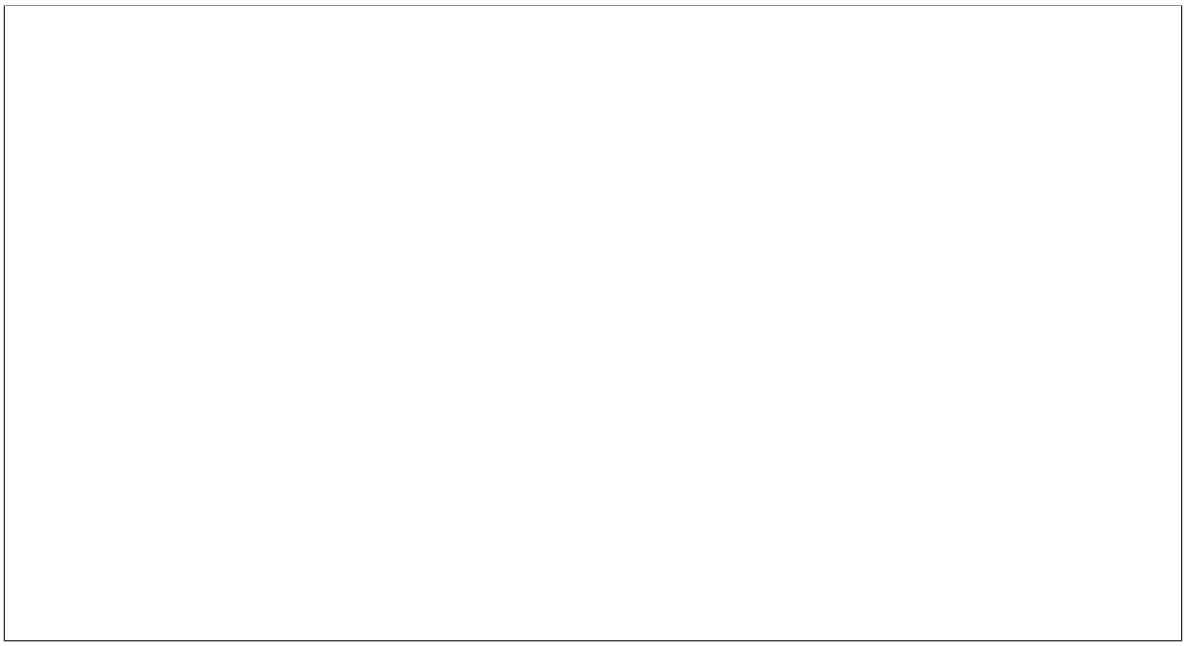
$$Y_i = \beta_1 + \beta_2 X_i + u_i$$

The estimator β_2 can be calculated by

$$\hat{\beta}_2 = \frac{\sum(X_i - \bar{X})(Y_i - \bar{Y})}{\sum(X_i - \bar{X})^2} = \frac{\sum x_i y_i}{\sum x_i^2}$$

In general, the variance of the estimator β_2 is computed by

$$Var(\hat{\beta}_2) = \frac{\sum x_i^2 \sigma_i^2}{(\sum x_i^2)^2} \tag{11.3}$$



This is totally different from the usual variance formula obtained under the assumption of homoscedasticity:

$$Var(\hat{\beta}_2) = \frac{\sigma^2}{\sum x_i^2}$$

The $\hat{\beta}_2$ is no longer BEST and the minimum variance. It is not BLUE.

What is BLUE in the presence of Heteroscedasticity? The answer is we have to apply the method of generalized least squares (GLS) instead.

11.0.3 The Method of Generalized Least Squares (GLS)

(Cont.) The Method of Generalized Least Squares (GLS)

11.0.4 Detection of Heteroscedaticity

The problem of heteroscedasticity is fairly severe since it causes the estimators, which identify the relationship of regressor and regressand, to lose the minimum variance or best property despite its unbiased property. Accordingly, the statistical inference studied in the previous chapters, such as confidence interval and hypothesis test, is not applicable. It is, thus, essential to detect the problem of heteroscedasticity. Again, there are a large number of methods for detection suggested by econometricians; yet, 4 approaches are discussed here.

1. *Finding the relationship of regressor and random disturbance term by graph*: the nature of the problem is that the conditional variance of disturbance term is not constant. Hence, we can detect the problem by plotting the diagram to show the relationship between $\hat{u_i}^2$ and the estimated Y_i from the regression line, $\hat{Y}_i$. Also, we may plot $\hat{u_i}^2$ against one of the explanatory variables (X_i) to detect this problem.

Figure 11.3: The relationship between u_i^2 and$\hat{Y}_i$ of the model suffering heteroscedasticity problem

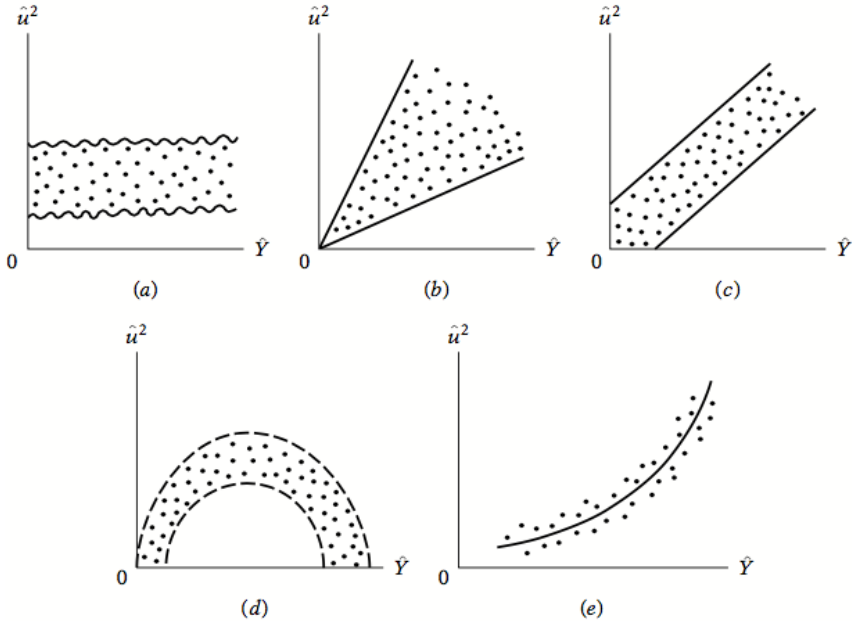
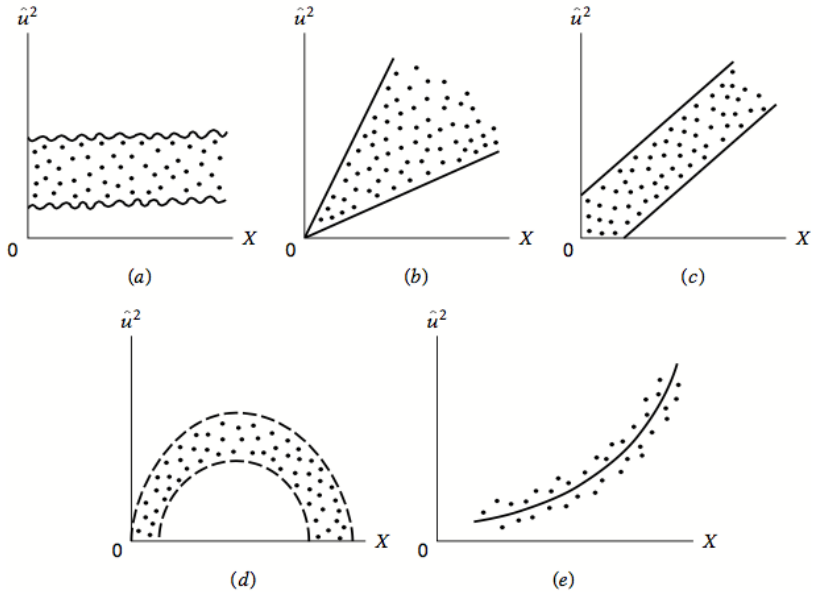


Figure 11.4: The relationship between u_i^2 and regressor X of the model suffering heteroscedasticity problem



2. **Park Test**[1]: the rationales is that, if we can construct the regression model that enables the regressor X to explains the volatility of the variance of disturbance term, that means the variance of disturbance term is not constant and depends on the regressor. The model could be constructed as Equation 11.4 and we can transform the model into the linear one as Equation 11.5

$$\sigma_i^2 = \sigma_u^2 X_i^{\beta} e^{v_i} \tag{11.4}$$

$$\ln \sigma_i^2 = \ln \sigma_u^2 + \beta \ln X_i + v_i \tag{11.5}$$

Practically, we would never know the true variance of the disturbance term in the model; so, we use $\hat{u}_i^2$ as the estimator and form the model similar to Equation 11.5, which is shown in Equation 11.6

$$\ln \hat{u}_i^2 = \ln \sigma_u^2 + \beta \ln X_i + v_i = \alpha + \beta \ln X_i + v_i \tag{11.6}$$

[1]Park, Rolla. E. (1966). “Estimation with Heteroscedasticity Error Terms," Econometrica, Vol.34, No.4.

After the establishment of the model in Equation 11.6, we, then, can perform the hypothesis test to examine whether the regressor X could explain the change in the regressand $\ln \hat{u}_i^2$. In this case, we test for the statistical significance of the coefficient associated with β by finding the t-statistic. If the regressor X is statistically significantly able to describe the regressand $\ln u_i^2$, we can conclude that the model faces the problem of heteroscedasticity. In short, the procedure for Park test is as follows.

Step 1: set up the model of interest to find the relationship of regressor X and regressand Y

Step 2: calculate $\hat{u}_i^2 = (Y_i - \hat{Y}_i)^2$ from the regression model in Step 1 to be the estimator of variance in Equation 11.6

Step 3: establish the model as in Equation 11.6 and perform the hypothesis test for the relationship between the regressor and the variance of disturbance term. The hypothesis can be set as

$$\begin{array}{lrcl} H_o: & \beta & = & 0 \\ H_a: & \beta & \neq & 0 \end{array}$$

If the null hypothesis is rejected, we can conclude that, the regressor X possess the explanatory power over the variance of disturbance term. In other word, the model suffers the heteroscedasticity prolem. Contrarily, if the null hypothesis cannot be rejected, it implies no heteroscedasticity problem in the model

3. **Breusch-Pagan Test**[2] or **LM Test**: consider the multiple regression model in Equation 11.7 and suppose that the variance of disturbance term has the linear relationship with the regressor as in Equation 11.8. To satisfy the homoscedasticity assumption such that the estimator is BLUE, all partial regression coefficients in Equation 11.8 must be zero.

$$Y_i = \beta_1 + \beta_2 X_{2i} + \beta_3 X_{3i} + \cdots + \beta_k X_{ki} + u_i \tag{11.7}$$

$$Var(u|X_2, \ldots, X_k) = E(u^2|X_2, \ldots, X_k) = \alpha_1 + \alpha_2 X_{2i} + \alpha_3 X_{3i} + \cdots + \alpha_k X_{ki} \tag{11.8}$$

The procedure of Breusch-Pagan test for heteroscedasticity is as follows.

Step 1: estimate the model as in Equation 11.7 by the method of OLS and calculate the value of $\hat{u}_i^2$

Step 2: establish the regression model as in Equation 11.8 to find the coefficient of determination $R^2_{\hat{u}_i^2}$

$$\hat{u}_i^2 = \alpha_1 + \alpha_2 X_{2i} + \alpha_3 X_{3i} + \cdots + \alpha_k X_{ki} + u_i \tag{11.9}$$

Step 3: compute the F-statistic by Equation 11.10 and perform hypothesis test to find out whether all the regressors X's can jointly explain the variance of the disturbance term. If they have the explanatory power, we can conclude that the model in Equation 11.7 would suffer heteroscedasticity problem.

[2] Breusch, Trevor. and Pagan, Adrian. (1979). "A Simple Test for Heteroscedasticity and Random Coefficient Variation," Econometrica, Vol.47, No.5, pp.1287-94

H_o : $\alpha_1 = \alpha_2 = \alpha_3 = ... = \alpha_k = 0$
H_a : otherwise

$$\hat{F} = \frac{R^2_{\hat{u}_i^2}/(k-1)}{(1-R^2_{\hat{u}_i^2})/(n-k)} \tag{11.10}$$

Furthermore, LM-statistic (**Lagrange Multiplier**) can be used to determine whether there is heteroscedasticity problem in the model and can be calculated by Equation 11.11

$$LM = nR^2_{\hat{u}_i^2} \tag{11.11}$$

The LM-statistic has the chi-square distribution with the degree of freedom $k-1$ or χ^2_{df}. We can use LM-statistic to test the following hypothesis.

H_o : Homoscedasticity
H_a : otherwise

If the LM-statistic obtained is greater than the critical value found the chi-square table, we can conclude that the model in Equation 11.7 faces the heteroscedasticity problem. Nevertheless, if the LM-statistic is less than the critical value, we can conclude that no heteroscedasticity problem exists.

4. **White Test**[3]: Consider the multiple regression model as in Equation 11.12.

$$Y_i = \beta_1 + \beta_2 X_{2i} + \beta_3 X_{3i} + u_i \tag{11.12}$$

White test has the different procedure from the third test only in the aspect of how the hypothesis is set. While Breusch-Pagan test states that the variance of disturbance term has the relationship with regressors, White test will cover the wider relationship between the variance of disturbance term and higher amount of regressors as in Equation 11.13 with the procedure as follows.

$$Var(u_i|X_2,X_3) = E(u_i^2|X_2,X_3) = \alpha_1 + \alpha_2 X_{2i} + \alpha_3 X_{3i} + \alpha_4 X_{2i}^2 + \alpha_5 X_{3i}^2 + \alpha_6 X_{2i} X_{3i} + \nu_i \tag{11.13}$$

Step 1: set up the model as in Equation 66 to obtain $\hat{u}_i^2$

Step 2: establish the regression model as Equation 11.13

$$\hat{u}_i^2 = \alpha_1 + \alpha_2 X_{2i} + \alpha_3 X_{3i} + \alpha_4 X_{2i}^2 + \alpha_5 X_{5i}^2 + \alpha_6 X_{2i} X_{3i} + \nu_i \tag{11.14}$$

Step 3: similar to Step-3 of Breusch-Pagan test, calculate F- or LM-statistic and set the null and alternative hypotheses. Then, compare the F- or LM-statistic with the critical value from the statistical table to test for heteroscedasticity.

[3]White, Halbert. (1980). "A Heteroscedasticity Consistent Covariance Matrix Estimator and a Direct Test of Heteroscedasticity," Econometrica, Vol.48, No.4, pp.817-38

Through the four methods stated above, the econometricians can test whether the model that is estimated by OLS method suffers the heteroscedasticity problem. The graphical test is nothing but an initial method without any statistical test at any level of significance. The other tests involve the formulation of hypothesis and statistical test at the chosen level of significance. Hence, either Park, Breusch-Pagan or White test concerns the level of significance at 0.01, 0.05 or 0.1, depending on the situations.

11.1 Remedial Measure for Heteroscedasticity

Even though the heteroscedasticity does not make the estimators biased, the variance of the estimators obtained from the regression model will not be minimum. The loss of the best property will impair the statistical inference in which the econometricians may be interested such as the confidence interval and hypothesis test of whether there is statistically significant relationship between explanatory and explained variables.

The remedial measure for heteroscedasticity will enable the researcher to better analyze and apply the statistical tools for the study of relationship between explanatory and explained variables in the model. There are two approaches to remediation which are:

First Case where the variance of each disturbance term (σ_i^2) is known and

Second Case where the variance of each disturbance term (σ_i^2) is unknown.

First Case: When σ_i^2 is Known: The method of Weighted Least Squares

If σ_i^2 is known, the most straightforward method of correcting heteroscedasticity is to apply the generalized least squares (GLS) as we learn in section 10.3. The estimators we obtained are BLUE.

$$\hat{\beta}_2^* = \frac{(\sum w_i)(\sum w_i X_i Y_i) - (\sum w_i X_i)(\sum w_i Y_i)}{(\sum w_i)(\sum w_i X_i^2) - (\sum w_i X_i)^2} \tag{11.15}$$

The variance of above estimator can be computed by :

$$Var(\hat{\beta}_2^*) = \frac{\sum w_i}{(\sum w_i)(\sum w_i X_i^2) - (\sum w_i X_i)^2} \tag{11.16}$$

where $w_i = \frac{1}{\sigma_i^2}$

The difference between OLS and GLS is that, for GLS, the estimators obtained will minimize the weighted sum of residual squared or $\sum w_i \hat{u}_i^2$. On the other hand, for OLS, the ones obtained will minimize the sum of residual squared $\sum \hat{u}_i^2$.

For OLS

$$\sum \hat{u}_i^2 = \sum (Y_i - \hat{\beta}_1 - \hat{\beta}_2 X_i)^2$$

For GLS

$$\sum w_i \hat{u}_i^2 = \sum w_i (Y_i - \hat{\beta}_1^* - \hat{\beta}_2^* X_i)^2$$

Another difference is that GLS will assign different weights to each observation of disturbance term ($\hat{u}_i^2$) according to its importance.

To be specific, if the error associated with the observation is large (that is, the variance is high), the value of the estimator in the model will greatly deviate from the value of true parameter. That observation should not be much of interest. Due to GLS method, the weight assigned will be inversely proportional to the variance of observation. Thus, that observation will be assigned a small weight of ($\frac{1}{\sigma_i^2}$). For those from a population with smaller σ_i will get proportionately larger weight in minimizing the $\sum w_i \hat{u}_i^2$.

In brief, the larger weight will be assigned to the observations that concentrate in their mean (namely, lower variance); whereas, smaller weight will be assigned to the observations that deviate from their mean (namely, higher variance).

Generally, for estimation[4] it is desirable to establish the population regression model that describes the true relationship between explanatory and explained variables. Paying more attention to the observation clustering around its (population) mean is preferable to the observation diffusing from its mean. The practice of weight assignment of GLS is a special case of least square which is also known as **weighted least squares: WLS**. Contrarily, OLS assigns the same weight for all observations.

[4] In principle, the variance of each random disturbance term is known when we have an access to the whole population data. In that case, to reflect the true relationship on average, the establishment of the population regression model should focus more on the observation close to its mean than the one far from its mean.

11.1.1 Second Case: When σ_i^2 is not known

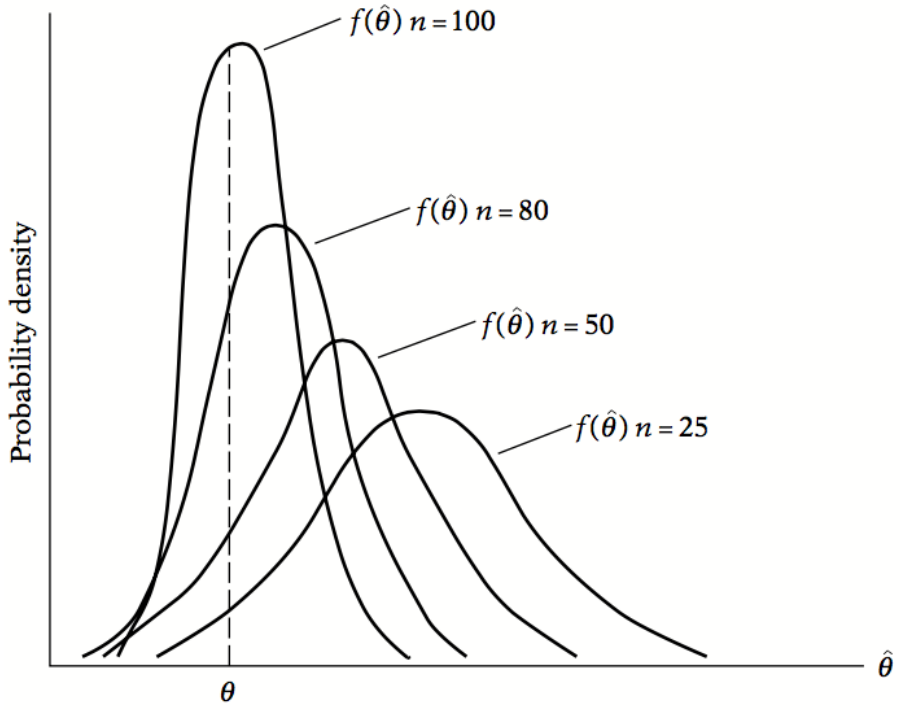
As noted earlier, if true σ_i^2 are known, we can use the WLS method to obtain BLUE estimators. However the true σ_i^2 are rarely known.

In this situation, we can apply the following methods to obtain the consistent (in the statistical sense) estimates of the variances and covariances of OLS estimators.

Consistency

θ is said to be a consistent estimator if it approaches the true value θ as the sample size gets larger and larger. Figure 11.5 illustrates this property. In this figure we have the distribution of θ based on sample sizes of 25, 50, 80, and 100. As the figure shows, θ based on n = 25 is biased since its sampling distribution is not centered on the true θ. But as n increases, the distribution of $\hat{\theta}$ not only tends to be more closely centered on θ (i.e., $\hat{\theta}$ becomes less biased) but its variance also becomes smaller. If in the limit (i.e., when n increases indefinitely) the distribution of $\hat{\theta}$ collapses to the single point θ, that is, if the distribution of $\hat{\theta}$ has zero spread, or variance, we say that $\hat{\theta}$ is a consistent estimator of θ.

Figure 11.5 : The Distribution of $\hat{\theta}$ as sample size increase



1 The distribution of $\hat{\theta}$ as sample size increases.

There are two methods to obtain consistent (in the statistical sense) estimates of the variances and covariances of OLS estimators:

1. *White's heteroscedasticity-consistent standard errors*: by the method of White to estimate unknown standard deviation, in principle, if the sample size is large enough, this estimator can be used to represent the true standard deviation. Moreover, econometricians can apply this estimator to further statistical test as if there is no heteroscedasticity problem. Nevertheless, if the sample size is not large enough, the estimators through White method will not have t-distribution and result in false statistical conclusion.

After all, it should be aware that if the model dose not suffer heteroscedasticity problem but econometricians still use White's estimator, the conclusion from the statistical analysis will be erroneous. Accordingly, the formal test for heteroscedasticity in the model should be conducted such that the existence of heteroscedasticity is verified.

2. *Some assumptions for the distribution of random disturbance term*: consider simple regression model

$$Y_i = \beta_1 + \beta_2 X_i + u_i$$

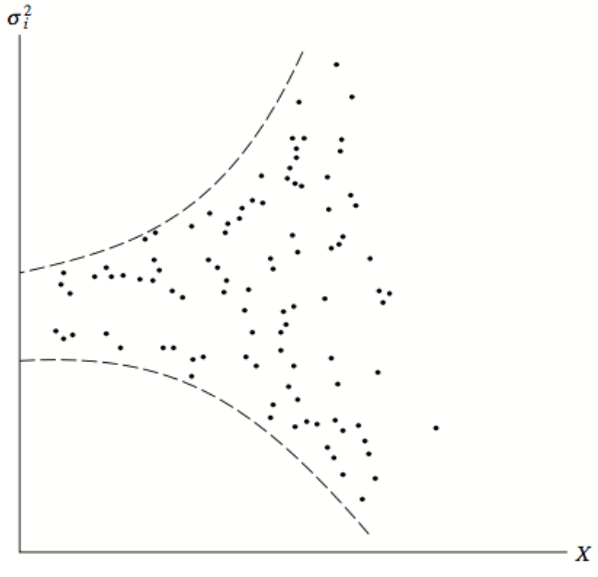
We can set some assumptions for the distribution of random disturbance term as follows.

Assumption 1: let the variance of random disturbance term be proportional to X_i^2

$$E(u_i^2) = \sigma^2 X_i^2$$

With this assumption, it can be found that the variance of disturbance term of the regression model will be constant and can be shown by

Figure 11.6 : Error variance proportional to X^2



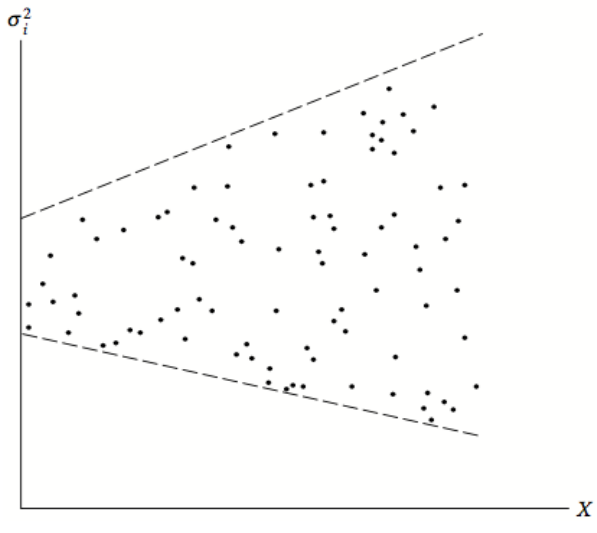
Error variance proportional to X^2.

Assumption 2: let the variance of random disturbance term be proportional to X_i

$$E(u_i^2) = \sigma^2 X_i$$

With this assumption, considering the variance of disturbance term of the regression model, it can be found that the variance will be constant.

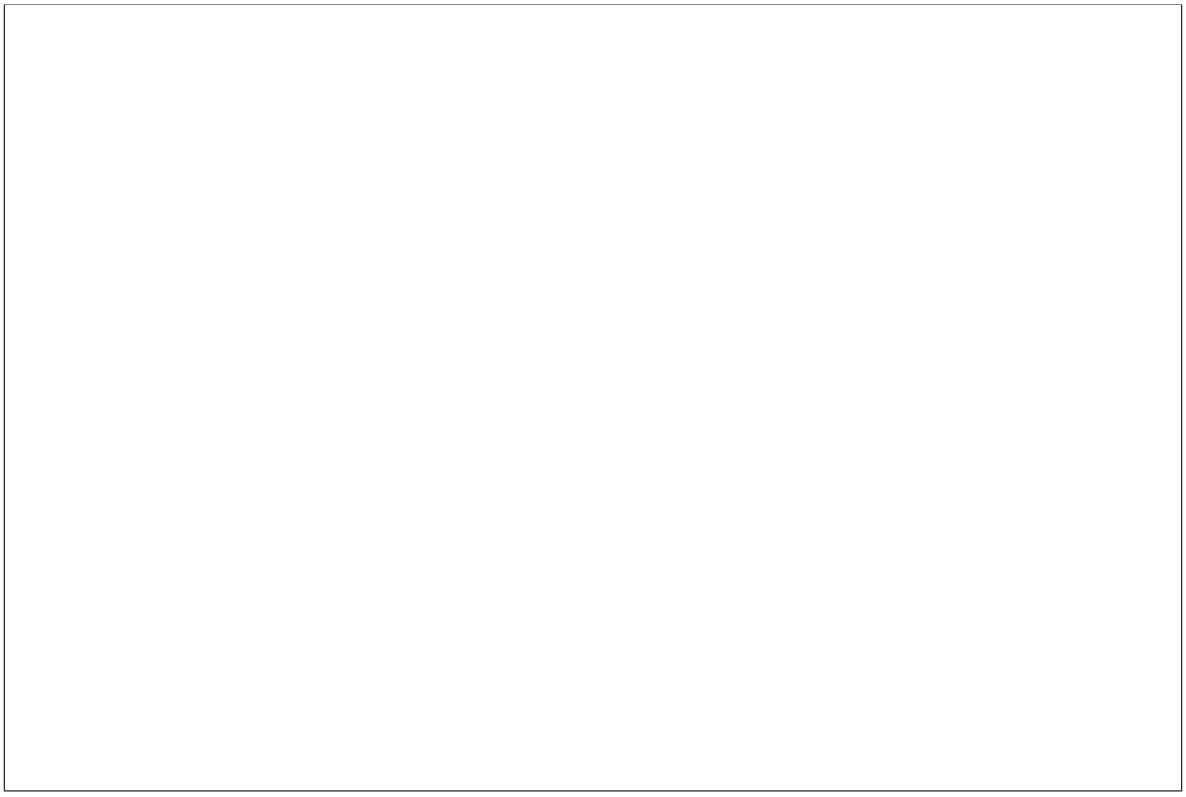
Figure 11.7: Error variance proportional to X



Error variance proportional to *X*.

Assumption 3: let the variance of random disturbance term be proportional to the mean of explanatory variable squared or $[E(Y_i)]^2$.

$$E(u_i^2) = \sigma^2[E(Y_i)]^2$$



Before the assumption about disturbance term is made, we need to identify whether the disturbance term has the relationship with other variables as in the assumption going to be made. To illustrate, the diagram depicting the relationship between the disturbance term and explanatory variable squared may be constructed to examine the validity of Assumption 1. After we explore that relationship, we, then, set the corresponding assumption to solve heteroscedasticity problem.

12. Autocorrelation

12.1 Nature of Autocorrelation

In Chapter 11 , we study the problem of heteroscedasticity. In sum, the problem of heteroscedasticity ruins the minimum variance property of estimators. In this Chapter, another problem of random disturbance term is considered. That problem is **autocorrelation** among disturbance term which violates one of the assumptions for classical linear regression model (CLRM)

The nature of autocorrelation is when there is correlation among disturbance terms or

$$cov(u_i, u_j | X_i, X_j) = E(u_i, u_j) \neq 0 \quad \text{where } i \neq j \tag{12.1}$$

For time series data, when the random disturbance terms are autocorrelated when the data in each period is correlated. For instance, the protest in a country that reduces the amount of export of goods and services in one month may also reduce the export of the following months. Hence, in this case, the random disturbance terms in these periods will be negative to reflect the fact that the amount of export tends to be below the mean.

For cross-sectional data, the problem of autocorrelation may occur. For example, the consumption expenditure of one family may reduce due to the great flood. Also, the flood influences other families in the same way. The consumption expenditure of these families tends to be positively correlated; hence, the random disturbance term from this set of data may also be positively correlated.

Figure 12-1 illustrates the pattern of random disturbance term when the random disturbance term faces autocorrelation problem with the increasing trend. Contrarily, Figure 12-2 depicts the case where the random disturbance term has no obvious systematic pattern, namely no autocorrelation.

Figure 12-1: Autocorrelation among disturbance term with increasing pattern

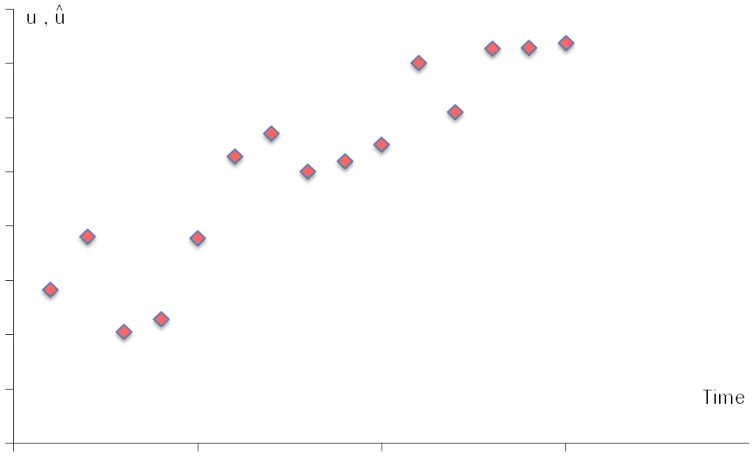


Figure 12-2: No autocorrelation among disturbance term

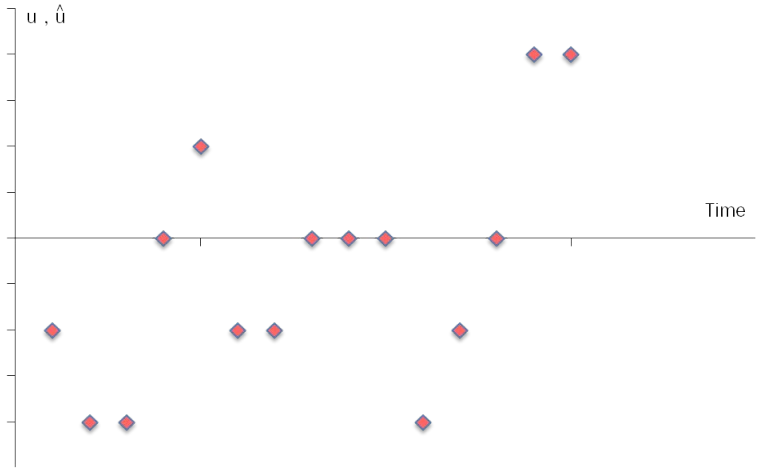
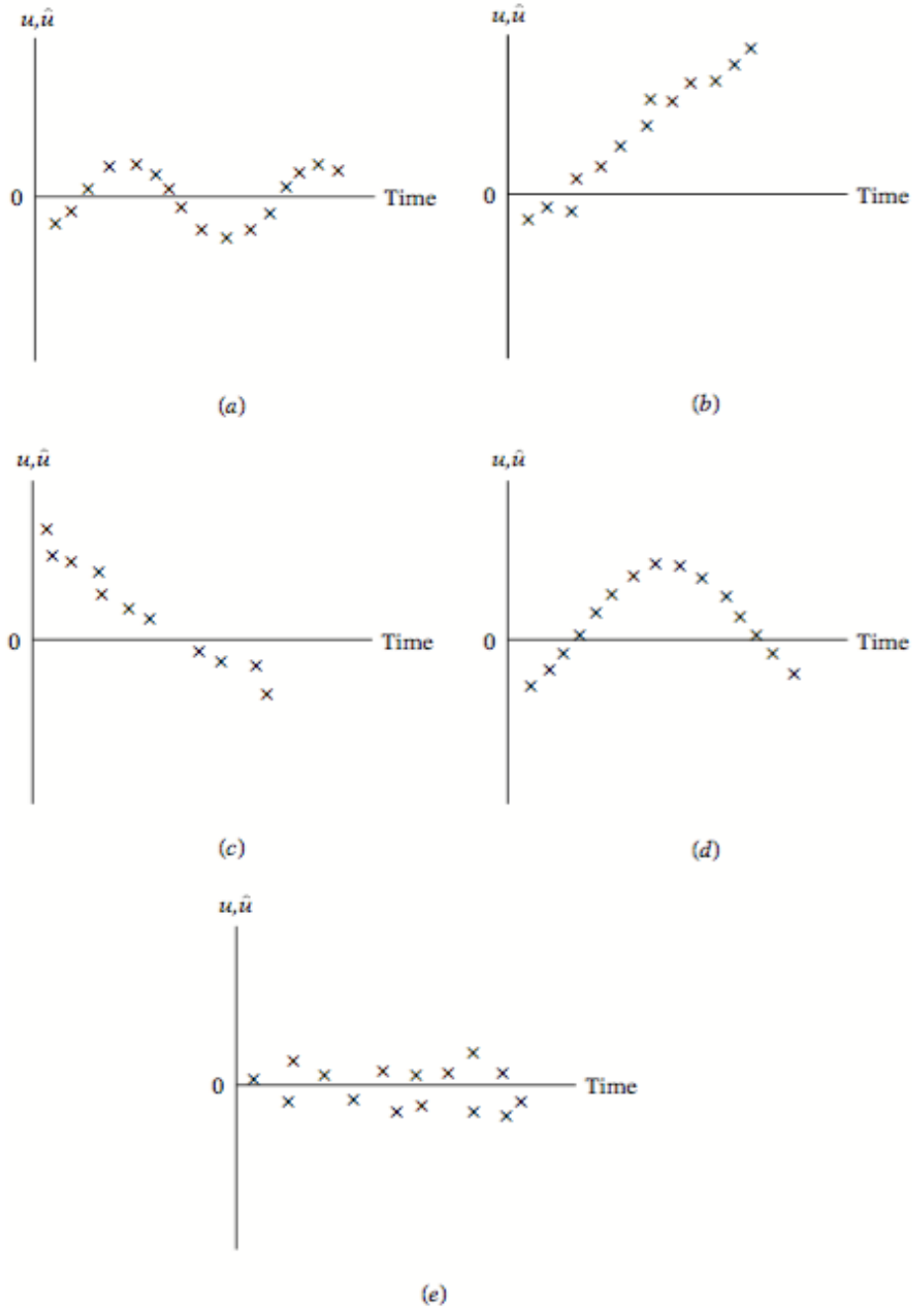


Figure 12-3: Patterns of autocorrelation and nonautocorrelation



among random disturbance terms stems from many factors. The main causes are when the model or data used in the model have the following properties.

1. The autocorrelation problem is more frequently found in the model where time series data is used than where cross-sectional one is used. The reason is that cross-sectional data involves a greater variety of observations, which tend to be independent from one another. The consumption expenditure of people in an entire country, for instance, is diverse. Any factors liable to cause an error may be negligible when the data of the entire country is employed. On the other hand, for time series data, the same sample is studied across time. Mostly, this fact results in the relationship among observations. To illustrate, macroeconomic data may indicate a positive sign in the recovery period and this trend may be prolonged until any external shock coming in.

2. **Model misspecification**, where the important regressors are omitted, could bring about the autocorrelation problem. For instance, consider the model explaining the demand for chicken with essential regressors including its price and the price of pork, as the substitute product.

$$Y_t = \beta_1 + \beta_2 X_{2t} + \beta_3 X_{3t} + u_t$$

where

Y_t = demand for chicken
X_{2t} = price of chicken
X_{3t} = price of pork

Unfortunately, suppose we wrongly specify the model such that the regressor X_{3t} is dropped and the model becomes

$$Y_t = \beta_1 + \beta_2 X_{2t} + \nu_t$$

where $\nu_t = \beta_3 X_{3t} + u_t$. It can be seen that the random disturbance term in this misspecified model (ν_t) incorporates the relationship of demand for chicken and the price of pork. This characteristic could result in significant pattern in disturbance term, leading to autocorrelation problem. The autocorrelation in this case is called **false autocorrelation** since the problem is not originated from the disturbance term itself but model misspecification instead.

3. **Model misspecification**, where the functional form is incorrect, could give rise to the autocorrelation problem as well. Consider the model of marginal cost which depends on the amount of goods produced.

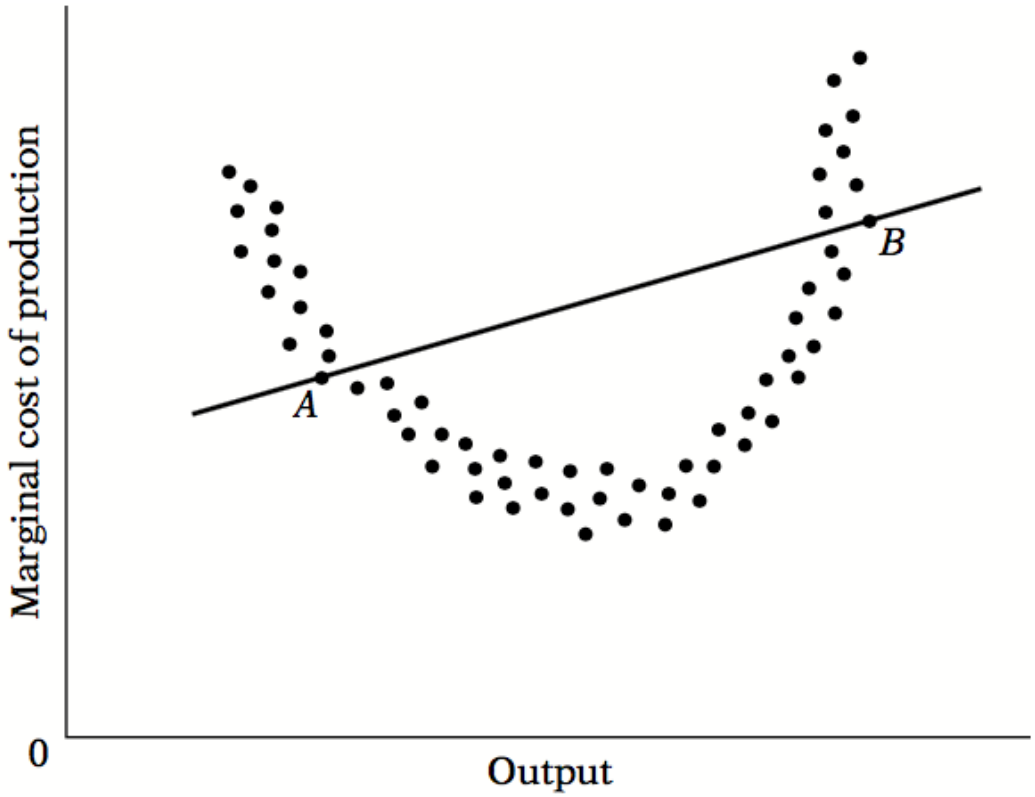
$$MC_i = \beta_1 + \beta_2 Output_i + \beta_3 Output_i^2 + u_i$$

However, suppose the model is mistakenly specified as

$$MC_i = \alpha_1 + \alpha_2 Output_i + \nu_i$$

In this case, the random disturbance term is $v_i = \beta_3 Output_i^2 + u_i$. The result occurring is be similar to the case where the crucial regressors are neglected from the model. That is, a systematic pattern can be observed in the random disturbance term. The resulting autocorrelation is also called false autocorrelation.

Figure 12-4: Specification Bias:Incorrect Functional Form



4. **Cobweb phenomenon** might be another cause. For instance, some economists believe that supply of agricultural product displays the cobweb pattern. That is, the supplier of agricultural product makes a decision based on the last-year price as the production process takes time to deploy. The farmers have to decide first which types of plant will be produced and then production process will be carried out. Hence, they tend to base their decision on the price in the period when the type of plants is chosen rather than the price when the product is marketed.

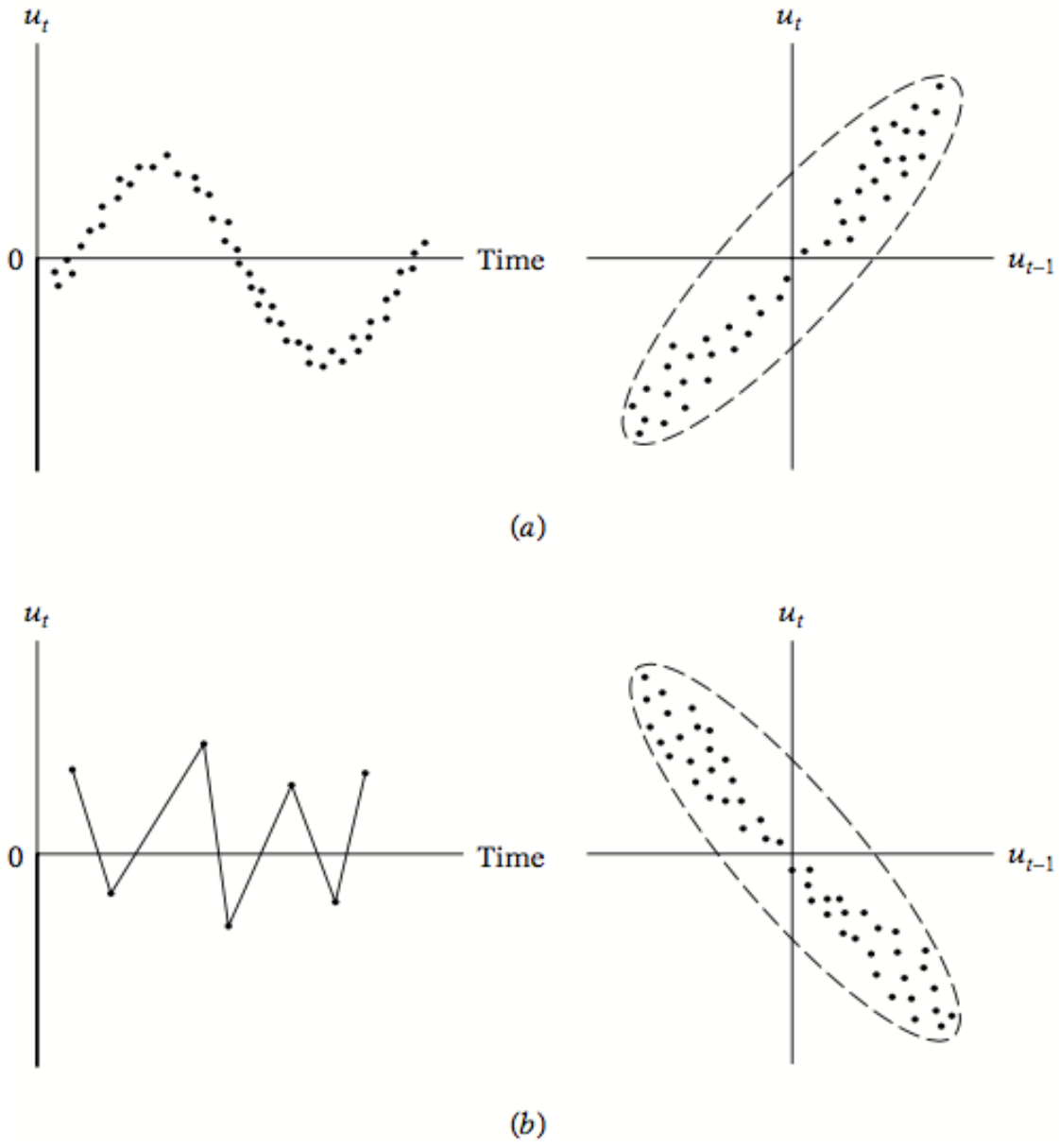
If the price of one plant in the last period is high, there will be a great incentive for farmers to produce that plant. The product will, then, flood the market, forcing its price to go down. Contrarily, if the price of that plant in the last period is low, that plant will become unprofitable to produce in the view of farmers. This probably results in deficiency of the product, raising the price of the plant. Accordingly, the current amount of agricultural product will rely on the price last year. With the predictable pattern

of regressor and regressand, the random disturbance term may display that systematic pattern, leading to the autocorrelation problem.

5. **The transformation of data** into first-difference form may inflict the autocorrelation problem on the model. Consider simple regression model. It is obvious that, when the model is established through first-difference transformation, autocorrelation in random disturbance term will result.

In sum, there are a variety of reasons why the error term in a regression model may be autocorrelated.
It should be noted also that autocorrelation can be positive as well as negative, although most economic time series generally exhibit positive autocorrelation because most of them ether move upward or downward over extended time periods and do not exhibit a constant up-and-down movement

Figure 12-5: (a) Positive autocorrelation and (b) negative autocorrelation



12.2 Consequence of Autocorrelation

Since the autocorrelation in random disturbance term violates classical linear regression model assumptions, the following result can be shown in this section. Consider the simple linear regression model with X as explanatory variable, Y as explained variable, and the random disturbance term at time t: u_t has the relationship with the one-period-lagged disturbance term u_{t-1}.

$$Y_t = \beta_1 + \beta_2 X_t + u_t \tag{12.2}$$

$$u_t = \rho u_{t-1} + \varepsilon_t, \ -1 < \rho < 1 \tag{12.3}$$

where ρ is the coefficient of autocovariance which specifies the degree of relationship between the disturbance term at one period and the term at lagged period. Let the value of ρ ranges between -1 and 1. According to Equation 12.3, this kind of relationship is called **first-order autoregressive: AR(1)**, namely the lag period is one. Also, if the maximum lag period is two, we call it AR(2). Generally, with the lag period of p, we can write the autoregressive model as Equation 12.4.

$$u_t = \rho_1 u_{t-1} + \rho_2 u_{t-2} + ... + \rho_p u_{t-p} + \varepsilon_t \tag{12.4}$$

ε is the **white noise error term** in the autoregressive model with the following properties.

$$\begin{aligned} E(\varepsilon_t) &= 0 \\ var(\varepsilon_t) &= \sigma_\varepsilon^2 \\ cov(\varepsilon_t, \varepsilon_{t+s}) &= 0 \quad where \quad s \neq 0 \end{aligned}$$

It can be seen that, if the coefficient of autocovariance ρ is equal to 1 or -1, the variance of the error term will be undefined. We have to specify the value of ρ between this range to make the disturbance term stationary. Otherwise, the disturbance term may deviate from what it should be, namely in non-stationary disturbance term, so that the regression analysis is inapplicable.

Consider simple regression model without first-order autoregressive disturbance term.

$$Y_t = \beta_1 + \beta_2 X_t + u_t$$

Through OLS method, the estimator has the following close form solution.

$$\hat{\beta}_2 = \frac{\sum(X_t - \bar{X})(Y_t - \bar{Y})}{\sum(X_t - \bar{X})^2} = \frac{\sum x_t y_t}{\sum x_t^2}$$

The variance of the estimator can be calculated by the following formula.

$$Var(\hat{\beta}_2) = \frac{\sigma^2}{\sum(X_t - \bar{X})^2} = \frac{\sigma^2}{\sum x_t^2}$$

Nevertheless, under AR(1) scheme, the variance of the estimator can be computed as:

The difference between the two above situation is that the variance under AR(1) scheme will be higher. In Equation 12.3, as ρ is equal to zero, or there is no autocorrelation in disturbance term, Equation 12.3 will converge to the usual formula of the variance of the estimator. Hence, due to autocorrelation problem, the OLS estimators will not possess best property, namely the estimator will not have minimum variance !!. But,the OLS estimators are still unbiased.

12.2.1 Detection of Autocorrelation

When the autocorrelation problem leads to some undesirable properties of the estimators, the conclusion drawn from hypothesis testing may be misleading. Therefore, to prevent this mistake, we should examine whether the model suffers this problem. There are many approaches used to detect this problem and some of them are discussed in this section.

1. *Finding the relationship among disturbance terms by graph*: as depicted in Figure 12-3, if there is autocorrelation, the pattern of disturbance term across time will have systematic form.

2. *t-test for autocorrelation*: we can examine AR(1) model or $u_t = \rho u_{t-1} + \varepsilon_t$ and the independent variables in the model have to be **strictly exogenous**. That is,

$$E(u_t|X_{2t}, X_{3t}, ..., X_{kt}) = 0$$

or

$$cov(u_t, X_{jt}) = 0 \text{ where } j = 2, 3, ..., k$$

When the independent variables are strictly exogenous, we can set the following hypothesis as

$$H_o : \rho = 0 \text{ and } H_a : \rho \neq 0$$

with the following test procedure.

Step 1: estimate the model of interest through OLS method to obtain $\hat{u}_t$

Step 2: construct AR(1) model with dependent variable u_t and independent variable u_{t-1} and, then, estimate the regression model to obtain the estimated value of ρ.

$$\hat{u}_t = \hat{\rho}\hat{u}_{t-1} + \hat{\varepsilon}_t$$

Step 3: calculate t-statistic of the estimator $\hat{\rho}$ and test for statistical significance. If we can reject the null hypothesis, that means there is the first order autocorrelation among disturbance terms.

This approach can be applied to the test for higher order of autoregressive model like $u_t = \rho u_{t-3} + \varepsilon_t$ which includes setting hypothesis, computing t-statistic, comparing it with the critical value in statistical table, and drawing the conclusion of whether the null hypothesis should be rejected.

3.*Durbin-Watson test*[1]: DW-statistic test is one of the popular methods used to detect first order autocorrelation problem. Six assumptions are required for DW-test.

Assumption 1: the model has to include the intercept

Assumption 2: explained variable X is non stochastic

Assumption 3: the relationship of the disturbance term has to be generated by AR(1) process.

Assumption 4: the disturbance term u_t is normally distributed.

Assumption 5: the model under examination does not include lagged regressand Y_{t-1} as the explanatory variable. That is, if the model follows equation below, we cannot apply DW-statistic to the test for autocorrelation.

$$Y_t = \beta_1 + \beta_2 X_{2t} + \alpha Y_{t-1} + u_t$$

Assumption 6: There is no missing data.

When all the assumptions are satisfied, DW-statistic can be computed by Equation 75.

$$DW = \frac{\sum_{t=2}^{n}(\hat{u}_t - \hat{u}_{t-1})^2}{\sum_{t=1}^{t=n}\hat{u}_t^2} \approx 2(1-\hat{\rho}) \tag{12.5}$$

$$\hat{\rho} = \frac{\sum \hat{u}_t \hat{u}_{t-1}}{\sum \hat{u}^2} \tag{12.6}$$

where $\hat{\rho}$ ranges from -1 to 1, causing DW-statistic to range from 0 to 4. Consider the following possible cases. If $\hat{\rho}$ approaches 0, DW will approach 2, that is no first-order autocorrelation among disturbance terms. If $\hat{\rho}$ approach 1, DW will approach 0, that is positive first-order autocorrelation among disturbance terms. If $\hat{\rho}$ approach -1, DW will approach 4, that is negative first-order autocorrelation among disturbance terms.

After DW-statistic is obtained, we can use it to test the following hypothesis.

$$H_o : \rho = 0 \text{ and } H_a : \rho \neq 0$$

[1]Durbin, James. and Watson, Geoffrey. (1951). “Testing for Serial Correlation in Least-Squares Regression," Biometrika, Vol.38 pp.159-171

The DW-statistic has to be compared with DW-statistical table invent in 1950 in order to draw the conclusion about the test. The critical value will include d_L and d_u which are the upper and lower bound respectively. The degree of freedom $k-1$ (which is the amount of explanatory variables excluding intercept term) and level of significance (α) of 0.05 and 0.01 will vary according to the circumstance. The comparison of DW-statistic to the critical value provides two beneficial insights.

If it turns out that the estimate of the coefficient of autocovariance is greater than 0, or equivalently DW-statistic is lower than 2, it can be the suspected that the random disturbance terms may have **positive autocorrelation**. Hence, the null and alternative hypothesis can be set as

$$H_o : \rho \leq 0 \text{ and } H_a : \rho > 0$$

We cannot reject the null hypothesis when calculated DW-statistic is greater than d_U. We reject the null hypothesis when calculated DW-statistic is lower than d_L. That is, disturbance term has no positive serial correlation at the level of significance α. Nevertheless, if DW-statistic lies between d_L and d_U ($d_L \leq DW \leq d_U$), we cannot conclude whether the disturbance term has positive serial correlation.

If, on the other hand, it appears that the estimate of the coefficient of autocovariance is less than 0, or equivalently DW-statistic is greater than 2, it can be suspected that the random disturbance terms may have **negative autocorrelation**. Hence, the null and alternative hypotheses can be set as

$$H_o : \rho \geq 0 \text{ and } H_a : \rho < 0$$

We cannot reject the null hypothesis when calculated DW-statistic is greater than $4-d_U$. We reject the null hypothesis when calculated DW-statistic is greater than $4-d_L$. That is, disturbance term has no positive serial correlation at the level of significance α. Nevertheless, if DW-statistic lies between $4-d_L$ and $4-d_U$ ($4-d_U \leq DW \leq 4-d_L$), we cannot conclude whether the disturbance term has negative serial correlation. Figure 12.6 illustrates the criterion whether reject or not reject the null hypothesis.

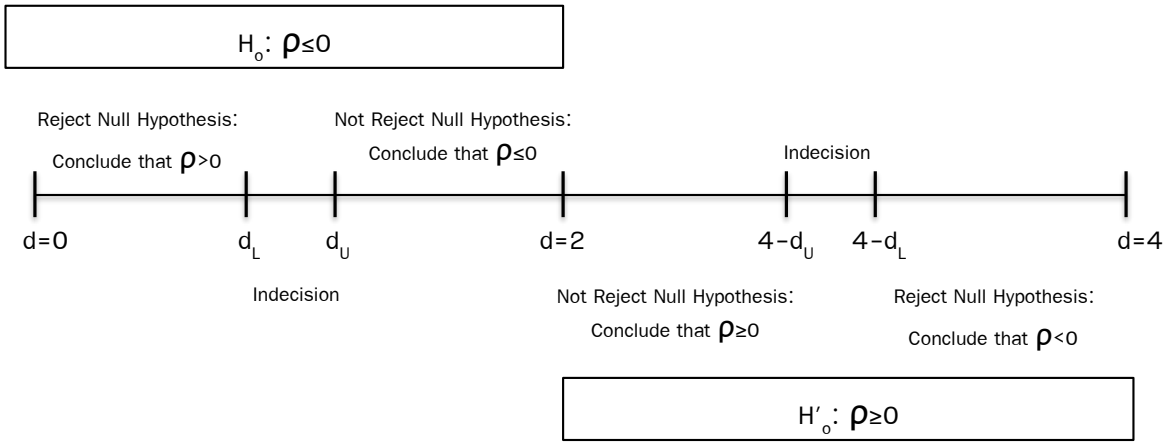
In short, the procedure to test for first order autocorrelation among disturbance terms with DW-statistic involves the following steps.

Step 1: estimate the model of interest to find $\hat{u}_t$

Step 2: calculate DW-statistic by the formula in Equation 12.5

Step 3: compare DW-statistic with the critical d from the table with the criterion shown in Figure 12.6 to achieve the conclusion about the relationship among disturbance terms.

Figure 12.6: Criterion for rejecting or not rejecting the null hypothesis with DW-statistic



4. **Breusch-Godfrey test**[2]: this method can be used to test for any order, from 1 or AR(1) to p or AR(p) of autocorrelation as illustrated in Equation 12.4 from the simple regression model

$$u_t = \rho_1 u_{t-1} + \rho_2 u_{t-2} + ... + \rho_p u_{t-p} + \varepsilon_t \tag{12.4}$$

We can form the hypothesis about the relationship among these disturbance terms as
H_o : $\rho_1 = \rho_2 = ... = \rho_p = 0$
H_a : otherwise
with the following procedure.

Step 1: establish the regression model of interest. The example shown is the simple one.

Step 2: establish another model to obtain the relationship among disturbance terms with $\hat{u}_t$ as the explained variable and X_{2t} and $\hat{u}_{t-1}$, $\hat{u}_{t-2}$ until $\hat{u}_{t-p}$ as the explanatory variables.

$$\hat{u}_t = \alpha_1 + \alpha_2 X_{2t} + \hat{\rho}_1 \hat{u}_{t-1} + \hat{\rho}_2 u_{t-2} + ... + \hat{\rho}_p u_{t-p} + \varepsilon_t \tag{12.7}$$

Step 3: if the sample size is large, LM-statistic can be computed by

$$LM = (n-p)R^2 \sim \chi^2_p \tag{12.8}$$

Step 4: compare LM-statistic with the critical value of chi-square table to conclude whether the null hypothesis should be rejected. Specifically, we reject the null hypothesis if $(n-p)R^2$ is greater than the critical chi-square at the chosen level of significance and conclude that there exists the autocorrelation problem.

[2]Breusch, Trevor .S. (1978) “Testing for Autocorrelation in Dynamic Linear Models", Australian Economic Papers, Vol.17, Issue.31, pp.334-355. Godfrey, Leslie G. (1978). “Testing Against General Autoregressive and Moving Average Error Models When the Regressor Includes Lagged Dependent Variables," Econometrica, Vol.46, No.6, pp.1293-1301.

12.2.2 Remedial Measure for Autocorrelation

After the problem is detected, to prevent the problem from making the variance of estimators so unreliable that the conclusion drawn from hypothesis test is misleading, the remedial measure is necessary. In this section, only the remedial measures for the first order autocorrelation are discussed.

Case when the ρ is known

1. *Generalized least squares (GLS)*: consider the simple regression model with the first order autocorrelation problem AR(1)

$$Y_t = \beta_1 + \beta_2 X_t + u_t$$

$$u_t = \rho u_{t-1} + \varepsilon_t$$

We can separate the procedure into the case where the coefficient of autocovariance (ρ) is known and unknown. For the case when the ρ **is known**, we can solve the problem by

$$\begin{aligned} Y_{t-1} &= \beta_1 + \beta_2 X_{t-1} + u_{t-1} \\ \rho Y_{t-1} &= \rho\beta_1 + \rho\beta_2 X_{t-1} + \rho u_{t-1} \end{aligned}$$

$$\begin{aligned} (Y_t - \rho Y_{t-1}) &= \beta_1(1-\rho) + \beta_2(X_t - \rho X_{t-1}) + (u_t - \rho u_{t-1}) \\ (Y_t - \rho Y_{t-1}) &= \beta_1(1-\rho) + \beta_2(X_t - \rho X_{t-1}) + \varepsilon_t \end{aligned}$$

$$Y_t^* = \beta_1^* + \beta_2^* X_t^* + \varepsilon_t$$

where

$$\begin{aligned} \varepsilon_t &= u_t - \rho u_{t-1} \\ Y_t^* &= Y_t - \rho Y_{t-1} \\ X_t^* &= X_t - \rho X_{t-1} \\ \beta_1^* &= \beta_1(1-\rho) \end{aligned}$$

According to the above property of ε_t, we find that the classical linear regression model assumption is satisfied. Hence, it can be concluded that, the estimators in the new model generated from the above procedure are best linear unbiased estimators (BLUE).

Notwithstanding, due to the above procedure, the regressor and regressand of the new model is in the difference form which means that one observation is lost. Thus, to mitigate the problem, the first observation on X and Y may be transformed to $X_1^* = X_1\sqrt{1-\rho^2}$ and $Y_1^* = Y_1\sqrt{1-\rho^2}$. We call this transformation process Prais-Winsten transformation.

Case when the ρ is unknown

In the case when the ρ **is unknown**, two solutions to the problem are available. The first one is *first difference method*. This approach is usually used when DW-statistic for the model of interest is less than R^2. The first-difference model is constructed as:

$$\begin{aligned} Y_t - Y_{t-1} &= \beta_1 - \beta_1 + \beta_2(X_t - X_{t-1}) + (u_t - u_{t-1}) \\ \Delta Y &= \beta_2 \Delta X_t + \varepsilon_t \end{aligned}$$

When the above regressor and regressand are obtained, we can apply the OLS method for non-intercept model to estimate this new model which solves the first order autocorrelation problem.

The other method is *to use estimator $\hat{\rho}$ obtained from the calculation of DW-statistic*. From Equation 75 and 76, when the sample size is large enough, we can compute the value of $\hat{\rho}$ through equation 79.

$$\hat{\rho} = 1 - \frac{d}{2} \tag{12.9}$$

After we obtain the estimate, we can apply it to the remedial measure stated above when the value of ρ is known.

2. *Heteroscedasticity and autocorrelation-consistent standard error*: when the sample size is large enough, Newey and West suggest the formula for the variance of estimators when the autocorrelation and heteroscedasticity problem occur. With this formula, the standard deviation required in statistical analysis, such as hypothesis test, confidence interval and so forth, will be applicable.

At the present time, most econometric computer packages provide this estimation of variance in the set of statistical results. Although this method does not lessen the degree of autocorrelation problem, the obtained standard error is fixed and applicable for further statistical analysis.

It is noteworthy that the difference between the method of White and Newey-West is recognized. The approach suggested by White can solve the specific problem of heteroscedasticity; whereas the one suggested by Newey-West is designed to tackle the problems of both heteroscedasticity and autocorrelation.