Question 1:

Given the equation for the production function

$$Q = f(K, L) = 18 * [0.2K^{-0.4} + 0.8L^{-0.4}]^{-2.5}$$

1.1 What type of constant return to scale does the production function exhibit?

1.2 Is the production function increasing with respect to K and L?

1.3 Use the implicit function rule to find the marginal rate of technical substitution (MRTS) of L for K.

1.4 Use the Hessian matrix. Proof that the production function is concave.

1.1 $Q_1 = f(K,L) = 18 * [0.2K^{-0.4} + 0.8L^{-0.4}]^{-2.5}$

$Q_1 = 18[0.2K + 0.8L]$

$Q_2 = f(zK, zL) = 18 * [0.2(zK)^{-0.4} + 0.8(zL)^{-0.4}]^{-2.5}$

$= 18 * [z^{-0.4}(0.2K) + z^{-0.4}(0.8L)^{-0.4}]^{-2.5}$

$= 18 * [2z^{-0.4}(0.2K^{-0.4} + 0.8L^{-0.4})]^{-2.5}$

$= 18[2z(0.2K + 0.8L)]$

when Add z on capital and Add z on labor production function increase for 2z this type is <u>increasing return to scale</u>

1.2 $Q_1 = f(K,L) = 18 * [0.2K^{-0.4} + 0.8L^{-0.4}]^{-2.5}$

$Q_1 = 18[0.2K + 0.8L]$

$= 3.6K + 14.4L$

$Q_2 = f(2K, 2L) = 18 * [0.2(2K)^{-0.4} + 0.8(2L)^{-0.4}]^{-2.5}$

$= 18[0.4K^{-0.4} + 1.6L^{-0.4}]^{-2.5}$

$= 18[0.4K + 1.6L]$

$= 7.2K + 28.8L$

∴ The production function is increase with the respect to K and L

1.3 $MRTS = \frac{MPL}{MPK}$

$Q = 18 * [0.2K + 0.8L]$

$Q = 3.6K + 14.4L$

$MPL = \frac{\partial Q}{\partial L} = 14.4$

$MPK = \frac{\partial Q}{\partial K} = 3.6$

$MRTS = \frac{14.4}{3.6}$

$MRTS = 4$

1.4 $Q = 3.6K + 14.4L$

$f'_K = 3.6$

$f'_L = 14.4$

Positive

$$H = \begin{bmatrix} 3.6 & 0 \\ 0 & 14.4 \end{bmatrix}$$

minimum

$|H| = 51.84$ Positive

Question 2: Define f(x,y) for all (x,y) by

$$f\ (x, y) = e^{x+y} + e^{x-y} - \frac{3}{2}x - \frac{1}{2}y$$

2.1 Derive the Hessian matrix of $f\ (x, y)$.

2.2 Show that f(x,y) is monotonically convex function. That's, the function is convex for the entire domain sets defining f(x,y).

2.3 Find the *global* extrema of f(x,y). What type of extrema is it?

2.1)

$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$$

$$f_x = \frac{df}{dx} = \left[\left(e^x \cdot \frac{de^y}{dy}\right) + \left(e^y \cdot \frac{de^x}{dx}\right)\right] + \left[\frac{\left(e^x \frac{de^x}{dx}\right) - \left(e^x \frac{de^y}{dx}\right)}{(e^y)^2}\right]$$

$$- \frac{3}{2} + 0$$

$$= e^{x+y} + \left(\frac{e^y e^x}{e^y}\right) - \frac{3}{2}$$

$$= e^{x+y} + e^{x-y} - \frac{3}{2}$$

$$f_{xx} = \frac{d f_x}{dx} = \left[\left(e^x \cdot \frac{de^y}{dy}\right) + \left(e^y \cdot \frac{de^x}{dx}\right)\right] + \left[\frac{\left(e^x \frac{de^x}{dx}\right) - \left(e^x \frac{de^y}{dx}\right)}{(e^y)^2}\right]$$

$$+ 0$$

$$= e^{x+y} + e^{x-y}$$

$$f_{xy} = \frac{df_x}{dy} = \left[\left(e^x \cdot \frac{de^y}{dy}\right) + \left(e^y \cdot \frac{de^x}{dy}\right)\right] + \left[\frac{\left(e^x \frac{de^x}{dy}\right) - \left(e^x \frac{de^y}{dy}\right)}{(e^y)^2}\right]$$

$$= (e^x e^y) - (e^y \cdot 0) + \left[\frac{(e^y \cdot 0) - (e^x \cdot e^y)}{(e^y)^2}\right]$$

$$= e^{x+y} - e^{x-y}$$

$$f_y = \frac{df}{dy} = \left[\left(e^x \cdot \frac{de^y}{dy}\right) + \left(e^y \cdot \frac{de^x}{dy}\right)\right] + \left[\frac{\left(e^x \frac{de^x}{dy}\right) - \left(e^x \frac{de^y}{dy}\right)}{(e^y)^2}\right]$$

$$+ 0 - \frac{1}{2}$$

$$= e^{x+y} - e^{x-y} - \frac{1}{2}$$

$$f_{yx} = \frac{df_y}{dx} = \left[\left(e^x \cdot \frac{de^y}{dx}\right) + \left(e^y \cdot \frac{de^x}{dx}\right)\right] + \left[\frac{\left(e^x \frac{de^x}{dx}\right) - \left(e^x \frac{de^y}{dx}\right)}{(e^y)^2}\right]$$

$$+ 0$$

$$= (e^x \cdot 0)(e^y e^x) - \left(\frac{(e^y e^x) - (e^x \cdot 0)}{e^{2y}}\right)$$

$$\therefore f_{yx} = e^{x+y} - e^{x-y}$$

$$f_{yy} = \left[\left(e^x \cdot \frac{de^y}{dy}\right) + \left(e^y \cdot \frac{de^x}{dy}\right)\right] + \left[\frac{\left(e^x \frac{de^x}{dy}\right) - \left(e^x \frac{de^y}{dy}\right)}{(e^y)^2}\right]$$

$$+ 0$$

$$= (e^x \cdot e^y) + (e^y \cdot 0) - \left(\frac{(e^y \cdot 0) - (e^x \cdot e^y)}{e^{2y}}\right)$$

$$\therefore f_{yy} = e^{x+y} + e^{x-y}$$

Hence $H = \begin{bmatrix} e^{x+y} + e^{x-y} & e^{x+y} - e^{x-y} \\ e^{x+y} - e^{x-y} & e^{x+y} + e^{x-y} \end{bmatrix}$

2.2) Monotonically convex $\rightarrow f''(x,y) > 0 \quad \therefore f_{xx} \,\&\, f_{yy} < 0$

$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} e^{x+y} + e^{x-y} & e^{x+y} - e^{x-y} \\ e^{x+y} - e^{x-y} & e^{x+y} + e^{x-y} \end{bmatrix}$$

$$|H_1| = |e^{x+y} + e^{x-y}| = e^{x+y} + e^{x-y}$$

$$|H_2| = \begin{vmatrix} e^{x+y} + e^{x-y} & e^{x+y} - e^{x-y} \\ e^{x+y} - e^{x-y} & e^{x+y} + e^{x-y} \end{vmatrix}$$

$$= (e^{x+y} + e^{x-y})(e^{x+y} + e^{x-y}) - (e^{x+y} - e^{x-y})(e^{x+y} - e^{x-y})$$

$$= \left[(e^{x+y})^2 + 2(e^{x+y})(e^{x-y}) + (e^{x-y})^2\right] - \left[(e^{x+y})^2 - 2(e^{x+y})(e^{x-y}) + (e^{x-y})^2\right]$$

$$= \left[e^{2x+2y} + 2\left(\frac{e^x \cdot e^y}{e^y} \cdot e^x\right) + e^{2x-2y}\right] - \left[e^{2x+2y} - 2\left(\frac{e^x \cdot e^y}{e^y} \cdot e^x\right) + e^{2x-2y}\right]$$

$$= 4e^{2x}$$

$$\therefore 4e^{2x} > 0 \qquad \forall x, \forall y$$

$\forall x, \forall y$ H is always positive definite

$$\therefore d^2 f > 0 \quad \forall x, \forall y$$

f is monotically convex

2.3) F.O.C

$$f_x = \frac{df}{dx} = \left[\left(e^x \cdot \frac{de^y}{dx}\right) + \left(e^y \cdot \frac{de^x}{dx}\right)\right] + \left[\frac{\left(e^y \cdot \frac{de^x}{dx}\right) - \left(e^x \cdot \frac{de^y}{dx}\right)}{(e^y)^2}\right]$$

$$- \frac{3}{2} + 0$$

from 2.1

$$\therefore f_x = e^{x+y} + e^{x-y} - \frac{3}{2} = 0 \quad \text{——} \quad ①$$

$$f_y = \frac{df}{dy} = \left[\left(e^x \cdot \frac{de^y}{dy}\right) + \left(e^y \cdot \frac{de^x}{dy}\right)\right] + \left[\frac{\left(e^y \cdot \frac{de^x}{dy}\right) - \left(e^x \cdot \frac{de^y}{dy}\right)}{(e^y)^2}\right]$$

$$+ 0 - \frac{1}{2}$$

$$\therefore f_y = e^{x+y} - e^{x-y} - \frac{1}{2} = 0 \quad \text{——} \quad ②$$

$① + ② = e^{x+y} = 1 \quad \therefore x+y = \ln 1 = 0$

$① - ② = e^{x-y} = 0.5 \quad \therefore x - y = \ln 0.5$

$x^* = 0.5 \ln 0.5$ and $y^* = -0.5 \ln 0.5$

$\therefore (x^*, y^*) = (0.5 \ln 0.5, -0.5 \ln 0.5)$ is global minimum

; f is monotically convex

Question 3:

A monopolist faces the market demand given by $P = Q^{-c}$ where "c" is a parameter with positive value, "P" is the price per unit output and "Q" is the amount of output. Suppose that monopolist's production technology is given by $Q = K^{\frac{1}{3}}L^{\frac{2}{3}}$ where "K" and "L" are the level of capital used and

the number of labor employed, respectively. Assume that the unit price of K and L are set equal to "r" and "w", respectively. Consider the following problems.

3.1) What type of the return to scale technology does the production function exhibit?

From now on, assume that c = ¼ . Consider the following problems.

3.2) Construct the profit function of the monopolist. (Hint: your profit function should be expressed in terms of K and L.)

3.3) The firm wants to maximize profit and seek for combination of the two factor inputs . Derive the demand for factor inputs, capital and labor.

3.4) How does the demand for labor vary with respect to w and r? Show your result by using partial derivative.

3.5) Confirm your answer with the second-order condition.

3.1) $Q = K^{\frac{1}{3}} L^{\frac{2}{3}}$

Degree of homogenity $= \alpha + \beta = \frac{1}{3} + \frac{2}{3} = 1$

$\therefore$ The function exhibits a constant return to scale.

3.2) $\pi = TR - TC$

$Q = f(k, L)$

; $\pi = (PQ) - (wL + rk)$

$= Q^{-c} \cdot Q - (wL + rk)$

$= Q^{-\frac{5}{4}} - (wL + rk)$

$= (k^{\frac{1}{3}} L^{\frac{2}{3}})^{-\frac{5}{4}} - (wL + rk)$

$= [k^{\frac{1}{3} \times -\frac{5}{4}} L^{\frac{2}{3} \times -\frac{5}{4}}] - (wL + rk)$

$= [k^{-\frac{5}{12}} L^{-\frac{5}{6}}] - wL - rk \rightarrow \pi(k, L)$ #

3.3) F.O.C $\frac{d\pi}{dk} = -\frac{5}{12} k^{-\frac{17}{12}} L^{-\frac{5}{6}} - r = 0$

$-\frac{5}{12} k^{-\frac{17}{12}} L^{-\frac{5}{6}} = r$

$k^{\frac{17}{12}} = \frac{-5}{12 r L^{\frac{5}{6}}}$

$k^{*} = \left(\frac{-5}{12 r L^{\frac{5}{6}}} \right)^{\frac{12}{17}}$

$\frac{d\pi}{dL} = -\frac{5}{6} k^{-\frac{5}{12}} L^{-\frac{11}{6}} - w = 0$

$$-\frac{5}{6} K^{\frac{5}{12}} L^{-\frac{11}{6}} = w$$

$$L^{\frac{11}{6}} = \frac{-5}{6wK^{\frac{5}{12}}}$$

$$L^{*} = \left(\frac{-5}{6wK^{\frac{5}{12}}}\right)^{\frac{6}{11}}$$

$$L^{*} = \left(\frac{-5}{6}\right)^{\frac{6}{11}} \cdot w^{\frac{6}{11}} \cdot K^{-\frac{5}{22}}$$

3.4)
$$\frac{dL^{*}}{dw} = -\frac{6}{11} w^{-\frac{6}{11}-\frac{11}{11}}$$

$$= -\frac{6}{11} w^{-\frac{17}{11}}$$

$$= \frac{-6}{11 w^{\frac{17}{11}}}$$

∴ w increase L decrease by $\frac{-6}{11w^{\frac{12}{11}}}$

Negative relationship between labor demand and w

$\frac{dL^{*}}{dr} = 0$ So no substitution effect

∴ No relationship between r and labor demand.

3.5) $H = \begin{bmatrix} \pi_{KK} & \pi_{KL} \\ \pi_{LK} & \pi_{LL} \end{bmatrix}$

Check $\pi_K = \frac{d\pi}{dK} = -\frac{5}{12} K^{-\frac{17}{12}} L^{-\frac{5}{6}} - r$

$\pi_{KK} = \frac{d\pi_K}{dK} = \left(-\frac{17}{12}\right)\left(-\frac{5}{12}\right) K^{-\frac{29}{12}} L^{-\frac{5}{6}} = \frac{85}{144 K^{\frac{29}{12}} L^{\frac{5}{6}}}$

$\pi_{KL} = \frac{d\pi_K}{dL} = \left(-\frac{5}{12}\right)\left(-\frac{5}{6}\right) K^{-\frac{17}{12}} L^{-\frac{11}{6}} = \frac{25}{72 K^{\frac{17}{12}} L^{\frac{11}{6}}}$

$\pi_L = \frac{d\pi}{dL} = -\frac{5}{6} K^{-\frac{5}{12}} L^{-\frac{11}{6}} - w$

$\pi_{LK} = \frac{d\pi_L}{dK} = \left(-\frac{5}{6}\right)\left(-\frac{5}{12}\right) K^{-\frac{17}{12}} L^{-\frac{11}{6}} = \frac{25}{72 K^{\frac{17}{12}} L^{\frac{11}{6}}}$

$\pi_{LL} = \frac{d\pi_L}{dL} = \left(-\frac{5}{6}\right)\left(-\frac{11}{6}\right) K^{-\frac{5}{12}} L^{-\frac{17}{6}} = \frac{55}{36 K^{\frac{5}{12}} L^{\frac{17}{6}}}$

$\therefore H = \begin{bmatrix} \frac{85}{144 K^{\frac{29}{12}} L^{\frac{5}{6}}} & \frac{25}{72 K^{\frac{17}{12}} L^{\frac{11}{6}}} \\ \frac{25}{72 K^{\frac{17}{12}} L^{\frac{11}{6}}} & \frac{55}{36 K^{\frac{5}{12}} L^{\frac{17}{6}}} \end{bmatrix}$

$|H_1| = \frac{85}{144 K^{\frac{29}{12}} L^{\frac{5}{6}}}$

$|H_2| = \left[\left(\frac{85}{144 K^{\frac{29}{12}} L^{\frac{5}{6}}}\right)\left(\frac{55}{36 K^{\frac{5}{12}} L^{\frac{17}{6}}}\right)\right] - \left[\left(\frac{25}{72 K^{\frac{17}{12}} L^{\frac{11}{6}}}\right)^2\right]$

$= \frac{4{,}010}{5184 K^{\frac{34}{12}} L^{\frac{22}{6}}}$

$$K^y = \left(\frac{-5}{12rL^{\frac{5}{6}}}\right)^{\frac{12}{17}} \qquad L^y = \left(\frac{-5}{6wK^{\frac{5}{12}}}\right)^{\frac{6}{11}}$$

$\downarrow$ Plug K^y, L^y to $|H_1|, |H_2|$

$|H_1| < 0$ at (K^y, L^y)

$|H_2| > 0$ at (K^y, L^y)

H is negative definite ; at K^y, L^y ; $d^2\pi < 0$

$\therefore$ Local maximizer

if $K = 1$, $L = 2$; $|H_1| < 0$, $|H_2| > 0$

$\therefore$ π is globally concave

Local maximum $\rightarrow$ Global maximum.