Last time

⇒ The Qualitative Analysis - Graphic Approach

(R1) anywhere above the hor. axis $\Rightarrow \frac{dy}{dt} > 0$

$\Rightarrow y$ is increasing

anywhere below the ——— $\Rightarrow \frac{dy}{dt} < 0$

$\Rightarrow y$ is decreasing

(R2) $\frac{dy}{dt} = 0 \Rightarrow y(t)$ is stationary over time.

if the equilibrium exists

$$Q = f(K, L) \quad ; \quad K, L > 0 \qquad - (1)$$

$$Q = \underline{\underline{L\,\phi(k)}} \quad ; \quad k = \frac{K}{L} \qquad - (2)$$

$$MP_K = \frac{\partial Q}{\partial K} = \frac{\partial L\,\phi(k)}{\partial K}$$

$$= \frac{\partial L\,\phi(k)}{\partial k} \cdot \frac{\partial k}{\partial K}$$

$$= L \cdot \phi'(k) \cdot \frac{1}{L}$$

$$MP_K > 0 = \boxed{\phi'(k) > 0}$$

$(+)$ $(+)$

$$f_{KK} = \frac{\partial MP_K}{\partial K} = \frac{\partial \phi'(k)}{\partial K}$$

$$= \frac{\partial \phi'(k)}{\partial k} \cdot \frac{\partial k}{\partial K}$$

$$f_{KK} = \phi''(k) \cdot \frac{1}{L}$$

$(+)$

$(-)$ $\boxed{\phi''(k) < 0}$

I = Savings

$$I(t) = \frac{dK}{dt} = sQ \quad \text{--- (3)}$$

$$(= K'(t))$$

s = a constant

MPS

$$\frac{1}{L}\frac{dL}{dt} = \lambda \quad \text{--- (4)}$$

$$\left(= \frac{1}{L} \cdot L'(t)\right)$$

$$Q = L\phi(k) \quad \text{--- (2)}$$

$$K'(t) = sQ \quad \text{--- (3)}$$

$$L'(t) = \lambda L \quad \text{--- (4)}$$

From $k = \frac{K}{L} \Rightarrow K = k.L$

$$K'(t) = k'(t) L + k.L'(t)$$

sub (3) on LHS
sub (4) on RHS

$$sQ = k'(t) L + k(\lambda L)$$

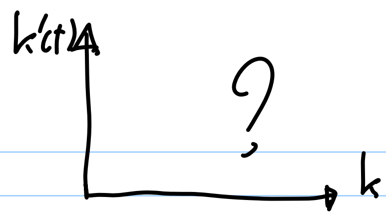
sub (2) on LHS

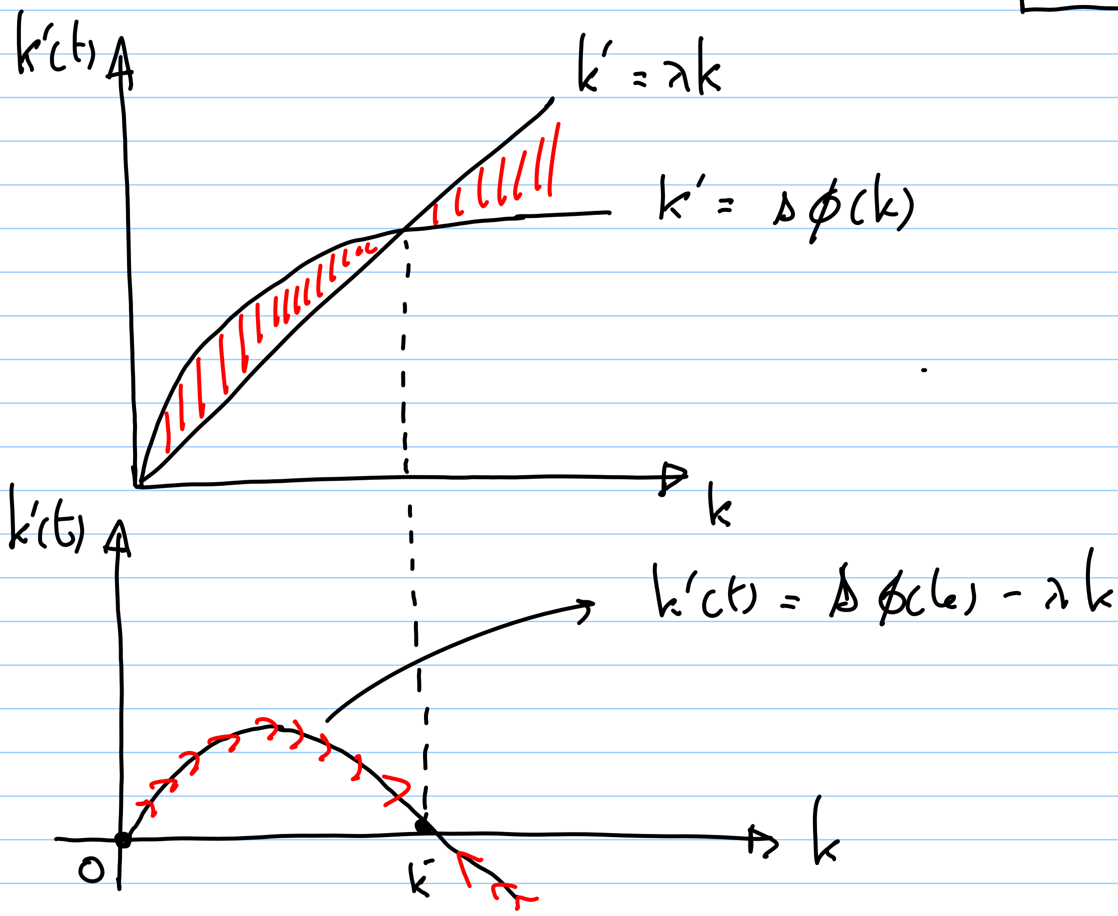
$$s(\cancel{L}\phi(k)) = k'(t)\cancel{L} + k\lambda\cancel{L}$$

$$\boxed{k'(t) = s\phi(k) - \lambda k} \quad \text{— (5)}$$

$$\frac{dy}{dt} = f(y)$$

A Qualitative - Graphic Analysis $\Rightarrow$





at $k=0 \Rightarrow \boxed{k' = s0 - \lambda 0 = 0}$

$k = \bar{k}$ $\quad k' = s\phi(\bar{k}) - \lambda \bar{k}$

$\Downarrow$ $\boxed{k' = 0}$

true ?

The phase line has $\boxed{\text{a} \ominus \text{slope @ } \bar{k}}$

the equilibrium is dynamically stable

$Q = L\,\boxed{\phi(k)}$ the growth rate of output $\to$ zero

$\to \oplus$

$\to \ominus$

output grows at the rate of 'λ'

A Quantitative Analysis

$$Q = K^{\alpha} L^{1-\alpha} \quad ; \quad 0 < \alpha < 1$$

$$= L \left(\frac{K}{L}\right)^{\alpha}$$

$$= L\,k^{\alpha}$$

$$Q = L.\phi(k) \quad ; \quad \phi(k) = k^{\alpha}$$

(5) becomes $k'(t) = s k^{\alpha} - \lambda k$ ✓✓

solve for $k(t)$ [the time path of k]

at $t=0$ $\quad k(0) = k_0 \rightarrow$ the initial value of k

$$\boxed{k(t)^{1-\alpha} = \left[k(0) - \frac{s}{\lambda}\right] e^{-(1-\alpha)\lambda t} + \frac{s}{\lambda}}$$

$$k'(t) + \lambda k = s k^{\alpha}$$

$$y'(t) + u(t)\, y(t) = w(t)\, [y(t)]^{r}$$

$$r = \alpha \quad , \quad u(t) = \lambda \quad , \quad w(t) = s$$

$$z(t) = [k(t)]^{1-\alpha}$$

$$\frac{1}{1-r} z'(t) + u(t)\, z(t) = w(t)$$

$$z'(t) + (1-\alpha)\lambda\, z(t) = (1-\alpha) s$$

$$z(t) = \left[z(0) - \frac{\cancel{(1-\alpha)} s}{\cancel{(1-\alpha)} \lambda}\right] e^{-(1-\alpha)\lambda t} + \frac{\cancel{(1-\alpha)} s}{\cancel{(1-\alpha)} \lambda}$$

$$z(t) = \left[z(0) - \frac{s}{\lambda}\right] e^{-(1-\alpha)\lambda t} + \frac{s}{\lambda}$$

$$k(t)^{1-\alpha} = \underbrace{\left[[k(0)]^{1-\alpha} - \frac{s}{\lambda}\right]} e^{-(1-\alpha)\lambda t} + \frac{s}{\lambda}$$

As $t \to \infty$ $\quad k(t)^{1-\alpha} \to \dfrac{\Delta}{\lambda}$

$$\bar{k} = \left[\frac{\Delta}{\lambda}\right]^{\frac{1}{1-\alpha}}$$

$\Delta \uparrow \Rightarrow \bar{k} \uparrow$

$\lambda \downarrow \Rightarrow \bar{k} \uparrow$

2.2 Higher-order Differential Equations

Focus $\Rightarrow$ 2^{nd} order, Linear

$$y''(t) + u_1(t)\,y'(t) + u_2(t)\,y(t) = w(t)$$

$$y'(t) + u(t)\,y(t) = w(t)$$

2.2.1 Second-order Linear Differential Equations with constant coefficients and constant term

$$\left.\begin{aligned} u_1(t) &= a_1 \\ u_2(t) &= a_2 \\ w(t) &= b \end{aligned}\right\} \text{constants}$$

$$y''(t) + a_1 y'(t) + a_2 y(t) = b \qquad \text{— (1)}$$

$b = 0 \Rightarrow$ homogeneous equation

$$y(t) = y_c + y_p$$

y_p = the intertemporal equilibrium value of y

y_c = the deviation of the time path $y(t)$ from the equilibrium

(c) <u>Find Particular Integral</u>

3 possible cases

case 1: $a_2 \neq 0$, $a_1 \neq 0$

case 2: $a_2 = 0$, $a_1 \neq 0$

case 3: $a_2 = 0$, $a_1 = 0$

<u>Case 1</u>: $a_2 \neq 0$, $a_1 \neq 0$

$y_p = k$ (a constant)

$y'(t) = 0$

$y''(t) = 0$

sub into (1) $\quad 0 + a_1 . 0 + a_2 k = b$

$$\boxed{k = \frac{b}{a_2}}$$

$$y_p = \frac{b}{a_2} \quad — (2)$$

Case 2 : $a_2 = 0$, $a_1 \neq 0$

$y_p = \frac{b}{a_2}$ is not defined

We try some constant form of $y_p \Rightarrow$

$$\begin{cases} y_p = kt \\ y'(t) = k \\ y''(t) = 0 \end{cases}$$

sub (1)

$$0 + a_1 . k + 0 . kt = b$$

$$a_1 k = b$$

$$k = \frac{b}{a_1}$$

$$\therefore \boxed{y_p = kt = \frac{b}{a_1} t} \quad — (3)$$

$a_2 = 0$, $a_1 \neq 0$

Case 3 $a_2 = 0$ $a_1 = 0$

$$\left.\begin{array}{l} y_p = \frac{b}{a_2} \quad — (2) \\ y_p = \frac{b}{a_1} t \quad — (3) \end{array}\right\} \text{are not defined}$$

$$y_p = kt^2$$

$$\left.\begin{aligned} y'(t) &= 2kt \\ y''(t) &= 2k \end{aligned}\right\}$$ sub into (1)

$$2k + 0 \cdot 2kt + 0\, kt^2 = b$$

$$2k = b$$

$$k = \frac{b}{2}$$

$$\boxed{y_p = kt^2 = \frac{b}{2}t^2} \quad — (4)$$

$$a_1 = 0 \ , \ a_2 = 0$$