

Solution: Optional Problems for Assignment 6

Optional Problems

1. Find the critical numbers of the given functions on $(-\infty, \infty)$.

(a) $f(x) = x^3 - 3x^2 + 3x - 1$

(b) $f(x) = \frac{x^2}{x^2+2}$

(c) $f(x) = e^{-x} + 2x$

(d) $f(x) = -x + \sin(x)$

(e) $f(x) = x^2 - 8 \ln(x)$

(a) $f(x) = x^3 - 3x^2 + 3x - 1$

Answer: $f'(x) = 3x^2 - 6x + 3$

$$f'(x) = 0 \Leftrightarrow 3(x^2 - 2x + 1) = 3(x - 1)^2 = 0 \Leftrightarrow (x - 1) = 0 \Leftrightarrow x = 1$$

Therefore, the only critical number of $f(x)$ is $x = 1$.

(b) $f(x) = \frac{x^2}{x^2+2}$

Answer:

$$f'(x) = \frac{(x^2 + 2)(2x) - x^2(2x)}{(x^2 + 2)^2} = \frac{2x^3 + 4x - x^3}{(x^2 + 2)^2} = \frac{x^3 + 4x}{(x^2 + 2)^2}$$

$$f'(x) = 0 \text{ when } x^3 + 4x = 0 \implies x(x^2 + 4) = 0 \implies x = 0.$$

Therefore, the only critical number of $f(x)$ is $x = 0$.

(c) $f(x) = e^{-x} + 2x$

Answer:

$$f'(x) = e^{-x} + 2x = -e^{-x} + 2$$

$$f'(x) = 0 \text{ when } -e^{-x} + 2 = 0 \implies e^{-x} = 2 \implies e^x = 1/2 \implies x = \ln(1/2) \text{ or } x = -\ln(2).$$

Therefore, the only critical number of $f(x)$ is $x = -\ln(2)$.

(d) $f(x) = -x + \sin(x)$

Answer:

$$f'(x) = -1 + \cos(x)$$

$$f'(x) = 0 \text{ when } -1 + \cos(x) = 0 \implies \cos(x) = 1 \implies x = 2\pi n \text{ for } n = 0, \pm 1, \pm 2, \dots$$

Therefore, the critical numbers of $f(x)$ are $x = 2\pi n$, where $n = 0, \pm 1, \pm 2, \dots$

(e) $f(x) = x^2 - 8 \ln(x)$

Answer:

$$f'(x) = 2x - \frac{8}{x}$$

$$f'(x) = 0 \text{ when } 2x - \frac{8}{x} = 0 \implies x = \frac{4}{x} \implies x^2 = 4 \implies x = \pm 2.$$

Therefore, the critical numbers of $f(x)$ are $x = -2$ and $x = 2$.

2. Determine the intervals on which the given function f is increasing and the interval on which f is decreasing.

(a) $f(x) = x^2 + 6x - 1$ (b) $f(x) = x^4 - 4x^3 + 9$ (c) $f(x) = x^2 e^{-x}$

Answer: The function $f(x)$ is increasing when $f'(x) > 0$ and $f(x)$ is decreasing when $f'(x) < 0$.

(a) $f(x) = x^2 + 6x - 1 \implies f'(x) = 2x + 6$

$$f'(x) = 2x + 6 > 0 \implies x > -3$$

so $f(x)$ is increasing on the interval $[-3, \infty)$, and

$$f'(x) = 2x + 6 < 0 \implies x < -3$$

so $f(x)$ is decreasing on the interval $(-\infty, -3]$.

(b) $f(x) = x^4 - 4x^3 + 9 \implies f'(x) = 4x^3 - 12x^2$

$$f'(x) = 4x^3 - 12x^2 > 0 \implies 4x^2(x - 3) > 0 \implies x > 3$$

so $f(x)$ is increasing on the interval $[3, \infty)$, and

$$f'(x) = 4x^3 - 12x^2 < 0 \implies 4x^2(x - 3) < 0 \implies x < 3$$

so $f(x)$ is decreasing on the interval $(-\infty, 3]$.

(c) $f(x) = x^2 e^{-x} \implies f'(x) = -x^2 e^{-x} + 2x e^{-x} \implies f'(x) = x e^{-x}(2 - x)$,
since $e^{-x} > 0$, $f'(x) = 0 \implies x = 0$ or $x = 2$:

	$x < 0$	$0 < x < 2$	$2 < x$
Sign of $f'(x) = x e^{-x}(2 - x)$	-	+	-
$(x)(e^{-x})(2 - x)$	$(-)(+)(+)$	$(+)(+)(+)$	$(+)(+)(-)$

so $f(x)$ is increasing ($f'(x) = +$) on the interval $[0, 2]$,

and $f(x)$ is decreasing ($f'(x) = -$) on the interval $(-\infty, 0] \cup [2, \infty)$.

3. Use the *First Derivative Test* to find the relative extrema of the given function.

(a) $f(x) = x^3 - 3x$ (b) $f(x) = x^3 + x - 3$ (c) $\frac{x^2+3}{x+1}$ (d) $f(x) = x^2 - 2|x|$

Answer:

(a) $f(x) = x^3 - 3x \implies f'(x) = 3x^2 - 3 = 3(x - 1)(x + 1)$,
 $f'(x) = 0 \implies x = \pm 1$ are critical numbers:

	$x < -1$	$-1 < x < 1$	$1 < x$
Sign of $f'(x) = 3(x - 1)(x + 1)$	+	-	+
$(x - 1)(x + 1)$	$(-)(-)$	$(+)(-)$	$(+)(+)$

Therefore, from the *First Derivative Test*,

- relative maximum occurs at $x = -1$ (“+” changes to “-” at $x = -1$)

and $f(-1) = (-1)^3 - 3(-1) = 2$ is the relative maximum

- relative minimum occurs at $x = 1$ (“-” changes to “+” at $x = 1$) and

$f(1) = 1^3 - 3(1) = -2$ is the relative minimum.

(b) $f(x) = x^3 + x - 3 \implies f'(x) = 3x^2 + 1 \neq 0, \forall x \in (-\infty, \infty).$

$f'(x)$ is well-defined for any x and $f'(x) \neq 0$ imply that there is *no critical number* for $f(x)$. Since a relative extremum can only occur at a critical number and there is no critical number, therefore, we conclude that **there is no relative extrema** for this function.

(c) $f(x) = \frac{x^2+3}{x+1}$ Note that the domain of this function is all real number except for $x = -1$:
 $D = \mathbb{R}/\{-1\}$
 $\implies f'(x) = \frac{(x+1)(2x) - (x^2+3)}{(x+1)^2} = \frac{x^2+2x-3}{(x+1)^2} = \frac{(x+3)(x-1)}{(x+1)^2},$
 $f'(x) = 0 \implies (x+3)(x-1) = 0 \implies x = -3, 1$ are the critical numbers.

	$x < -3$	$-3 < x < 1 (x \neq -1)$	$1 < x$
Sign of $f'(x)$	+	-	+
$(x+3)(x-1)$	$(-)(-)$	$(+)(-)$	$(+)(+)$

Therefore, from the *First Derivative Test*,

- relative maximum occurs at $x = -3$ (“+” changes to “-” at $x = -3$)

and $f(-3) = \frac{(-3)^2+3}{-3+1} = -6$ is the relative maximum

- relative minimum occurs at $x = 1$ (“-” changes to “+” at $x = 1$) and

$f(1) = \frac{1^2+3}{1+1} = 2$ is the relative minimum.

Notice that it is possible to have a “relative” minimum *larger* than a “relative” maximum.

(d) Note that we can write $f(x) = x^2 - 2|x|$ as:

- For $x < 0$, $f(x) = x^2 + 2x$
- For $x \geq 0$, $f(x) = x^2 - 2x$.

(i) The critical numbers:

- For $x < 0$, $f'(x) = 2x + 2$
- For $x > 0$, $f'(x) = 2x - 2$,

and $f'(x) = 0$ when $2x + 2 \Rightarrow x = -1$ or $2x - 2 = 0 \Rightarrow x = 1$. Note that $f'(x)$ does not exist when $x = 0$. Hence, the critical numbers are $x = -1, 1, 0$.

(ii) Relative extrema: Relative extrema can occur only at the critical points: $x = -1, 1, 0$. So we consider the subintervals:

	$x < -1$	$-1 < x < 0$	$0 < x < 1$	$1 < x$
	$f'(x) = 2x + 2$	$f'(x) = 2x + 2$	$f'(x) = 2x - 2$	$f'(x) = 2x - 2$
Sign of $f'(x)$	-	+	-	+

Therefore, from the *First Derivative Test*,

- relative maximum occurs at $x = 0$ (“+” changes to “-” at $x = 0$)

and $f(0) = 0$ is the relative maximum

- relative minima occurs at $x = -1, 1$ (“-” changes to “+” at $x = -1, 1$) and

$f(-1) = 1^2 - 2|-1| = -1$ and $f(1) = 1^2 - 2|1| = -1$ are the relative minima.

That is, -1 is the relative minimum, which occurs at $x = \pm 1$.

4. For each given function,
- (i) use the Second Derivative Test, when applicable, to find the relative extrema;
 - (ii) find intercepts and points of inflection, when possible;
 - (iii) find the intervals on which it is increasing and the interval on which it is decreasing;
 - (iv) find the intervals on which it is concave up and the intervals on which it is concave down;
 - and (v) sketch the graph.

(a) $f(x) = x^3 + 3x^2 + 3x + 1$

(b) $f(x) = 6x^5 - 10x^3$

(c) $f(x) = \frac{x}{x^2+2}$

(d) $f(x) = 2x - x \ln(x)$

Answer:

Note that there are 2 steps

(1) Find critical numbers c_j

(2) Check sign of $f''(c_j)$:

- If $f''(c_j) < 0$, $f(c_j)$ is a relative maximum.

- If $f''(c_j) > 0$, $f(c_j)$ is a relative minimum.

(If $f''(c_j) = 0$, no conclusion – use the First Derivative Test.)

(a) $f(x) = x^3 + 3x^2 + 3x + 1$

$$f'(x) = 3x^2 + 6x + 3, \quad f''(x) = 6x + 6.$$

(i) Critical numbers:

$$f'(x) = 3x^2 + 6x + 3 = 0 \Rightarrow 3(x+1)^2 = 0 \Rightarrow x = -1 \text{ is the only critical number.}$$

From the Second Derivative Test,

$$f''(-1) = 6(-1) + 6 = 0$$

so we cannot conclude anything. The First derivative will be used instead,

	$x \in (-\infty, -1)$	$x \in (-1, \infty)$
Sign of $f'(x) = 3(x+1)^2$	+	+

Therefore, from the table above, the *First Derivative Test* implies that there is **no relative extremum** since the sign of $f'(x)$ is always positive.

(ii) Find intercepts and points of inflection:

- The y-intercept($x=0$) is at $y = f(0) = 0^3 + 3 \cdot 0^2 + 3(0) + 1 = 1 \Rightarrow y = 1$.

- The x-intercept($y=0$) is at $0 = f(x) = (x+1)^3 \Rightarrow x = -1$.

- To find the inflection points, we first solve $f''(x) = 0$ and see the sign change of $f''(x)$:

$$f''(x) = 6x + 6 = 0 \Rightarrow x = -1.$$

	$x \in (-\infty, -1)$	$x \in (-1, \infty)$
Sign of $f''(x) = 6(x+1)$	-	+

Since the sign of $f''(x)$ changes at the $x = -1$, it gives an inflection point. When $x = -1$, $y = f(-1) = (-1)^3 + 3(-1)^2 + 3(-1) + 1 = 0$. Hence, the inflection point is $(-1, 0)$.

(iii) Since $f'(x) = 3(x+1)^2 > 0$ for all $x \neq -1$, and $f'(x) = 0$ at $x = -1$, $f(x)$ is increasing on $(-\infty, \infty)$.

(iv) Concavity:

$$f''(x) = 6x + 6 > 0 \Rightarrow x > -1 \text{ and } f''(x) = 6x + 6 < 0 \Rightarrow x < -1.$$

That is, $f(x)$ is concave up on $(-1, \infty)$ and $f(x)$ is concave down on $(-\infty, -1)$.

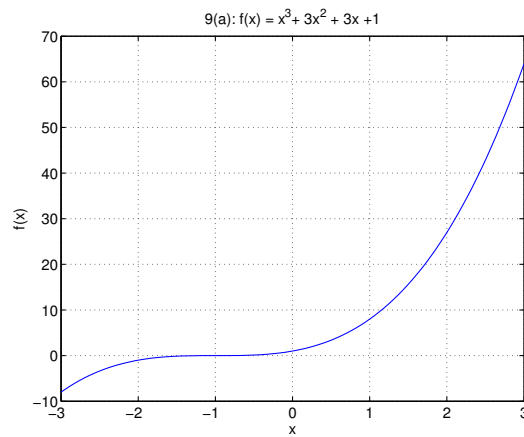


Figure 1: (a): $f(x) = x^3 + 3x^2 + 3x + 1$

(b) $f(x) = 6x^5 - 10x^3$

(i) Critical numbers:

$$f'(x) = 30x^4 - 30x^2 = 30x^2(x^2 - 1) = 30x^2(x - 1)(x + 1) = 0 \Rightarrow$$

$x = 0, -1, 1$ are the only critical numbers. From

$$f''(x) = 120x^3 - 60x = 60x(2x^2 - 1)$$

and from the Second Derivative Test,

$f''(0) = 0 \Rightarrow$, no conclusion

$f''(-1) = -60 < 0 \Rightarrow$, $f(-1) = -6 + 10 = 4$ is a relative maximum

$f''(1) = 60 > 0 \Rightarrow$, $f(1) = 6 - 10 = -4$ is a relative minimum.

The First Derivative Test will be used to identify the critical number $x = 0$ as shown in the table. Since the sign of $f'(x)$ does not change at $x = 0$, $x = 0$ does not give an extremum.

	$x \in (-\infty, -1)$	$x \in (-1, 0)$	$x \in (0, 1)$	$x \in (1, \infty)$
Sign of $f'(x) = 30x^2(x - 1)(x + 1)$	+	-	-	+
$x^2(x - 1)(x + 1)$	(+)(-)(-)	(+)(-)(+)	(+)(-)(+)	(+)(+)(+)

(ii) Find intercepts and points of inflection:

- The y-intercept($x=0$) is at $y = f(0) = 6(0)^5 - 10(0)^3 = 0 \Rightarrow y = 0$.

- The x-intercept($y=0$) occurs when $0 = f(x) = x^3(6x^2 - 10)$

$\Rightarrow$ at $x = 0, x = -\sqrt{5/3}, x = \sqrt{5/3}$.

- To find the inflection points, we first solve $f''(x) = 0$ and see the sign change of $f''(x)$:

$$f''(x) = 60x(2x^2 - 1) = 0 \Rightarrow x = 0, -\sqrt{1/2}, \sqrt{1/2}.$$

	$x \in (-\infty, -\sqrt{1/2})$	$x \in (-\sqrt{1/2}, 0)$	$x \in (0, \sqrt{1/2})$	$x \in (\sqrt{1/2}, \infty)$
Sign of $f''(x) = 60x(2x^2 - 1)$	-	+	-	+
$(x)(x - \sqrt{1/2})(x + \sqrt{1/2})$	(-)(-)(+)	(-)(-)(+)	(+)(-)(+)	(+)(+)(+)

Since the sign of $f''(x)$ changes at the $x = 0, -\sqrt{1/2}, \sqrt{1/2}$, they give inflection points.

Since $f(0) = 0$, $f(-\sqrt{1/2}) = 7\sqrt{2}/4$, $f(\sqrt{1/2}) = -7\sqrt{2}/4$,

the inflection points are $(0, 0), (-\sqrt{1/2}, 7\sqrt{2}/4), (\sqrt{1/2}, -7\sqrt{2}/4)$.

(iii) From the table in (i), we see that

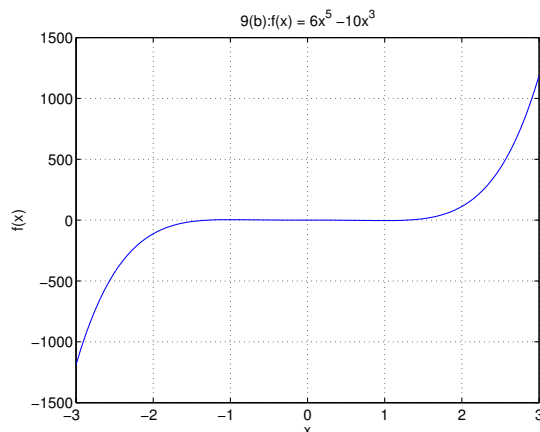
$f(x)$ is increasing ($f'(x) > 0$) on $(-\infty, -1) \cup (1, \infty)$ and

$f(x)$ is decreasing ($f'(x) < 0$) on $(-1, 1)$.

(iv) From the table in (ii), we see that

$f(x)$ is concave up ($f''(x) > 0$) on $(-\sqrt{1/2}, 0) \cup (\sqrt{1/2}, \infty)$ and

$f(x)$ is concave down ($f''(x) < 0$) on $(-\infty, -\sqrt{1/2}) \cup (0, \sqrt{1/2})$.

Figure 2: (b): $f(x) = 6x^5 - 10x^3$

(c) $f(x) = \frac{x}{x^2+2}$

(i) Critical numbers:

$$f'(x) = \frac{(x^2 + 2) - 2x^2}{(x^2 + 2)^2} = \frac{2 - x^2}{(x^2 + 2)^2} = 0 \Rightarrow 2 - x^2 = 0$$

$x = -\sqrt{2}, \sqrt{2}$ are the only critical numbers. From

$$f''(x) = \frac{2x(x^2 + 2)(x^2 - 6)}{(x^2 + 2)^4} = \frac{2x(x^2 - 6)}{(x^2 + 2)^3}$$

and from the Second Derivative Test,

$$f''(-\sqrt{2}) = \frac{\sqrt{2}}{8} > 0 \Rightarrow f(-\sqrt{2}) = \frac{-\sqrt{2}}{(-\sqrt{2})^2 + 2} = -\sqrt{2}/4 \text{ is a relative minimum}$$

$$f''(\sqrt{2}) = -\frac{\sqrt{2}}{8} < 0 \Rightarrow f(\sqrt{2}) = \frac{\sqrt{2}}{(\sqrt{2})^2 + 2} = \sqrt{2}/4 \text{ is a relative maximum.}$$

(ii) Find intercepts and points of inflection:

- The y-intercept ($x=0$) is at $y = f(0) = 0 \Rightarrow y = 0$.
- The x-intercept ($y=0$) occurs when $0 = f(x) \Rightarrow$ at $x = 0$.
- To find the inflection points, we first solve $f''(x) = 0$ and see the sign change of $f''(x)$:

$$f''(x) = \frac{2x(x^2 - 6)}{(x^2 + 2)^3} = 0 \Rightarrow x = 0, -\sqrt{6}, \sqrt{6}.$$

Since the denominator $(x^2 + 2)^3 > 0$, the sign of $f''(x)$ depends on the numerator $2x(x^2 - 6) = 2x(x - \sqrt{6})(x + \sqrt{6})$ as shown in the table below.

	$x \in (-\infty, -\sqrt{6})$	$x \in (-\sqrt{6}, 0)$	$x \in (0, \sqrt{6})$	$x \in (\sqrt{6}, \infty)$
Sign of $f''(x) = \frac{2x(x^2-6)}{(x^2+2)^3}$	-	+	-	+
$(2x)(x - \sqrt{6})(x + \sqrt{6})$	$(-)(-)(+)$	$(-)(-)(+)$	$(+)(-)(+)$	$(+)(+)(+)$

Since the sign of $f''(x)$ changes at the $x = 0, -\sqrt{6}, \sqrt{6}$, they give inflection points. Since $f(0) = 0$, $f(-\sqrt{6}) = \frac{-\sqrt{6}}{(-\sqrt{6})^2+2} = -\sqrt{6}/8$, $f(\sqrt{6}) = \frac{\sqrt{6}}{(\sqrt{6})^2+2} = \sqrt{6}/8$, the inflection points are $(0, 0)$, $(-\sqrt{6}, -\sqrt{6}/8)$, $(\sqrt{6}, \sqrt{6}/8)$.

(iii) From

$$f'(x) = \frac{2-x^2}{(x^2+2)^2} = \frac{(\sqrt{2}-x)(\sqrt{2}+x)}{(x^2+2)^2},$$

the denominator is always positive, so the sign of $f'(x)$ depends on the numerator: $(\sqrt{2}-x)(\sqrt{2}+x)$.

	$x \in (-\infty, -\sqrt{2})$	$x \in (-\sqrt{2}, \sqrt{2})$	$x \in (\sqrt{2}, \infty)$
Sign of $f'(x) = \frac{(\sqrt{2}-x)(\sqrt{2}+x)}{(x^2+2)^2}$	—	+	—
$(\sqrt{2}-x)(\sqrt{2}+x)$	$(+)(-)$	$(+)(+)$	$(-)(+)$

Hence, $f(x)$ is increasing ($f'(x) > 0$) on $(-\sqrt{2}, \sqrt{2})$ and $f(x)$ is decreasing ($f'(x) < 0$) on $(-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)$.

(iv) From the table in (ii), we see that

$f(x)$ is concave up ($f''(x) > 0$) on $(-\sqrt{6}, 0) \cup (\sqrt{6}, \infty)$ and $f(x)$ is concave down ($f''(x) < 0$) on $(-\infty, -\sqrt{6}) \cup (0, \sqrt{6})$.

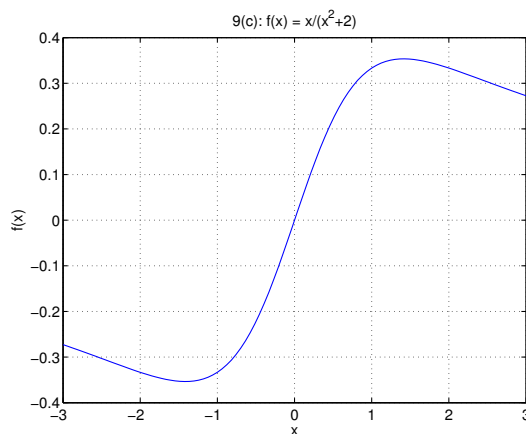


Figure 3: (c): $f(x) = \frac{x}{x^2+2}$

(d) $f(x) = 2x - x \ln(x)$

First note that the domain of f is

$$\mathcal{D} = (0, \infty),$$

since $\ln(x)$ is only well-defined for $x > 0$.

(i) Critical numbers: $f'(x) = 2 - \left[x \frac{1}{x} + \ln(x)\right] = 1 - \ln(x)$.

$$f'(x) = 1 - \ln(x) = 0 \Rightarrow \ln(x) = 1 \Rightarrow x = e^1.$$

$x = e$ is the only critical number. From

$$f''(x) = -\frac{1}{x}$$

and from the Second Derivative Test,

$$f''(e) = -\frac{1}{e} < 0 \Rightarrow f(e) = 2e - e \ln(e) = 2e - e = e \text{ is a relative maximum.}$$

(ii) Find intercepts and points of inflection:

- There is no y-intercept, since $x = 0$ is not in the domain of the function $f(x) = 2x - x \ln(x)$ ($\ln(x)$ is not well-defined at $x = 0$).

- The x-intercept ($y=0$) occurs when $0 = f(x) \Rightarrow 2x - x \ln(x) = x(2 - \ln(x)) = 0 \Rightarrow x = 0$ or $\ln(x) = 2 \Rightarrow$ at $x = 0, x = e^2$.

- To find the inflection points, we first solve $f''(x) = 0$ and see the sign change of $f''(x)$: notice that

$$f''(x) = -\frac{1}{x} \neq 0 \quad \Rightarrow \quad \text{There is no inflection point.}$$

(iii) From $f'(x) = 1 - \ln(x)$,

- $f'(x) > 0$ when $1 - \ln(x) > 0 \Leftrightarrow \ln(x) < 1 \Leftrightarrow x < e \Rightarrow f(x)$ is **increasing** on $(-\infty, e]$.
- $f'(x) < 0$ when $1 - \ln(x) < 0 \Leftrightarrow \ln(x) > 1 \Leftrightarrow x > e \Rightarrow f(x)$ is **decreasing** on $[e, \infty)$.

(iv) From $f''(x) = -\frac{1}{x}$,

$f(x)$ is concave down ($f''(x) < 0 \Leftrightarrow x > 0$) on $(0, \infty)$. Note that $f(x)$ is not concave up, since $f''(x) > 0 \Leftrightarrow x < 0$, but $x < 0$ is not in the domain of $f(x)$: $\mathcal{D} = (0, \infty)$.

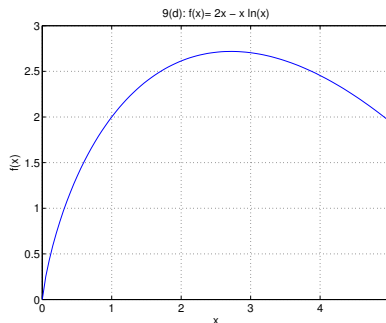


Figure 4: (d): $f(x) = 2x - x \ln(x)$

5. Let $X, Y_1, Y_2 \subseteq \mathbb{R}$. Consider the relations f_1, f_2 defined by

$$f_1 = \left\{ (x, y) \in X \times Y_1 \mid y = \sqrt{\frac{x+1}{2x-3}} \right\}, \quad f_2 = \left\{ (x, y) \in X \times Y_2 \mid (x, -y) \in f_1 \right\}.$$

Sketch the curve of the relation $f = f_1 \cup f_2$.

Answer:

(Details to be added...symmetry/critical numbers/increasing& decreasing/ concavity...)

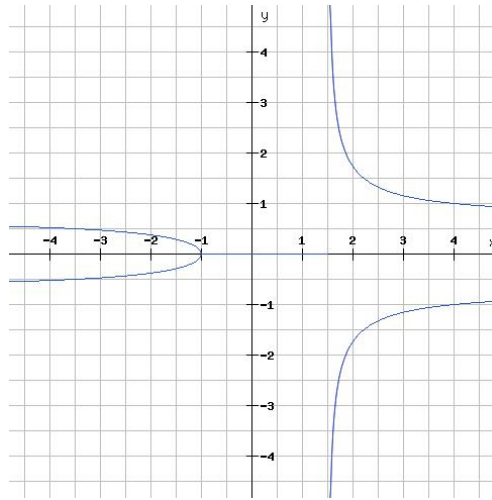


Figure 5: $f = f_1 \cup f_2 = \{(x, y) \mid y^2 = \frac{x+1}{2x-3}\}$

6. Find the vertical and horizontal asymptotes (if any) of $f(x) = \frac{x^2-1}{2x}$ and show that $y = \frac{x}{2}$ is a slant asymptote of f .

Answer:(Details to be added...)

- Vertical asymptote: $x = 0$
- Horizontal asymptote: None
- Slant asymptote $y = \frac{x}{2}$. We have to show that $\lim_{x \rightarrow \infty} [f(x) - \frac{x}{2}] = 0$.

$$\lim_{x \rightarrow \infty} [f(x) - \frac{x}{2}] = \lim_{x \rightarrow \infty} \left(\frac{x^2-1}{2x} - \frac{x}{2} \right) = \lim_{x \rightarrow \infty} \left(\frac{x^2-1-x^2}{2x} \right) = \lim_{x \rightarrow \infty} \frac{-1}{2x} = 0$$