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CHAPTER 2

The Logic of Compound Statements

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2.1 Logical Form and Logical Equivalence

The central concept of deductive logic is the concept of argument form. An argument is a sequence of statements aimed at demonstrating the truth of an assertion. The assertion at the end of the sequence is called the *conclusion,* and preceding statements are called *premises.* To have confidence in the conclusion that you draw from an argument, you must be sure that the premises are acceptable on their own merits or follow from other statements that are know to be true.

2.1 Logical Form and Logical Equivalence

Consider the following two arguments,

Argument 1 If the program syntax is faulty or if program execution results in division by zero, then the computer will generate an error message. Therefore, if the computer does not generate an error message, then the program syntax is correct and program execution does not result in division by zero.

2.1 Logical Form and Logical Equivalence

Argument 2 If x is a real number such that $x < -2$ or $x > 2$, then $x^2 > 4$. Therefore, if $x^2 \ngtr 4$, then $x \nless -2$ and $x \ngtr 2$.

To illustrate the logical form of these arguments, we use letters of the alphabet (such as p, q,and r) to represent the component sentences and the expression "not p" to refer to the sentence "It is not the case that p." Then the *common logical form* of both the previous arguments is as follows:

If p or q, then r.

Therefore, If not r, then not p and not q.

Example 2.1.1
Identifying Logical Form

Fill in the blanks below so that argument (b) has the same form as argument (a). Then represent the common form of the arguments using letters to stand for component sentences.

a. If Jane is a math major or Jane is a computer science major, then Jane will take Math 150.

Jane is a computer science major.

Therefore, Jane will take Math 150

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Example 2.1.1
Identifying Logical Form

b. If logic is easy or ________, then________.

I will study hard.

Therefore, I will get an A in this course.

Statements

• Definition

A **statement** (or **proposition**) is a sentence that is true or false but not both.

For example, “Two plus two equals four” and “Two plus two equals five” are both statements, the first because it is true and the second because it is false. On the other hand, the truth or falsity of “He is a college student” depends on the reference for the pronoun *he*. For some values of *he* the sentence is true; for others it is false.

Similarly, “$x + y > 0$” is not a statement because for some values of x and y the sentence is true; if $x = -1$ and $y = 0$, the sentence is false.

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Compound Statements

We now introduce three symbols that are used to build more complicated logical expressions out of simpler ones. The symbol $\backsim$ denotes *not,*
$\wedge$ denotes *and,*
and $\vee$ denotes *or*.

Given a statement p, the sentence "$\backsim p$" is read "not p" or "It is not the case that p" and is called the **negation of p**. Given another statement q, the sentence "$p \wedge q$" is read "p and q" and is called the **conjunction of** p and q. The sentence "$p \vee q$" is read "p or q" and is called the **disjunction of** p and q.

Example 2.1.2 Translating from English to Symbols: But and Neither-Nor

Write each of the following sentences symbolically, letting $h =$ "It is hot" and $s =$"It is sunny."

a. It is not hot but it is sunny.

b. It is neither hot nor sunny.

The notation for inequalities involves *and* and *or* statements. For instance, if x, a, and b are particular real numbers, then

$x \le a$	means	$x < a$	or	$x = a$
$a \le x \le b$	means	$a \le x$	and	$x \le b$.

Example 2.1.3 *And, Or,* and Inequalities.

Suppose x is a particular real number. Let $p, a,$ and r symbolize "$0 < x$," "$x < 3$," and "$x = 3$," respectively. Write the following inequalities symbolically:

a. $x \leq 3$

b. $0 < x < 3$

c. $0 < x \leq 3$

Truth Values

The negation of a statement is a statement that exactly expresses what it would mean for the statement to be false.

• Definition

If p is a statement variable, the **negation** of p is "not p" or "It is not the case that p" and is denoted $\sim p$. It has opposite truth value from p: if p is true, $\sim p$ is false; if p is false, $\sim p$ is true.

The truth values for negation are summarized in a *truth table.*

Truth Table for $\sim p$

p	$\sim p$
T	F
F	T

Truth Values

• Definition

If p and q are statement variables, the **conjunction** of p and q is "p and q," denoted $p \wedge q$. It is true when, and only when, both p and q are true. If either p or q is false, or if both are false, $p \wedge q$ is false.

Truth Table for $p \wedge q$

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Truth Values

• Definition

If p and q are statement variables, the **disjunction** of p and q is "p or q," denoted $p \vee q$. It is true when either p is true, or q is true, or both p and q are true; it is false only when both p and q are false.

Here is the truth table for disjunction:

Truth Table for $p \vee q$

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Evaluating the Truth of More General Compound Statements

• Definition

A **statement form** (or **propositional form**) is an expression made up of statement variables (such as p, q, and r) and logical connectives (such as $\sim$, $\wedge$, and $\vee$) that becomes a statement when actual statements are substituted for the component statement variables. The **truth table** for a given statement form displays the truth values that correspond to all possible combinations of truth values for its component statement variables.

Example 2.1.4

Truth Table for Exclusive Or

Construct the truth table for the statement form $(p \vee q) \wedge \sim(p \wedge q)$.

Example 2.1.5

Truth Table for $(\boldsymbol{p} \wedge \boldsymbol{q}) \vee \sim \boldsymbol{r}$

Construct a truth table for the statement
$(p \wedge q) \vee \sim r$

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Logical Equivalence

- **Definition**

Two *statement forms* are called **logically equivalent** if, and only if, they have identical truth values for each possible substitution of statements for their statement variables. The logical equivalence of statement forms P and Q is denoted by writing $P \equiv Q$.

Two *statements* are called **logically equivalent** if, and only if, they have logically equivalent forms when identical component statement variables are used to replace identical component statements.

p	q	$p \wedge q$	$q \wedge p$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	F	F

$p \wedge q$ and $q \wedge p$ always have the same truth values, so they are logically equivalent

Example 2.1.6

Double Negative Property: $\sim(\sim \boldsymbol{p}) \equiv \boldsymbol{p}$

Construct a truth table to show that the negation of the negation of a statement is logically equivalent to the statement, annotating the table with a sentence of explanation.

Example 2.1.7
Showing Nonequivalence

Show that the statement forms $\sim(p \wedge q)$ and $\sim p \wedge \sim q$ are not logically equivalent.

Example 2.1.8
Negations of *And* and *Or*:
De Morgan's Laws

The negation of the conjunction of two statements is logically equivalent to the disjunction of their negations. That is, statements of the forms $\sim(p \wedge q)$ and $\sim p \wedge \sim q$ are logically equivalent. Check this using truth tables.

De Morgan's Laws

The negation of an *and* statement is logically equivalent to the *or* statement in which each component is negated.

The negation of an *or* statement is logically equivalent to the *and* statement in which each component is negated.

Example 2.1.9
Applying De Morgan's Laws

Write negations for each of the following statements:

a. John is 6 feet tall and he weighs at least 200 pounds.

b. The bus was late or Tom's watch was slow.

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Example 2.1.10
Applying De Morgan's Laws

Use De Morgan's laws to write the negation of

$$-1 < x \leq 4.$$

Tautologies and Contradictions

- **Definition**

A **tautology** is a statement form that is always true regardless of the truth values of the individual statements substituted for its statement variables. A statement whose form is a tautology is a **tautological statement.**

A **contradication** is a statement form that is always false regardless of the truth values of the individual statements substituted for its statement variables. A statement whose form is a contradication is a **contradictory statement.**

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Example 2.1.12

Tautologies and Contradictions

Show that the statement form $p \vee \sim p$ is a tautology and that the statement form $p \wedge \sim p$ is a contradiction.

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Example 2.1.13
Logical Equivalence Involving Tautologies and Contradictions.

If $\mathbf{t}$ is a tautology and $\mathbf{c}$ is a contradiction, show that $p \wedge \mathbf{t} \equiv p$ and $p \wedge \mathbf{c} \equiv \mathbf{c}$

Summary of Logical Equivalences

Theorem 2.1.1 Logical Equivalences

Given any statement variables p, q, and r, a tautology $\mathbf{t}$ and a contradiction $\mathbf{c}$, the following logical equivalences hold.

1.	*Commutative laws:*	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
2.	*Associative laws:*	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
3.	*Distributive laws:*	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
4.	*Identity laws:*	$p \wedge \mathbf{t} \equiv p$	$p \vee \mathbf{c} \equiv p$
5.	*Negation laws:*	$p \vee {\sim}p \equiv \mathbf{t}$	$p \wedge {\sim}p \equiv \mathbf{c}$
6.	*Double negative law:*	${\sim}({\sim}p) \equiv p$	
7.	*Idempotent laws:*	$p \wedge p \equiv p$	$p \vee p \equiv p$
8.	*Universal bound laws:*	$p \vee \mathbf{t} \equiv \mathbf{t}$	$p \wedge \mathbf{c} \equiv \mathbf{c}$
9.	*De Morgan's laws:*	${\sim}(p \wedge q) \equiv {\sim}p \vee {\sim}q$	${\sim}(p \vee q) \equiv {\sim}p \wedge {\sim}q$
10.	*Absorption laws:*	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
11.	*Negations of* **t** *and* **c**:	${\sim}\mathbf{t} \equiv \mathbf{c}$	${\sim}\mathbf{c} \equiv \mathbf{t}$

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Example 2.1.14
Simplifying Statement Forms

Use Theorem 2.1.1 to verify the logical equivalence

$$\sim(\sim p \wedge q) \wedge (p \vee q) \equiv p$$

2.2 Conditional Statements

When you make a logical inference or deduction, you reason *from* a hypothesis *to* a conclusion. Your aim is to be able to say, "*If* such and such is known, *then* something or other must be the case."

Let p and q be statements. A sentence of the form "If p then q" is denoted by symbolically by "$p \rightarrow$ q"; p is called the *hypothesis* and q is called the *conclusion*. For instance, consider the following statement:

$$\text{If } \underbrace{4{,}686 \text{ is divisible by } 6}_{\text{hypothesis}}, \text{ then } \underbrace{4{,}686 \text{ is divisible by } 3}_{\text{conclusion}}$$

Such a sentence is called *conditional* because the truth of statement q is conditioned on the truth of statement p

2.2 Conditional Statements

Truth Table for $p \to q$

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

• Definition

If p and q are statement variables, the **conditional** of q by p is "If p then q" or "p implies q" and is denoted $p \to q$. It is false when p is true and q is false; otherwise it is true. We call p the **hypothesis** (or **antecedent**) of the conditional and q the **conclusion** (or **consequent**).

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Example 2.2.2

Truth table for $\boldsymbol{p \vee \sim q \rightarrow \sim p}$

Construct a truth table for statement form $p \vee \sim q \rightarrow \sim p$

Example 2.2.3 Division into Cases: Showing that $p \vee q \to r \equiv (p \to r) \wedge (q \to r)$

Use truth tables to show the logical equivalence of the statement forms $p \vee q \to r$ and $(p \to r) \wedge (q \to r)$.

Representation of If-Then As Or

$$p \rightarrow q \equiv \sim p \vee q$$

The logical equivalence of "if p then q" and "not p or q" is occasionally used in everyday speech. Here is one instance.

Example 2.2.4 Application of the Equivalence between $\sim p \vee q$ and $p \rightarrow q$

Rewrite the following statement in if-then form.

Either you get to work on time or you are fired.

The Negation of a Conditional Statement

The negation of "if p then q" is logically equivalent to "p and not q."

This can be restated symbolically as follows:

$$\sim(p \to q) \equiv p \wedge \sim q$$

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Example 2.2.5
Negations of *If-Then* Statements

Write negations for each of the following statements:

a. If my car is in the repair shop, then I cannot get to class.

b. If Sara lives in Athens, then she lives in Greece.

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The Contrapositive of a Conditional Statement

- **Definition**

The **contrapositive** of a conditional statement of the form "If p then q" is

$$\text{If } {\sim}q \text{ then } {\sim}p.$$

Symbolically,

$$\text{The contrapositive of } p \to q \text{ is } {\sim}q \to {\sim}p.$$

Example 2.2.6
Writing the Contrapositive

Write each of the following statements in its equivalent contrapositive form:

a. If Howard can swim across the lake, then Howard can swim to the island.

b. If today is Easter, then tomorrow is Monday.

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The Converse and Inverse of a Conditional Statement

• Definition

Suppose a conditional statement of the form “If p then q” is given.

1. The **converse** is “If q then p.”
2. The **inverse** is “If $\sim p$ then $\sim q$.”

Symbolically,

$$\text{The converse of } p \to q \text{ is } q \to p,$$

and

$$\text{The inverse of } p \to q \text{ is } \sim p \to \sim q.$$

Example 2.2.7
Writing the Converse and the Inverse

Write the converse and inverse of each of the following statements:

a. If Howard can swim across the lake, then Howard can swim to the island.

b. If today is Easter, then tomorrow is Monday.

1. A conditional statement and its converse are *not* logically equivalent.
2. A conditional statement and its inverse are *not* logically equivalent.
3. The converse and the inverse of a conditional statement are logically equivalent to each other.

Only If and the Biconditional

• Definition

It p and q are statements,

p **only if** q means "if not q then not p,"

or, equivalently,

"*if* p then q."

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Example 2.2.8

Converting *Only If* to *If-Then*

Rewrite the following statement in if-then form in two ways, one of which is the contrapositive of the other.

John will break the world's record for the mile run only if he runs the mile in under four minutes.

• Definition

Given statement variables p and q, the **biconditional of p and q** is "p if, and only if, q" and is denoted $p \leftrightarrow q$. It is true if both p and q have the same truth values and is false if p and q have opposite truth values. The words *if and only if* are sometimes abbreviated **iff.**

The biconditional has the following truth table:

Truth Table for $p \leftrightarrow q$

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Order of Operations for Logical Operators

1. $\sim$	Evaluate negations first.
2. $\wedge, \vee$	Evaluate $\wedge$ and $\vee$ second. When both are present, parentheses may be needed.
3. $\rightarrow, \leftrightarrow$	Evaluate $\rightarrow$ and $\leftrightarrow$ third. When both are present, parentheses may be needed.

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Necessary and Sufficient Conditions

• Definition

If r and s are statements:

r is a **sufficient condition** for s means "if r then s."

r is a **necessary condition** for s means "if not r then not s."

The occurrence of r is necessary to obtain the occurrence of s. Note that because of the equivalence between a statement and its contrapositive,

r is a necessary condition for s also means "if s then r."

Consequently,

r is a necessary and sufficient condition for s means "r if, and only if, s."

2.3 Valid and Invalid Arguments

• Definition

An **argument** is a sequence of statements, and an **argument form** is a sequence of statement forms. All statements in an argument and all statement forms in an argument form, except for the final one, are called **premises** (or **assumptions** or **hypotheses**). The final statement or statement form is called the **conclusion.** The symbol ∴, which is read "therefore," is normally placed just before the conclusion.

To say that an *argument form* is **valid** means that no matter what particular statements are substituted for the statement variables in its premises, if the resulting premises are all true, then the conclusion is also true. To say that an *argument* is **valid** means that its form is valid.

2.3 Valid and Invalid Arguments

Testing an Argument Form for Validity

1. Identify the premises and conclusion of the argument form.
2. Construct a truth table showing the truth values of all the premises and the conclusion.
3. A row of the truth table in which all the premises are true is called a **critical row**. If there is a critical row in which the conclusion is false, then it is possible for an argument of the given form to have true premises and a false conclusion, and so the argument form is invalid. If the conclusion in *every* critical row is true, then the argument form is valid.

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Example 2.3.1
Determining Validity or Invalidity

Determine whether the following argument form is valid or invalid.

$$p \to q \vee \sim r$$
$$q \to p \wedge r$$
$$\therefore \quad p \to r$$

Modus Ponens and Modus Tollens

An argument form consisting of two premises and a conclusion is called a **syllogism.** The first and second premises are called the **major premise** and **minor premise,** respectively.

The most famous form of syllogism in logic is called **modus ponens.** It has the following form:

If p then q.

p

$\therefore q$

Modus Ponens and Modus Tollens

Here is an argument of this form:

If the sum of the digits of 371,487 is divisible by 3, then 371,487 is divisible by 3.

The sum of the digits of 371,487 is divisible by 3. ∴ 371,487 is divisible by 3.

Modus Ponens and Modus Tollens

		premises		conclusion	
p	q	$p \to q$	p	q	
T	T	T	T	T	← critical row
T	F	F	T		
F	T	T	F		
F	F	T	F		

The first row is the only one in which both premises are true, and the conclusion in that row is also true. Hence the argument form is valid.

Now consider another valid argument form called **modus tollens.** It has the following form:

Modus Ponens and Modus Tollens

If p then q.

$\sim q$

$\therefore \sim p$

Here is an example of modus tollens:

If Zeus is human, then Zeus is mortal.

Zeus is not mortal.

∴Zeus is not human.

Example 2.3.2
Recognizing Modus Ponens and Modus Tollens

Use modus ponens or modus tollens to fill in the blanks of the following arguments so that they become valid inferences.

a. If there are more pigeons than there are pigeonholes, then at least two pigeons roost in the same hole.

   There are more pigeons that there are pigeonholes.

   ∴____ .

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Example 2.3.2
Recognizing Modus Ponens and Modus Tollens

b. If 870,232 is divisible by 6, then it is divisible by 3.
870,232 is not divisible by 3.

∴_______________________________________.

Additional Valid Argument Forms: Rules of Inference

A **rule of inference** is a form of argument that is valid. Thus modus ponens and modus tollens are both rules of inference. The following are additional examples of rules of inference that are frequently used in deductive reasoning.

Valid Arguments

The following argument forms are valid:

a.

$$p$$
$$\therefore p \vee q$$

b.

$$q$$
$$\therefore p \vee q$$

c.

$$p \wedge q$$
$$\therefore q$$

d.

$$p \wedge q$$
$$\therefore p$$

e.

$$p \vee q$$
$$\sim q$$
$$\therefore p$$

f.

$$p \vee q$$
$$\sim p$$
$$\therefore q$$

Valid Arguments

g. $p \to q$

$q \to r$

$\therefore p \to r$

h. $p \vee q$

$p \to r$

$q \to r$

$\therefore r$

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Example 2.3.8

Application: A More Complex Deduction

You are about to leave for school in the morning and discover that you don’t have your glasses. You know the following statements are true.

a. If I was reading the newspaper in the kitchen, then my glasses are on the kitchen table.

b. If my glasses are on the kitchen table, then I saw them at breakfast.

c. I did not see my glasses at breakfast.

Example 2.3.8

Application: A More Complex Deduction

d. I was reading the news paper in the living room or I was reading the newspaper in the kitchen.

e. If I was reading the newspaper in the living room then my glasses are on the coffee table.

Where are the glasses?

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Fallacies

For an argument to be valid, every argument of the same form whose premises are all true must have a true conclusion. It follows that for an argument to be invalid means that there is an argument of that form whose premises are all true and whose conclusion is false.

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Example 2.3.9 Converse Error

Show that the following argument is valid.

If Zach is a cheater, then Zach sits in the back row.

Zach sits in the back row.

∴Zach is a cheater.

$$p \rightarrow q$$
$$q$$
$$\therefore p$$

The fallacy underlying this invalid argument form is called the **converse error** because the conclusion of the argument would follow from the premises if the premise $p \rightarrow q$ were replaced by its converse.

Example 2.3.10 Inverse Error

Consider the following argument:

If interest rates are going up, stock market prices will go down.

Interest rates are not going up.

∴Stock market prices will not go down.

Note that this argument has the following form:

$$p \to q$$

$$\sim p$$

$$\therefore \sim q$$

The fallacy underlying this invalid argument form is called the **inverse error** because the conclusion of the argument would follow from the premises if the premise $p \rightarrow q$ were replaced by its inverse.

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Contradictions and Valid Arguments

Contradiction Rule

If you can show that the supposition that statement p is false leads logically to a contradiction, then you can conclude that p is true.

Example 2.3.13 Contradiction Rule

Show that the following argument form is valid:

$\sim p \rightarrow \boldsymbol{c}$, where $\boldsymbol{c}$ is a contradiction

$\therefore p$

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Summary of Rules of Inference

Table 2.3.1 Valid Argument Forms

Modus Ponens	$p \to q$ p $\therefore q$	**Elimination**	**a.** $p \lor q$ $\sim q$ $\therefore p$ **b.** $p \lor q$ $\sim p$ $\therefore q$
Modus Tollens	$p \to q$ $\sim q$ $\therefore \sim p$	**Transitivity**	$p \to q$ $q \to r$ $\therefore p \to r$
Generalization	**a.** p $\therefore p \lor q$ **b.** q $\therefore p \lor q$	**Proof by Division into Cases**	$p \lor q$ $p \to r$ $q \to r$ $\therefore r$
Specialization	**a.** $p \land q$ $\therefore p$ **b.** $p \land q$ $\therefore q$		
Conjunction	p q $\therefore p \land q$	**Contradiction Rule**	$\sim p \to c$ $\therefore p$