

Macroeconomics

Lecture 5

Stochastic Control Problem

$$\text{Max} \quad E_0 \sum_{t=0}^{\infty} \beta^t U(x_t, c_t), \quad 0 < \beta < 1 \quad (17)$$

$$\text{s.t.} \quad x_{t+1} = g(x_t, c_t, \varepsilon_{t+1}), \quad x_0 \text{ given} \quad (18)$$

ε_{t+1} is a sequence of i.i.d. (independently and identically distributed) random variables.

ε_{t+1} is realized at $t+1$ after c_t has been decided at time t .

Bellman's equation,

$$V(x) = \max_c \left[U(x, c) + \beta E \left[V \left[g(x, c, \tilde{\varepsilon}) \right] \mid x \right] \right] \quad (19)$$

The 1st – order condition for the problem on the RHS of (19) is

$$\frac{\partial U(x, c)}{\partial c} + \beta E \left[\frac{\partial g(x, c, \tilde{\varepsilon})}{\partial c} V' \left[g(x, c, \tilde{\varepsilon}) \right] \mid x \right] = 0 \quad (20)$$

Stochastic Control Problem

The value function must also satisfy (this is analogous to eq. (9))

$$V'(x) = \frac{\partial U[x, h(x)]}{\partial x} + \frac{\partial U[x, h(x)]}{\partial h(x)} \cdot \frac{\partial h(x)}{\partial x} + \beta E \left\{ \frac{\partial g(x, h(x), \tilde{\varepsilon})}{\partial x} V'(g(x, h(x), \tilde{\varepsilon})) \mid x \right\} \quad (21)$$

Consumption with a Random Return

$$\max E_0 \sum_{t=0}^{\infty} \beta^t U(c_t), \quad 0 < \beta < 1$$

$$s.t. \quad A_{t+1} = R_t (A_t - c_t), \quad t \geq 0, \quad A_0 \text{ given,}$$

and assumed that $U'(c) > 0$, $U''(c) < 0$.

\therefore Bellman's equation is

$$V(A_t, R_{t-1}) = \max_{s_t} \{U(A_t - s_t) + \beta E_t V(s_t R_t, R_t)\},$$

where $s_t = A_t - c_t$,

The 1st - order condition (w.r.t. s_t)

$$-U'(c_t) + \beta E_t \left[\frac{\partial V(s_t R_t, R_t)}{\partial s_t R_t} \cdot R_t \mid A_t, R_{t-1} \right] = 0. \quad (22)$$

Consumption with a Random Return

Then from (21),

$$\begin{aligned}
 V'(A_t, R_{t-1}) &= \frac{\partial U[A_t - h(A_t, R_{t-1})]}{\partial c_t} \cdot \frac{\partial c_t}{\partial A_t} \\
 &\quad + \beta E \left\{ \frac{\partial h(A_t, R_{t-1}) \cdot R_t}{\partial A_t} V'(h(A_t, R_{t-1}) \cdot R_t, R_t) \mid A_t, R_{t-1} \right\} \\
 &= \frac{\partial U[c_t]}{\partial c_t} \left[1 - \frac{\partial h(A_t, R_{t-1})}{\partial A_t} \right] \\
 &\quad + \frac{\partial h(A_t, R_{t-1})}{\partial A_t} \left[\beta E \left\{ R_t V'(h(A_t, R_{t-1}) \cdot R_t, R_t) \mid A_t, R_{t-1} \right\} \right] \\
 &= \frac{\partial U(c_t)}{\partial c_t} - \frac{\partial h(A_t, R_{t-1})}{\partial A_t} \left[\frac{\partial U(c_t)}{\partial c_t} - \left[\beta E \left\{ R_t \cdot V'(h(A_t, R_{t-1}) \cdot R_t, R_t) \mid A_t, R_{t-1} \right\} \right] \right] \\
 &= \frac{\partial U(c_t)}{\partial c_t}. \quad (\because \text{1st-order condition w.r.t. } s_t)
 \end{aligned}$$

Hence, $E_t \{V'(A_{t+1}, R_t)\} = E_t \{U'(c_{t+1})\}$.

Substituting the above equation into equation (22), one has

$$U'(c_t) = \beta E_t [U'(c_{t+1}) \cdot R_t] \quad (23)$$

\Rightarrow an optimal saving policy function $s_t = h(A_t, R_{t-1})$

Let $U(c_t) = \ln c_t$,

Assume that R_t be an i.i.d. random process such that $1 \leq ER_t < (1/\beta^2)$

If $c_t = \gamma A_t$, then (23) becomes

$$\frac{1}{\gamma A_t} = \beta E_t \left[\frac{R_t}{\gamma A_{t+1}} \right] = \beta E_t \left[\frac{R_t}{\gamma R_t (A_t - \gamma A_t)} \right], \quad (\because A_{t+1} = R_t (A_t - c_t))$$

Hence,
$$\frac{1}{\gamma A_t} = \beta E \left[\frac{1}{\gamma (A_t - \gamma A_t)} \right],$$

$$\gamma (A_t - \gamma A_t) = \beta \gamma A_t,$$

$$(1 - \beta) \gamma A_t = \gamma^2 A_t,$$

$$\therefore \gamma = 1 - \beta,$$

Then, $c_t = (1 - \beta) A_t.$

Note that $A_1 = R_0 [A_0 - c_0]$

$$\therefore A_1 = R_0 (1 - \gamma) A_0, \quad (\because c_t = \gamma A_t)$$

$$A_2 = R_1 (\beta) A_1 = R_1 \beta R_0 (1 - \gamma) A_0 = \beta^2 R_1 R_0 A_0$$

:

$$A_t = \beta^t \left[\prod_{j=0}^{t-1} R_j \right] A_0, \quad t = 1, 2, \dots$$

$$\therefore c_t = \gamma A_t = (1 - \beta) \beta^t \left[\prod_{j=0}^{t-1} R_j \right] A_0, \quad t = 1, 2, \dots$$