Consumer's problem

Constrained optimization
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❖ Key Terms

Economics

- Giffen good (p. 428)
- income effect (p. 428)
- indifference curve (p. 413)
- indifference mapping (p. 414)
- law of diminishing marginal utility (p. 412)
- marginal rate of substitution (p. 415)
- substitution effect (p. 428)

❖ Problems

1. For each of the following utility functions, where $0 < a < 1$:
   i) $U = x^a y^{1-a}$ ii) $U = a \ln x + (1 - a) \ln y$ iii) $U = [a x^{-2} + (1 - a) y^{-2}]^{-1/2}$
   a) Show that the marginal utilities for x and y are positive but diminishing.
   b) Derive the marginal rate of substitution.

2. Given a household's utility is a function of its consumption of two goods, x and y, where $U = U(x,y) = 2 \ln x + \ln y$, its budgeted income is \$36, and the unit prices of the two goods are $P_x = \$2$ and $P_y = \$4$.
   a) Show that the consumption of each good provides positive, but diminishing, marginal utility to the household.
   b) Find the utility-maximizing combination of the two goods. What is the total utility of the household at this optimum?
   c) Find the marginal utility of income when the household's utility is maximized.
   d) Check the second-order conditions.
   e) Find the new utility-maximizing combination of goods x and y if the household increases its budgeted income from \$36 to \$36.30. Find the change in total utility associated with this change in income. Compare this with the optimum value of the Lagrange multiplier.

3. Given the utility function, $U = 10[.4x^{-3} + .6y^{-2}]^{-1/3}$, a budgeted income of $I =$ \$100, and unit prices of the two goods, $P_x = \$6$ and $P_y = \$2$.
   a) Find the utility-maximizing combination of the two goods, x and y, subject to the income constraint.
   b) Determine the marginal utility of income and the maximum utility.
   c) Check the second-order conditions.

4. Given an individual, seeking to maximize his utility, which is a function of his consumption of goods and services per week, X, and his leisure hours per week, F, subject to a budget constraint. His utility function is: $U = X^a F^b$ $(0 < a < 1, 0 < b < 1)$. The budget constraint of this individual is given by: $P_X X = M + w \cdot (T - F)$, where

   X = quantity of goods and services consumed per week

   F = leisure hours consumed per week

   P_X = price index for goods and services

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M = nonwage income of the individual

w = wage rate per hour the individual earns

T = number of discretionary hours in the week to be allocated between work and leisure.

a) Set up the Lagrangian function, then derive and interpret the first-order conditions.
b) Check the second-order conditions.
c) Find and sign (if possible):
i) $\delta \bar{F}/\delta M$ and ii) $\delta \bar{F}/\delta w$.

Discuss whether these comparative static results make economic sense.

5. Given the following linear programming problem for utility maximization:

$$\text{maximize: } U = 5x + y$$

$$\begin{aligned} \text{subject to: } x + y &\le 8 && \text{(time constraint)} \\ 3x + y &\le 12 && \text{(budget constraint)} \\ x &\le 3 && \text{(ration constraint)} \\ x, y &\ge 0 \end{aligned}$$

where x and y are the quantities of the two goods under consideration and the individual is seeking to maximize her total utility (U) subject to three constraints:

- Up to 8 hours of time available for the consumption of the two goods. A unit of x and a unit of y each require 1 hour for consumption.
- Up to \$12 of income is budgeted for the consumption of the two goods. The unit prices are P_x = \$3 and P_y = \$1.
- A maximum of 3 units of x are available under the allotment of ration coupons.

a) Graph the feasible region. Find the solution by testing the extreme points. Plot the highest indifference curve the individual can attain given the constraints.
b) Solve the linear programming problem using the simplex method. Which constraints are binding? Find the shadow prices associated with time, income, and ration coupons.
c) Set up the dual, that is, the counterpart minimization problem. Solve using the simplex method. Relate the solutions for the maximization and minimization problems using Duality Theorems I and II.
d) What simplifying assumption is implicit in the linear objective function for the primal? Is this consistent with the usual assumptions we make about consumer behavior?

6. Given a household seeking to maximize its total utility, $U = x^{.5}y^{.4}$, subject to an income constraint. The budgeted income is: I = \$500 per month and the unit prices of the two goods are P_x = \$50 and P_y = \$20.

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a) Set up the Lagrangian and then find the utility-maximizing combination of the two goods, x and y, subject to the income constraint.
b) Determine the marginal utility of income and the maximum utility.
c) Check the second-order conditions.

Now add a second constraint, where the maximum available time for the consumption of these two goods is $T = 40$ hours per month and the unit time costs are $t_x = 5$ hours and $t_y = 1$ hour. Assume now that the budgeted income of \$500 is a maximum that the household is willing to spend each month on the consumption of goods x and y.

d) Set up the nonlinear programming problem.
e) Graph the feasible region.
f) Write out the Lagrangian function and the Kuhn-Tucker conditions.
g) Find the utility-maximizing quantities of the two goods and the maximum total utility.
h) Determine the marginal utilities of income and time. Determine whether there is any unspent income or unused time.
i) Determine, if possible, whether the utility-maximizing combination of the two goods constitutes an absolute maximum.

7. Refer to Problem 6 above. Suppose that the time cost of y increases from $t_y = 1$ to $t'_y = 1.25$ (hours). Repeat steps *d)*–*h)*.

8. Refer to Problem 6 above. Suppose that the maximum budgeted income for the consumption of these two goods increases from $I = \$500$ to $I' = \$600$ per month. Repeat steps *d)*–*h)*.

❖ Answers to Practice Problems

12.1 a) $U_x = .7x^{-.3}\,y^{.2} > 0$ and $U_{xx} = -.21x^{-1.3}\,y^{.2} < 0$
$U_y = .2x^{.7}\,y^{-.8} > 0$ and $U_{yy} = -.16x^{.7}\,y^{-1.8} < 0$
b) $MRS = -U_x/U_y = -3.5y/x < 0$
c) $dMRS/dx = d(-3.5y/x)/dx = 15.75y/x^2 > 0$

12.2 a) The Lagrangian is: $\mathcal{L}(\lambda, x, y) = 2x^{.6}\,y^{.3} + \lambda \cdot (90 - 4x - y)$. The utility-maximizing combination of the two goods is: $\bar{x} = 15$ and $\bar{y} = 30$.
b) The marginal utility of income is: $\bar{\lambda} = .28$. The maximum utility is: $\bar{U} = 28.17$.
c) The bordered Hessian equals: $|\bar{H}| = .23 > 0$, consistent with a maximum.

12.3 $\delta\bar{I}/\delta w = \dfrac{(T - \bar{F})(-U_{FF} + wU_{FI}) + w\bar{\lambda}}{|\bar{H}|}$ can't be signed; the sign of U_{FI} is indeterminate.

12.4 a) The utility-maximizing combination is: $\bar{x} = 6$ and $\bar{y} = 12$. The maximum utility is $\bar{U} = 6.36$.
b) The marginal utilities of income and time are respectively: $\bar{\lambda}_1 = .005$ and $\bar{\lambda}_2 = .152$.

Producer's Problem

Constrained Optimization

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In the next chapter we continue our analysis of constrained optimization for the firm with the technique of mathematical programming and multiple constraints. Here the optimization occurs under conditions where not all of the constraints need be binding.

❖ KEY TERMS

Economics

- CES production function (p. 343)
- Cobb-Douglas production function (p. 343)
- constant returns to scale (p. 344)
- decreasing returns to scale (p. 344)
- economically efficient (p. 349)
- elasticity of substitution (p. 359)
- fixed coefficients production function (p. 359)
- increasing returns to scale (p. 344)
- isocost line (p. 350)
- isoquant (p. 345)
- law of diminishing returns (p. 347)
- marginal rate of technical substitution (p. 346)
- output-expansion path (p. 356)
- production function (p. 342)
- technically efficient (p. 343)
- user cost of capital (p. 349)
- user cost of labor (p. 349)

Mathematics

- bordered Hessian (p. 338)
- homogeneous of degree r (p. 344)
- Lagrange multiplier (p. 330)
- Lagrangian (p. 330)
- linearly homogeneous (p. 344)
- overdetermined (p. 341)

❖ PROBLEMS

1. Given the objective function, $z(x,y) = 40x - 2x^2 + xy - 3y^2 + 20y - 40$:
   a) Find the critical values for the variables x and y.
   b) Find the optimum value for z.
   c) Check the second-order conditions to establish the type of relative extremum.
   d) Now add the constraint, $x + y = 10$, and find the new critical values for x and y.
   e) Find the constrained optimum value for z.
   f) Find the value of the Lagrange multiplier.
   g) Check the second-order condition for the constrained optimization.
2. Repeat parts a)–c) of Problem 1 for the objective function, $z(x,y) = -25x + 5x^2 - 2xy + 2y^2 - 15y + 12$. Then add the constraint, $3x + y = 15$, and repeat parts d)–g).
3. Repeat parts d)–g) in Problem 2 if, instead, the constraint is $3x + y = 18$.
4. Given the production functions:
   i) $Q = 2K^{.4}L^{.6}$ ii) $Q = 2\,(.4K^{-2} + .6L^{-2})^{-1/2}$ iii) $Q = .4K + .6L$
   for each find:
   a) the marginal products of labor and capital. Are labor and capital each subject to diminishing returns?

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b) the returns to scale
c) the elasticity of substitution
d) the equation for the output-expansion path, given that the user costs of labor and capital are respectively: $w_0 = \$10$ and $r_0 = \$20$.

5. Given a firm with a production function, $Q = 1.5K^{.4}L^{.6}$, seeking the cost-minimizing combination of capital and labor for producing $Q_0 = 200$ units of output, assume that the user costs of capital and labor to the firm are, respectively, $r_0 = \$20$ and $w_0 = \$12$.
   a) Find the cost-minimizing combination of K and L.
   b) Determine the marginal cost of production.
   c) Determine the minimum total cost.
   d) Check the second-order condition.
   e) If the firm decreases its output to $Q_1 = 190$, find the new cost-minimizing combination of K and L.
   f) If the user cost of capital declines to $r_2 = \$15$, find the new cost-minimizing combination of K and L for producing $Q_0 = 200$.

6. Repeat the analysis in Problem 5, parts a)–c), only where the firm's production function is given by $Q = 1.5(.4K^{-2} + .6L^{-2})^{-1/2}$.

7. A firm with a Cobb-Douglas production function, $Q = AK^aL^b$, $(0 < a < 1, 0 < b < 1)$, where A is an index of technology, facing user costs of labor and capital equal to w_0 and r_0, respectively, seeks to find the cost-minimizing combination of capital and labor for producing a selected level of output equal to Q_0.
   a) Set up the Lagrangian function.
   b) Derive the first-order conditions and solve for the optimal values of K and L.
   c) Solve for and interpret the optimal value of the Lagrange multiplier.
   d) Check the second-order conditions.
   e) Find and sign (if possible) the following comparative static results.
      i) $\delta\bar{L}/\delta A$ ii) $\delta\bar{L}/\delta w_0$ iii) $\delta\bar{L}/\delta Q_0$

   Do these results make economic sense? Discuss.

   Note: For this problem you do not have to use the Implicit Function Theorem to do comparative statics since you can solve explicitly for the equilibrium values of the endogenous variables.

8. A firm with a general production function, $Q = A \cdot Q(K, L)$, where A is an index of technology, facing user costs of labor and capital equal to w_0 and r_0, respectively, seeks to find the cost-minimizing combination of capital and labor for producing a selected level of output of Q_0.
   a) Set up the Lagrangian function.
   b) Derive and interpret the first-order conditions.
   c) Interpret the optimal value of the Lagrange multiplier.
   d) Check the second-order conditions.

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e) Using the Implicit Function Theorem and Cramer's rule, find and sign (if possible) the following comparative static results.
i) $\delta\overline{L}/\delta A$ ii) $\delta\overline{L}/\delta w_0$ iii) $\delta\overline{L}/\delta Q_0$

Do these results make economic sense? Discuss.

❖ Answers to Practice Problems

10.1 a) The critical values are: $\overline{x}_0 = 48.3$ and $\overline{y}_0 = 40.0$.
b) The optimal value of z is: $\overline{z}_0 = 1{,}608.\overline{3}$.
c) $|H_1| = -6 < 0$ and $|H_2| = 12 > 0$, consistent with a maximum.
d) The critical values are: $\overline{x}_1 = 43.8$ and $\overline{y}_1 = 36.2$.
e) The constrained optimum value of z is: $\overline{z}_1 = 1{,}592.3$.
f) The value of the Lagrange multiplier is: $\overline{\lambda}_1 = 3.85$.

10.2 a) The critical values are: $\overline{x}_0 = 1.\overline{6}$ and $\overline{y}_0 = .8\overline{3}$.
b) The optimal value of z is: $\overline{z}_0 = 9.72$.
c) $|H_1| = 10 > 0$ and $|H_2| = 24 > 0$, consistent with a minimum.
d) The critical values are: $\overline{x}_1 = 1.2$ and $\overline{y}_1 = .6$.
e) The constrained optimum value of z is: $\overline{z}_1 = 5.8$.
f) The value of the Lagrange multiplier is: $\overline{\lambda}_1 = -2.8$.

10.3 The bordered Hessian is $|\overline{H}| = 26 > 0$, which is consistent with a constrained maximum.

10.4 The bordered Hessian is $|\overline{H}| = -10 < 0$, which is consistent with a constrained minimum.

10.5 a) $Q = .5K^{.7}L^{.3}$ is characterized by constant returns to scale, $r = 1.0$.
b) $Q = [.5K^{-2} + .5L^{-2}]^{-.6}$ is characterized by increasing returns to scale, $r = 1.2$.

10.6 a) The marginal products are: $Q_L = 1.2K^{.5}L^{-.6} > 0$ and $Q_K = 1.5K^{-.5}L^{.4} > 0$.
b) $Q_{LL} = -.72K^{.5}L^{-1.6} < 0$ and $Q_{KK} = -.75K^{-1.5}L^{.4} < 0$.
c) $Q_{KL} = .6K^{-.5}L^{-.6} = Q_{LK} > 0$
d) $dMRTS/dL = 1.44K/L^2 > 0$

10.7 a) The cost-minimizing combination of capital and labor is: $\overline{K}_0 = 22.6$ and $\overline{L}_0 = 84.8$.
b) The marginal cost of output is: $\overline{\lambda}_0 = \$8.5$.
c) The minimum total cost is: $\overline{C} = \$847.8$.
d) The bordered Hessian equals $|\overline{H}| = -.55 < 0$, which is consistent with a constrained minimum.

10.8 a) $\overline{K} = 10.\overline{6}\,\overline{L}$.
b) $\overline{K} = 1.8\overline{L}$.

10.9 a) The elasticity of substitution is: $\sigma = 1$.
b) The elasticity of substitution is: $\sigma = 1/4$.