GMM using Moment Equations

In the study of interest rate structure, the continuous time model can be specified as:

$$r_{t+\Delta t} - r_t = (\alpha + \beta r_t)\Delta t + \varepsilon_{t+\Delta t} \quad (1)$$

where: $E[\varepsilon_{t+\Delta t}] = 0$ and $E[\varepsilon^2_{t+\Delta t}] = \sigma^2 r_t^{2\gamma} \Delta t$

Then, the model can be transformed to be discrete time model by setting $\Delta t = 1$. The discrete time model can be stated as:

$$r_{t+1} - r_t = \Delta r_t = \alpha + \beta r_t + \varepsilon_{t+1} \quad (2)$$

where: $E[\varepsilon_{t+1}] = 0$ and $E[\varepsilon^2_{t+1}] = \sigma^2 r_t^{2\gamma}$

The above model consists of four parameters including $\alpha, \beta, \sigma^2, \gamma$. The model can be estimate using GMM. Four moment condition equations can be stated as:

Zero mean condition:	$E(\varepsilon_{t+1}) = 0$
Orthogonality condition:	$E(\varepsilon_{t+1} r_t) = 0$
Variance condition:	$E(\varepsilon^2_{t+1} - \sigma^2 r_t^{2\gamma}) = 0$
Zero covariance condition:	$E\left(\left(\varepsilon^2_{t+1} - \sigma^2 r_t^{2\gamma}\right) r_t\right) = 0$

The above model can be claimed as unrestricted model for other eight interest rate structure models which can be indicated as follows:

Model	α	β	σ^2	γ
Unrestricted				
Merton		0		0
Vasicek				0
CIR SR				0.5
Dothan	0	0		1
GBM	0			1
Brennan & Schwartz				1
CIR VR	0	0		1.5
CEV	0			

Unrestricted model has four unknown parameters to be estimated and four moment condition equations, thus, method of estimate will be method of moment since it is exact identify equations model. Other eight restricted models will be estimated using GMM.

GMM can be performed by using the following command:

<u>Unrestricted model</u>:

```
. gmm (dr-{alpha}-{beta}*r) ((dr-{alpha}-{beta}*r)*r) ((dr-{alpha}-{beta}*r)^2-
{sigma2}*r^(2*{gamma})) (((dr-{alpha}-{beta}*r)^2-{sigma2}*r^(2*{gamma}))*r)
winitial(identity)

warning: 1 missing value returned for equation 1 at initial values
warning: 1 missing value returned for equation 2 at initial values
warning: 1 missing value returned for equation 3 at initial values
warning: 1 missing value returned for equation 4 at initial values

Step 1
numerical derivatives are approximate
flat or discontinuous region encountered
Iteration 0:   GMM criterion Q(b) =  .00005673
Iteration 1:   GMM criterion Q(b) =  .00005192  (backed up)
Iteration 2:   GMM criterion Q(b) =  .00004097
Iteration 3:   GMM criterion Q(b) =  3.562e-06
Iteration 4:   GMM criterion Q(b) =  2.174e-07
Iteration 5:   GMM criterion Q(b) =  1.644e-07

Step 2
Iteration 0:   GMM criterion Q(b) =  .01229538
Iteration 1:   GMM criterion Q(b) =   .0055188
Iteration 2:   GMM criterion Q(b) =  .00045234
Iteration 3:   GMM criterion Q(b) =  8.141e-07
Iteration 4:   GMM criterion Q(b) =  2.461e-12
Iteration 5:   GMM criterion Q(b) =  1.451e-23

GMM estimation

Number of parameters =   4
Number of moments    =   4
Initial weight matrix: Identity                     Number of obs  =    1429
GMM weight matrix:     Robust

------------------------------------------------------------------------------
             |               Robust
             |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
      /alpha |  -.0009679   .0012837    -0.75   0.451    -.0034839    .0015481
       /beta |   .0014035   .0005181     2.71   0.007     .0003882    .0024189
     /sigma2 |   .0006419   .0001156     5.55   0.000     .0004153    .0008686
      /gamma |   .1273995   .0720773     1.77   0.077    -.0138695    .2686685
------------------------------------------------------------------------------
Instruments for equation 1: _cons
Instruments for equation 2: _cons
Instruments for equation 3: _cons
Instruments for equation 4: _cons

. estat overid

  Test of overidentifying restriction:

  Hansen's J chi2(0) = 2.1e-20 (p =     .)

. est store Unrestricted
```

<u>Merton model</u>:

```
. gmm (dr-{alpha}) ((dr-{alpha})*r) ((dr-{alpha})^2-{sigma2}) (((dr-{alpha})^2-
{sigma2})*r) winitial(identity)

warning: 1 missing value returned for equation 1 at initial values
...
warning: 1 missing value returned for equation 4 at initial values
```

```
Step 1
Iteration 0:   GMM criterion Q(b) =  .00005673
Iteration 1:   GMM criterion Q(b) =  6.386e-07
Iteration 2:   GMM criterion Q(b) =  6.382e-07

Step 2
Iteration 0:   GMM criterion Q(b) =  .00760718
Iteration 1:   GMM criterion Q(b) =  .00652536
Iteration 2:   GMM criterion Q(b) =  .00652532

GMM estimation

Number of parameters =   2
Number of moments    =   4
Initial weight matrix: Identity                       Number of obs  =    1429
GMM weight matrix:     Robust

------------------------------------------------------------------------------
             |               Robust
             |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
      /alpha |   .0021651   .0007043     3.07   0.002     .0007847    .0035456
     /sigma2 |   .0008218   .0000853     9.63   0.000     .0006546     .000989
------------------------------------------------------------------------------
Instruments for equation 1: _cons
Instruments for equation 2: _cons
Instruments for equation 3: _cons
Instruments for equation 4: _cons

. estat overid

  Test of overidentifying restriction:

  Hansen's J chi2(2) = 9.32468 (p = 0.0094)

. est store Merton
```

Vasicek model:

```
. gmm (dr-{alpha}-{beta}*r) ((dr-{alpha}-{beta}*r)*r) ((dr-{alpha}-{beta}*r)^2-
{sigma2}) (((dr-{alpha}-{beta}*r)^2-{sigma2})*r) winitial(identity)

warning: 1 missing value returned for equation 1 at initial values
...
warning: 1 missing value returned for equation 4 at initial values

Step 1
Iteration 0:   GMM criterion Q(b) =  .00005673
Iteration 1:   GMM criterion Q(b) =  3.000e-09
Iteration 2:   GMM criterion Q(b) =  2.107e-09

Step 2
Iteration 0:   GMM criterion Q(b) =  .00263124
Iteration 1:   GMM criterion Q(b) =  .00226079
Iteration 2:   GMM criterion Q(b) =  .00226074

GMM estimation

Number of parameters =   3
Number of moments    =   4
Initial weight matrix: Identity                       Number of obs  =    1429
GMM weight matrix:     Robust

------------------------------------------------------------------------------
             |               Robust
             |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
      /alpha |  -.0003515   .0012376    -0.28   0.776     -.002777    .0020741
       /beta |   .0012479   .0005106     2.44   0.015     .0002471    .0022487
     /sigma2 |   .0007951   .0000862     9.23   0.000     .0006262    .0009641
------------------------------------------------------------------------------
Instruments for equation 1: _cons
Instruments for equation 2: _cons
Instruments for equation 3: _cons
Instruments for equation 4: _cons
```

```
. estat overid

  Test of overidentifying restriction:

  Hansen's J chi2(1) = 3.23059 (p = 0.0723)

. est store Vasicek
```

CIR SR model:

```
. gmm (dr-{alpha}-{beta}*r) ((dr-{alpha}-{beta}*r)*r) ((dr-{alpha}-{beta}*r)^2-
{sigma2}*r) (((dr-{alpha}-{beta}*r)^2-{sigma2}*r)*r) winitial(identity) nolog

warning: 1 missing value returned for equation 1 at initial values
...
warning: 1 missing value returned for equation 4 at initial values

Final GMM criterion Q(b) =  .0117512

GMM estimation

Number of parameters =   3
Number of moments    =   4
Initial weight matrix: Identity                      Number of obs  =    1429
GMM weight matrix:     Robust

------------------------------------------------------------------------------
             |               Robust
             |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
      /alpha |  -.0031028   .0011644    -2.66   0.008     -.005385   -.0008207
       /beta |   .0018442   .0005066     3.64   0.000     .0008514    .0028371
     /sigma2 |   .0002351   .0000278     8.45   0.000     .0001805    .0002896
------------------------------------------------------------------------------
Instruments for equation 1: _cons
Instruments for equation 2: _cons
Instruments for equation 3: _cons
Instruments for equation 4: _cons

. estat overid

  Test of overidentifying restriction:

  Hansen's J chi2(1) = 16.7924 (p = 0.0000)

. est store CIR_SR

. est table Unrestricted Merton Vasicek CIR_SR, star(0.1 0.05 0.01)

------------------------------------------------------------------------------
    Variable | Unrestricted       Merton          Vasicek         CIR_SR
-------------+----------------------------------------------------------------
alpha        |
       _cons | -.00096793      .00216513***   -.00035145      -.00310284***
-------------+----------------------------------------------------------------
beta         |
       _cons |  .00140354***                   .00124786**     .00184424***
-------------+----------------------------------------------------------------
sigma2       |
       _cons |  .00064191***    .00082182***    .00079513***    .00023507***
-------------+----------------------------------------------------------------
gamma        |
       _cons |  .12739949*
------------------------------------------------------------------------------
                                           legend: * p<.1; ** p<.05; *** p<.01

. *Test Unrestricted vs Merton

. test (_b[/beta]=0) (_b[/gamma]=0)

 ( 1)  [beta]_cons = 0
 ( 2)  [gamma]_cons = 0

           chi2(  2) =    8.96
         Prob > chi2 =    0.0113
```

GMM Examples Simulated Data

Example 1

```
set obs 500
capture drop y x z u
mat mCorr = (1, .7, .5\ .7, 1, 0 \ .5, 0, 1)
corr2data x z u, mean(10 10 0) sd(3 4 20) corr(mCorr) n(500) clear
corr
g y=1+2*x+u
reg y x
ivregress gmm y (x=z)
```

```
. set obs 500
number of observations (_N) was 500, now 500

. capture drop y x z u

. mat mCorr = (1, .7, .5\ .7, 1, 0 \ .5, 0, 1)

. corr2data x z u, mean(10 10 0) sd(3 4 20) corr(mCorr) n(500) clear
(obs 500)

. corr
(obs=500)

             |        x        z        u
-------------+---------------------------
           x |   1.0000
           z |   0.7000   1.0000
           u |   0.5000   0.0000   1.0000


. g y=1+2*x+u

. reg y x

      Source |       SS           df       MS      Number of obs   =       500
-------------+----------------------------------   F(1, 498)       =    424.96
       Model |  127744.002         1  127744.002   Prob > F        =    0.0000
    Residual |  149700.001       498  300.602411   R-squared       =    0.4604
-------------+----------------------------------   Adj R-squared   =    0.4593
       Total |  277444.003       499  556.000006   Root MSE        =    17.338

------------------------------------------------------------------------------
           y |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
           x |   5.333333   .2587168    20.61   0.000     4.825022    5.841644
       _cons |  -32.33333    2.70086   -11.97   0.000    -37.63982   -27.02685
------------------------------------------------------------------------------

. ivregress gmm y (x=z)

Instrumental variables (GMM) regression          Number of obs   =       500
                                                 Wald chi2(1)    =     19.64
                                                 Prob > chi2     =    0.0000
                                                 R-squared       =    0.2806
GMM weight matrix: Robust                        Root MSE        =     19.98

------------------------------------------------------------------------------
             |               Robust
           y |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
           x |          2   .4513481     4.43   0.000     1.115374    2.884626
       _cons |   .9999991    4.62114     0.22   0.829    -8.057269    10.05727
------------------------------------------------------------------------------
Instrumented:  x
Instruments:   z
```

Example 2

```
set obs 500
capture drop y x z* u
mat mCorr = (1, .7, .7, .5\ .7, 1, .4, 0 \ .7, .4, 1, 0\ .5, 0, 0, 1)
corr2data x z1 z2 u, mean(10 20 15 0) sd(3 4 5 20) corr(mCorr) n(500) clear
corr
g y=1+2*x+u
reg y x
ivregress gmm y (x=z1)
ivregress gmm y (x=z1 z2)
ivregress 2sls y (x=z1 z2)
```

```
. set obs 500
number of observations (_N) was 0, now 500

. capture drop y x z* u

. mat mCorr = (1, .7, .7, .5\ .7, 1, .4, 0 \ .7, .4, 1, 0\ .5, 0, 0, 1)

. corr2data x z1 z2 u, mean(10 20 15 0) sd(3 4 5 20) corr(mCorr) n(500) clear
(obs 500)

. corr
(obs=500)

             |        x       z1       z2        u
-------------+------------------------------------
           x |   1.0000
          z1 |   0.7000   1.0000
          z2 |   0.7000   0.4000   1.0000
           u |   0.5000  -0.0000  -0.0000   1.0000


. g y=1+2*x+u

. reg y x

      Source |       SS           df       MS      Number of obs   =       500
-------------+----------------------------------   F(1, 498)       =    424.96
       Model |  127743.999         1  127743.999   Prob > F        =    0.0000
    Residual |      149700       498  300.602411   R-squared       =    0.4604
-------------+----------------------------------   Adj R-squared   =    0.4593
       Total |  277443.999       499  555.999998   Root MSE        =    17.338

------------------------------------------------------------------------------
           y |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
           x |   5.333333   .2587168    20.61   0.000     4.825022    5.841644
       _cons |  -32.33333    2.70086   -11.97   0.000    -37.63982   -27.02685
------------------------------------------------------------------------------

. ivregress gmm y (x=z1)

Instrumental variables (GMM) regression          Number of obs   =        500
                                                 Wald chi2(1)    =      24.63
                                                 Prob > chi2     =     0.0000
                                                 R-squared       =     0.2806
GMM weight matrix: Robust                        Root MSE        =      19.98

------------------------------------------------------------------------------
             |               Robust
           y |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
           x |          2   .4030215     4.96   0.000     1.210092    2.789908
       _cons |   .9999999   4.004532     0.25   0.803    -6.848738    8.848738
------------------------------------------------------------------------------
Instrumented:  x
Instruments:   z1
```

```
. ivregress gmm y (x=z1 z2)

Instrumental variables (GMM) regression           Number of obs   =        500
                                                  Wald chi2(1)    =      35.10
                                                  Prob > chi2     =     0.0000
                                                  R-squared       =     0.2806
GMM weight matrix: Robust                         Root MSE        =      19.98

------------------------------------------------------------------------------
             |               Robust
           y |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
           x |          2   .3375669     5.92   0.000     1.338381    2.661619
       _cons |          1   3.373732     0.30   0.767    -5.612393    7.612393
------------------------------------------------------------------------------
Instrumented:  x
Instruments:   z1 z2

. estat overid

  Test of overidentifying restriction:

  Hansen's J chi2(1) = 1.1e-14 (p = 1.0000)

. ivregress 2sls y (x=z1 z2)

Instrumental variables (2SLS) regression          Number of obs   =        500
                                                  Wald chi2(1)    =      31.50
                                                  Prob > chi2     =     0.0000
                                                  R-squared       =     0.2806
                                                  Root MSE        =      19.98

------------------------------------------------------------------------------
           y |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
           x |          2   .3563483     5.61   0.000      1.30157     2.69843
       _cons |          1   3.673801     0.27   0.785    -6.200517    8.200518
------------------------------------------------------------------------------
Instrumented:  x
Instruments:   z1 z2
```

Example 3

```
set obs 500
capture drop y x z* u
mat mCorr = (1, .7, .3, .5\ .7, 1, .4, .2 \ .3, .4, 1, .3\ .5, .2, .3, 1)
corr2data x z1 z2 u, mean(10 20 15 0) sd(3 4 5 20) corr(mCorr) n(5000) clear
corr
g y=1+2*x+u
reg y x
ivregress gmm y (x=z1)
ivregress gmm y (x=z1 z2)
estat overid
ivregress 2sls y (x=z1 z2)
```

```
. set obs 500
number of observations (_N) was 0, now 500

. capture drop y x z* u

. mat mCorr = (1, .7, .3, .5\ .7, 1, .4, .2 \ .3, .4, 1, .3\ .5, .2, .3, 1)

. corr2data x z1 z2 u, mean(10 20 15 0) sd(3 4 5 20) corr(mCorr) n(5000) clear
(obs 5,000)

. corr
(obs=5,000)

             |        x       z1       z2        u
-------------+------------------------------------
           x |   1.0000
          z1 |   0.7000   1.0000
          z2 |   0.3000   0.4000   1.0000
           u |   0.5000   0.2000   0.3000   1.0000
```

```
. g y=1+2*x+u

. reg y x

      Source |       SS           df       MS      Number of obs   =     5,000
-------------+----------------------------------   F(1, 4998)      =   4264.96
       Model |  1279743.99         1  1279743.99   Prob > F        =    0.0000
    Residual |     1499700     4,998  300.060025   R-squared       =    0.4604
-------------+----------------------------------   Adj R-squared   =    0.4603
       Total |     2779444     4,999         556   Root MSE        =    17.322

------------------------------------------------------------------------------
           y |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
           x |   5.333333    .081666    65.31   0.000     5.173232    5.493435
       _cons |  -32.33333    .852611   -37.92   0.000    -34.00483   -30.66184
------------------------------------------------------------------------------

. ivregress gmm y (x=z1)

Instrumental variables (GMM) regression          Number of obs   =      5,000
                                                 Wald chi2(1)    =    1044.34
                                                 Prob > chi2     =     0.0000
                                                 R-squared       =     0.4274
GMM weight matrix: Robust                        Root MSE        =     17.841

------------------------------------------------------------------------------
             |               Robust
           y |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
           x |   3.904762   .1208297    32.32   0.000      3.66794    4.141584
       _cons |  -18.04762   1.229494   -14.68   0.000    -20.45738   -15.63785
------------------------------------------------------------------------------
Instrumented:  x
Instruments:   z1

. ivregress gmm y (x=z1 z2)

Instrumental variables (GMM) regression          Number of obs   =      5,000
                                                 Wald chi2(1)    =    1103.18
                                                 Prob > chi2     =     0.0000
                                                 R-squared       =     0.4315
GMM weight matrix: Robust                        Root MSE        =     17.776

------------------------------------------------------------------------------
             |               Robust
           y |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
           x |   3.997248   .1203476    33.21   0.000     3.761371    4.233125
       _cons |  -19.00297   1.224456   -15.52   0.000    -21.40286   -16.60308
------------------------------------------------------------------------------
Instrumented:  x
Instruments:   z1 z2

. estat overid

  Test of overidentifying restriction:

  Hansen's J chi2(1) = 293.132 (p = 0.0000)

. ivregress 2sls y (x=z1 z2)

Instrumental variables (2SLS) regression         Number of obs   =      5,000
                                                 Wald chi2(1)    =    1100.94
                                                 Prob > chi2     =     0.0000
                                                 R-squared       =     0.4305
                                                 Root MSE        =     17.792

------------------------------------------------------------------------------
           y |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
           x |    3.97411    .119773    33.18   0.000     3.739359    4.208861
       _cons |   -18.7411   1.223875   -15.31   0.000    -21.13985   -16.34235
------------------------------------------------------------------------------
Instrumented:  x
Instruments:   z1 z2
```