Solving Inequalities

We are interested in finding the solution set of $x \in \mathbb{R}$ for each of the following inequalities

$$f(x) < 0, \qquad f(x) > 0, \qquad f(x) \leq 0, \qquad f(x) \geq 0,$$

where $f(x)$ is a function in one the following forms.

1. $f(x)$ is a polynomial

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n,$$

   where $a_0, a_1, \ldots, a_n$ are some constants and n is a positive integer.

2. $f(x)$ is a rational function, i.e. $f(x)$ is in the form:

$$f(x) = \frac{P(x)}{Q(x)},$$

   where $P(x)$ and $Q(x)$ are polynomials, $Q(x) \not\equiv 0$.

1 Properties of Inequalities

The following are important properties of inequities that we will use without proof.

Theorem 1.1 (Properties of Inequalities)**.** Let a, b, and c be some real numbers.

(1) If $a < b$, then $a + c < b + c$ and $a - c < b - c$.

(2) If $a < b$ and $c > 0$ then $ac < bc$.
But if $a < b$ and $c < 0$ then $ac > bc$.
Note that mulitplication by a negative number flips the inequality.

(3) If $a < b$ and $b < c$, then $a < c$.

(4) If $a > 0$, then $\frac{1}{a} > 0$.

(5) If $0 < a < b$, then $\frac{1}{b} < \frac{1}{a}$

(6) If $0 < a < b$, then $\frac{1}{b^n} < \frac{1}{a^n}$ for any positive integer n.

(7) If $0 \leq a < b$, then $a^n < b^n$ for any positive integer n.

(8) If $0 < a < b$ and $0 < c < d$, then $0 < ac < bd$.

Example 1.1. Find the set of all real numbers x that satisfies the following inequalities.

1. $x + 5 \leq x - 2$

2. $5 - 2x > -5 - 2x$

3. $5 - 2x < 5 - 2x$

4. $5 + 2x < x - 1$

2 Inequality with quadratic polynomial

When we want to solve inequalities that involve quadratic polynomial (polynomial of degree two), the following theorem can be used to determine whether this quadratic polynomial can be factored in a simpler form. If so, it gives the factorization formula of the polynomial into linear terms.

Theorem 2.1. Consider the inequality involves quadratic polynomial

$$f(x) = ax^2 + bx + c,$$

for some constants $a \neq 0$, b, c.

- If $b^2 - 4ac \geq 0$ it is useful to use the following factorization:

$$ax^2 + bx + c = a(x - K_1)(x - K_2),$$

$$\boxed{K_1 = \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right) \; K_2 = \left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right)}$$

- If $b^2 - 4ac < 0$, we cannot factor the polynomial in terms of real numbers. Moreover,
  - if $a > 0$, then we always have $ax^2 + bx + c > 0$, $\forall x \in \mathbb{R}$.
  - if $a < 0$, then we always have $ax^2 + bx + c < 0$, $\forall x \in \mathbb{R}$.

Strategy for Solving a Quadratic Inequality with a Sign Diagram

1. Write the inequality with 0 on the right so that it is one of the following form:
   $f(x) < 0$, $\quad f(x) > 0$, $\quad f(x) \leq 0$, $\quad f(x) \geq 0$.
2. Solve for the roots of $f(x) = 0$. Use these **roots** to divide the number line into intervals.
3. Determine the sign on each interval.

Example 2.1. Find all possible values of x that satisfy the inequality.

$$x^2 + 8 > 6x$$

Example 2.2. Solve the following inequalities.

1. $3x^2 - x + 5 < 0$

2. $x^3 + x^2 \geq 2x$

3. $4x^4 - x^5 < 3x^3$

Example 2.3. Solve the following inequality.

$$x^3 - 7x > 36$$

Solution:
First use the long division to factor $x^3 - 7x - 36 = (x - 4)(x^2 + 4x + 9)$.

3 Inequality with rational function

Consider inequality with rational function $f(x)$:

$$f(x) < 0, \qquad f(x) > 0, \qquad f(x) \leq 0, \qquad f(x) \geq 0$$

where

$$f(x) = \frac{P(x)}{Q(x)},$$

$P(x)$ and $Q(x)$ are polynomials.

The solution set can be found by using a similar technique as in the case of polynomial inequalities. However, because rational expressions have denominators (and therefore may have places where they're not defined), you have to be a little more careful in finding your solutions.

1. Factor $P(x)$ and $Q(x)$ and find the roots for each of them.
   - The roots of $P(x)$ are the **zeros** of the rational function $f(x)$.
   - The roots of $Q(x)$ are the **undefined points** of $f(x)$.
2. Use these **zeroes** and **undefined points** to divide the number line into intervals.
3. Determine the sign of the rational on each interval.

Example 3.1. Find the solution set for the following inequality.

$$\frac{1}{x+4} \; < \; \frac{1}{x}$$

Example 3.2. Find the solution set for the following inequality.

$$\frac{x^3 - x^2}{x+1} + 2x \; > \; 0$$

Example 3.3. Find the solution set for the following inequality.

$$\frac{x^2}{-x^2+x-4} \;\le\; -1$$

Example 3.4. Find the solution set for the following inequality.

$$\frac{20x^2-17x-63}{e^x(x^2+1)} \;<\; 0$$

Example 3.5. Find a positive value of k so that $(-\infty, -3/2] \cup [4, \infty)$ is the solution set of the following inequality:

$$-2x^2 + \frac{5}{2}xk + 3k^2 \leq 0.$$

Example 3.6. Find **all possible values** of k so that $x \in (-\infty, -3/2] \cup [4, \infty)$ satisfies the following inequality:

$$-2x^2 + \frac{5}{2}xk + 3k^2 \leq 0.$$

Solution: In this problem, it is possible to find many different values of k. Note that, in order to have $x \in (-\infty, -3/2] \cup [4, \infty)$ satisfying the given inequality, we may have a larger solution set that contains $(-\infty, -3/2] \cup [4, \infty)$.
The formula for factoring the quadratic polynomial can be used to get

$$-2x^2 + \frac{5}{2}xk + 3k^2 = -2(x + 3/4k)(x - 2k).$$

So, solving $-2x^2 + \frac{5}{2}xk + 3k^2 \leq 0$ is equivalent to solving $-2(x + 3/4k)(x - 2k) \leq 0$ or

$$2(x + 3/4k)(x - 2k) \geq 0$$

which can be done by considering
$2(x + \frac{3}{4}k)(x - 2k) = 0$ or $x = -\frac{3}{4}k$ or $x = 2k$ for subdividing the intervals.

- Case I: for $k > 0$, we have $-\frac{3}{4}k < 2k$ the solution $x \in (-\infty, -\frac{3}{4}k] \cup [2k, \infty)$, which will contain $(-\infty, -\frac{3}{2}] \cup [4, \infty)$ when

$$-\frac{3}{4}k \geq -\frac{3}{2} \qquad \text{and} \qquad 2k \leq 4 \qquad \Rightarrow \qquad k \leq 2.$$

That is, $k \in (0, 2]$ in this case.

- Case II: for $k = 0$, the inequality becomes $-2x^2 \leq 0$ or $x^2 \geq 0$, which is always true for any real number x. I.e., the solution set is $\mathbb{R}$, which contains $(-\infty, -\frac{3}{2}] \cup [4, \infty)$.
That is, $k = 0$ is a possible value for this problem.

- Case III: for $k < 0$, we have $-\frac{3}{4}k > 2k$ the solution $x \in (-\infty, 2k] \cup [-\frac{3}{4}k, \infty)$, which will contain $(-\infty, -\frac{3}{2}] \cup [4, \infty)$ when

$$2k \geq -\frac{3}{2} \qquad \text{and} \qquad -\frac{3}{4}k \leq 4 \qquad \Rightarrow \qquad k \geq -\frac{3}{4}$$

That is, since $k < 0$, then $k \in [-\frac{3}{4}, 0)$ in this case.
From the 3 cases I, II, and III, the possible values of k are in the interval $[-\frac{3}{4}, 0) \cup \{0\} \cup (0, 2] = [-\frac{3}{4}, 2]$.
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Example 3.7. Find the solution set for the following inequality.

$$x^5 + 3x^4 - 23x^3 - 51x^2 + 94x + 120 \geq 0$$

Note that $x^5 + 3x^4 - 23x^3 - 51x^2 + 94x + 120 = (x+5)(x+3)(x+1)(x-2)(x-4)$.

Example 3.8. Find the solution set for the following inequality.

$$4x^3 - 12x^2 \ \geq \ 0$$

4 Applications

Example 4.1. Kate made a 76 on the midterm exam in math. To get a B, the average of her midterm and her final must be between 80 and 90. For what range of scores on the final exam will she get a B?

Solution: Between 84 and 104.
Let x be the final exam score.

$$80 < \frac{x+76}{2} < 90$$

Example 4.2. A store's daily profit for selling x magazine subscriptions is determined by the formula

$$P(x) = -x^2 + 80x - 1500.$$

For what values of x is the store's profit positive?

Solution: Solve $P(x) > 0$. $x \in (30, 50)$.