Lecture 4
Optimal Risky Portfolios

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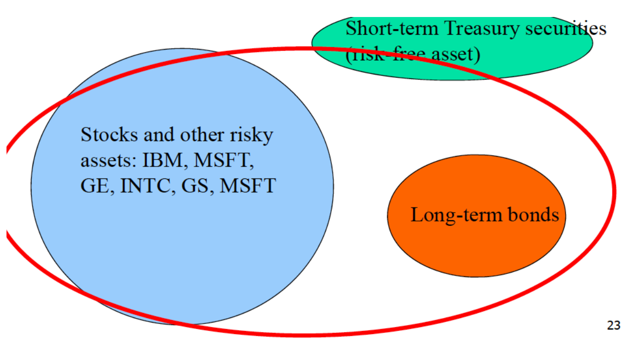
Optimal Risky Portfolios

- When choosing the optimal allocation between a risk-free asset and a risky portfolio, we have assumed that we have already selected the optimal risky portfolio
- In this section, we learn how to determine the optimal risky portfolio



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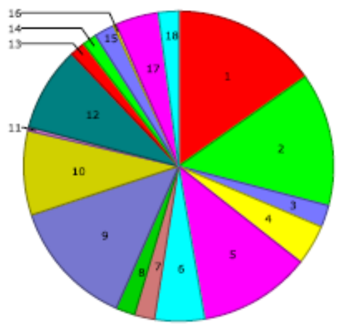
Example

Portfolio composition by Asset Allocation⁺



1	US Equity	15.2%
2	Europe Equity	13.9%
3	Asia Pac ex-Japan Equity	2.3%
4	Global Emerging Market Equity	4.2%
5	Global Aggregate Bond	11.7%
6	Global High Yield	5.2%
7	Hedge Fund	2.1%
8	Gold	2.0%
9	UK Equity	13.3%
10	Japan Equity	8.8%
11	Russia Equity	0.4%
12	UK Gilt	8.8%
13	Emerging Market Debt	1.6%
14	Property	1.5%
15	Commodity	2.1%
16	Private Equity	0.3%
17	Absolute Return	4.5%
18	Cash	2.1%
	Total	**100.0%**

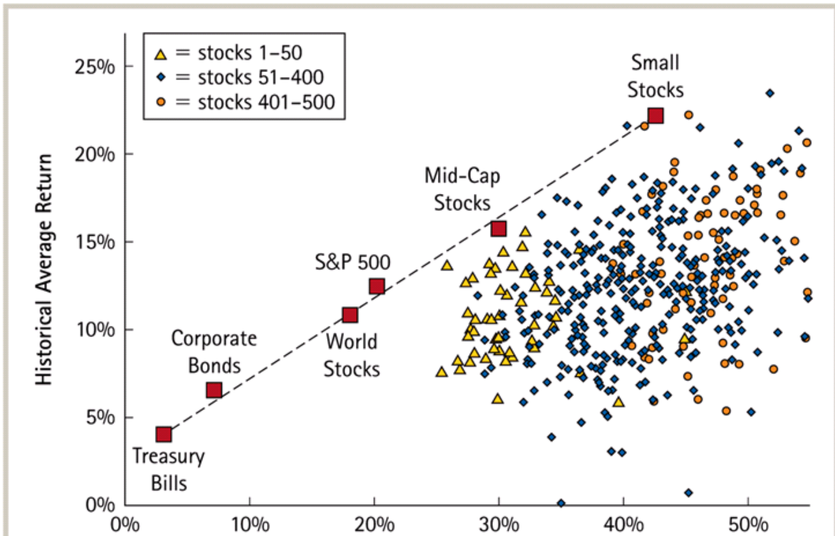
⁺ These were the target portfolio allocations as at 31 March 2011

Modern Portfolio Theory

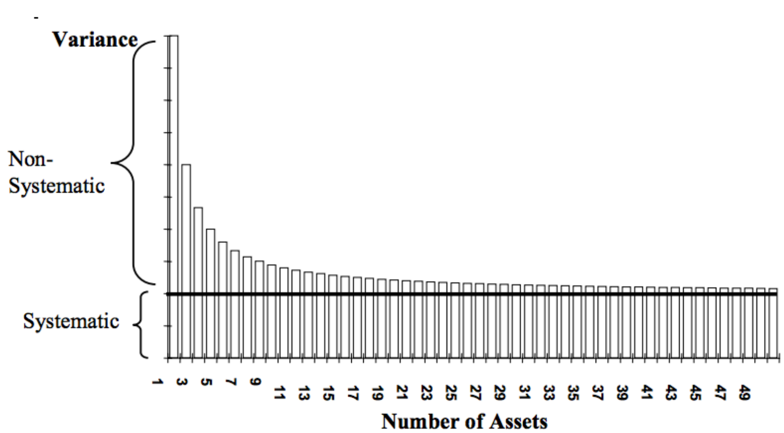
- Harry Markowitz
- Adjunct Professor of Finance
- UC San Diego

- An economist who devised the modern portfolio theory: 'Portfolio Selection', *The Journal of Finance*, 1952 (Published while a graduate student at the University of Chicago at 25 years old)
- Best known for his work in modern portfolio theory, emphasizing the importance of diversification
- Markowitz won the Nobel Prize in 1990, while a professor at CUNY
- A Markowitz Efficient Portfolio is one where no added diversification can lower the portfolio's risk for a given expected return

Recall the Concept of Diversification



Source: Berk & DeMarzo (2007) 5



- In practice, there is a limit to which you can reduce portfolio volatility
- Because there are some risks that cannot be completely diversified away!

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Mean Variance Analysis

- Up until the mean-variance analysis of Markowitz became known, an investment advisor might have given you the following advice:
  - If you are young you should put money into a couple of good growth stocks, maybe even into a few small stocks. Now is the time to take risk
  - If you are close to retirement, you should put all of your money into bond and safe stocks, and nothing into risky stocks. Don't take risks with your portfolio at this stage in your life!

Mean-Variance Analysis: Preview

- We will show that the optimal portfolio of risky assets is exactly the same for everyone, <u>no matter what their tolerance for risk</u>
  - Investors should control the risk of their portfolio not by reallocating among risky assets, but through the split between risky and risk-free assets
  - The portfolio of risky assets should contain a large number of assets - it should be a well diversified portfolio

A Portfolio of Two Risky Assets

Real world relevance

- Client looking to diversify a single concentrated holding in one particular asset
- Portfolio manager looking to add an additional asset to a pre-existing portfolio

- What are the characteristics of a portfolio that is composed of these two assets?

Portfolio Characteristics (n=2 case)

- The portfolio's expected return is a weighted sum of the expected returns of assets 1 and 2

- The variance is the square-weighted sum of the variance plus twice the cross-weighted covariance

Example: n=2 case

Asset	$E(\tilde{r})$	σ
A	25%	75%
B	10%	25%

- How does correlation affect the risk? Derive the minimum variance frontier under 3 different assumptions

1. $\rho_{AB} = 1$
2. $\rho_{AB} = -1$
3. $\rho_{AB} = 0$

- Under each assumption we can plot out a set of expected returns and standard deviations for different combinations of the assets
- *Minimum Variance Frontier* is the set of portfolios with the lowest variance for a given expected return

Case $\rho_{AB} = 1$

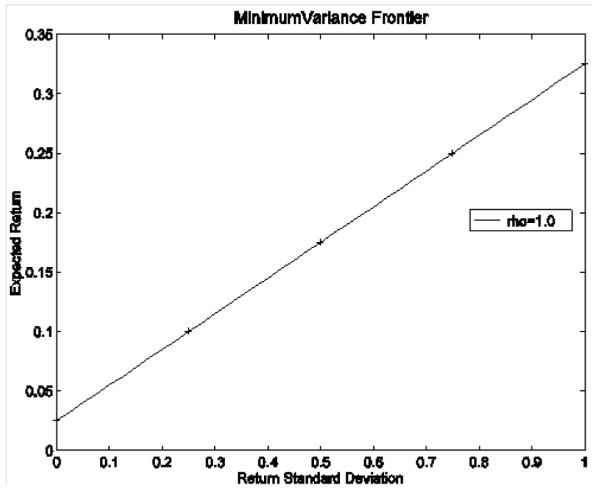
- Plug these numbers into these two equations:

$$\begin{aligned} E(\tilde{r}_p) &= 0.25w + 0.10 \cdot (1-w) \\ &= 0.15w + 0.10 \\ \sigma_p &= 0.75w + 0.25 \cdot (1-w) \\ &= 0.50w + 0.25 \end{aligned}$$

- In *Excel*, we can build a table with various possible w's:

w	$E(\tilde{r}_p)$	σ_p
-.5	2.5%	0.0%
0	10%	25%
0.5	17.5%	50.0%
1	25.0%	75.0%
1.5	32.5%	100.0%

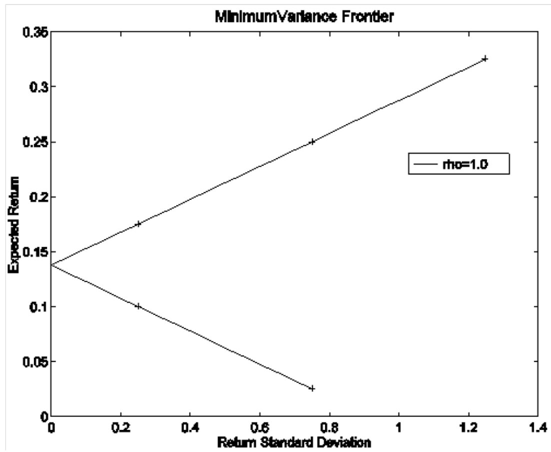
No Diversification Benefit



Case $\rho_{AB} = -1$

- When $\rho = -1$ we can again simplify the variance equation:

- Again, if we create a table of the expected returns and variances for different weights and plot these, we get: (here for $-0.5 \leq w \leq 1.5$):

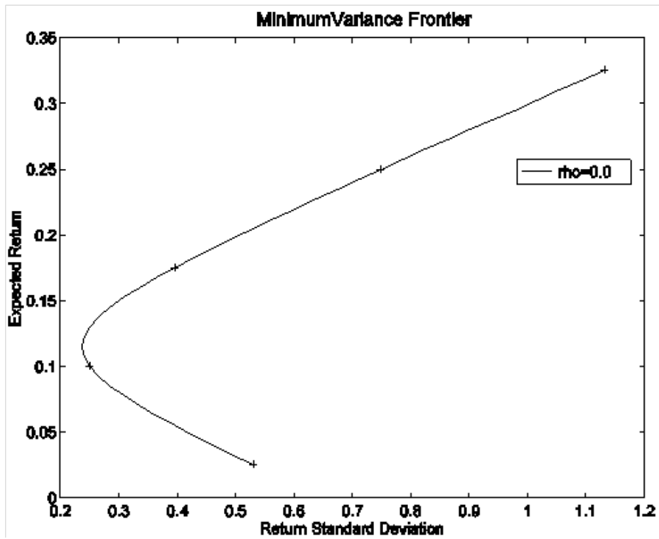


Perfect Hedge

- Perfect Hedge is a hedge that gives a portfolio with zero risk
- It is possible to find a perfect hedge with these 2 securities

- We have created a 'synthetic' risk-free security!

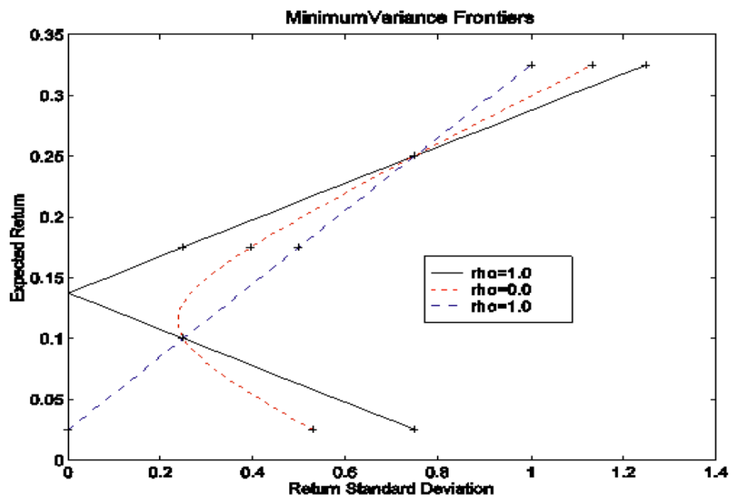
Two Risky assets, $\rho_{AB} = 0$



Diversification and Portfolio Effect

- The diversification effect
  - The reduction of portfolio standard deviation, compared with a simple linear combination of the standard deviations, that comes from holding two or more assets in the portfolio (provided their returns are not perfectly positively correlated)
  - The size of the diversification effect depends on the degree of correlation among asset's returns
  - Generally, the more different the assets are, the greater the diversification

All cases together



Portfolio Characteristics (General n asset case)

- The portfolio expected return is always the share-weighted sum of the expected returns for the assets included in the portfolio

Portfolio Variance (n>2)

- Given a vector of portfolio weights and the matrix of variances and covariances, the portfolio variance is computed by adding all cells by the product of the row weight, the column weight and the cell variance or covariances

	x_1	x_2	x_3		x_n
x_1	σ_{11}	σ_{12}	σ_{13}	…	σ_{1n}
x_2	σ_{21}	σ_{22}	σ_{23}	…	σ_{2n}
x_3	σ_{31}	σ_{32}	σ_{33}	…	σ_{3n}
	.	.	.		.
	.	.	.		.
	.	.	.		.
x_n	σ_{n1}	σ_{n2}	σ_{n3}	…	σ_{nn}

Set of All Portfolios of Risky Assets

Effect of Increasing # of Assets in an Equally Weighted Portfolio

1. Start with our equation for variance:

$$\sigma_p^2 = \sum_{i=1}^{N} w_i^2 \sigma_i^2 + \sum_{i=1}^{N} \sum_{\substack{j=1 \\ i \neq j}}^{N} w_i w_j cov(\tilde{r}_i, \tilde{r}_j)$$

2. Then make the simplifying assumption that $w_i = 1/N$ for all assets:

$$\sigma_p^2 = \left(\frac{1}{N^2}\right) \sum_{i=1}^{N} \sigma_i^2 + \sum_{i=1}^{N} \left(\frac{1}{N^2}\right) \sum_{\substack{j=1 \\ i \neq j}}^{N} cov(\tilde{r}_i, \tilde{r}_j)$$

3. The average variance and covariance of the securities are:

$$\overline{\sigma^2} = \left(\frac{1}{N}\right) \sum_{i=1}^{N} \sigma_i^2 \qquad \overline{cov} = \frac{1}{N(N-1)} \sum_{\substack{j=1 \\ i \neq j}}^{N} cov(\tilde{r}_i, \tilde{r}_j)$$

1. Plugging these into our equation gives:

$$\sigma_p^2 = \left(\frac{1}{N}\right) \overline{\sigma^2} + \left(\frac{N-1}{N}\right) \overline{cov}$$

2. What happens as N becomes large?

$$\left(\frac{1}{N}\right) \to 0 \text{ and } \left(\frac{N-1}{N}\right) \to 1$$

3. *Only the average covariance matters for large portfolios.*
4. If the average covariance is zero, then the portfolio variance is close to zero for large portfolios.

- The component of risk that can be diversified away we call the *diversifiable* or *non-systematic* risk.
- Empirical Facts
  - ↪ The average (annual) return standard deviation is 49%
  - ↪ The average (annual) covariance between stocks is 0.037, and the average correlation is about 39%.
- Since the average covariance is positive, even a very large portfolio of stocks will be risky. We call the risk that cannot be diversified away the *systematic risk.*

What if a risk-free asset is available?

- Recall that the capital allocation line is the straight line through the risk-free asset and the risky asset
- Adding the risk-free asset generates a number of CAL choices
  – Which CAL to choose?

CAL with two risky assets

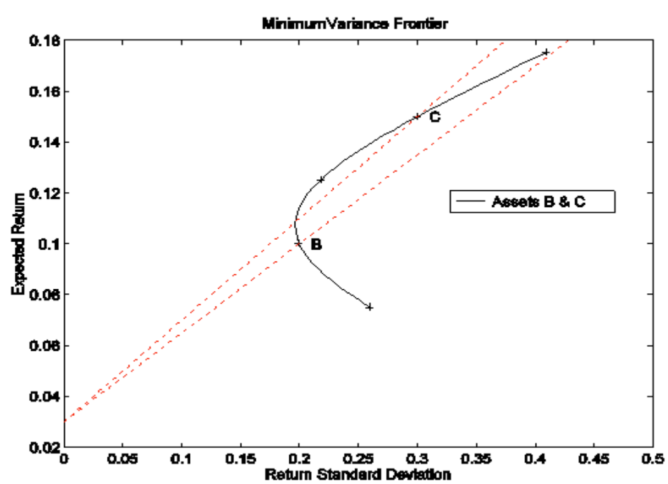
- Assume we can only trade in risk-free asset with 3% return and risk assets B and C where

Asset	$E(r)$	σ
B	10%	20%
C	15%	30%

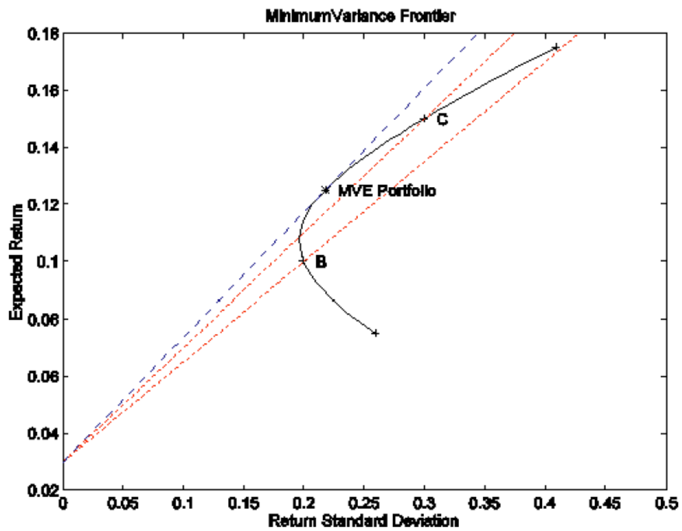
with correlation coefficient 0.5

- We can compute the minimum variance frontier created by combinations of B and C
  - The risk-free asset plus B, or the risk-free asset plus C gives us two possible CALs

CAL with two risky assets



The Mean-Variance Efficient (MVE) Portfolio



Finding the MVE Portfolio (Algebra)

Optimal Risky Portfolio

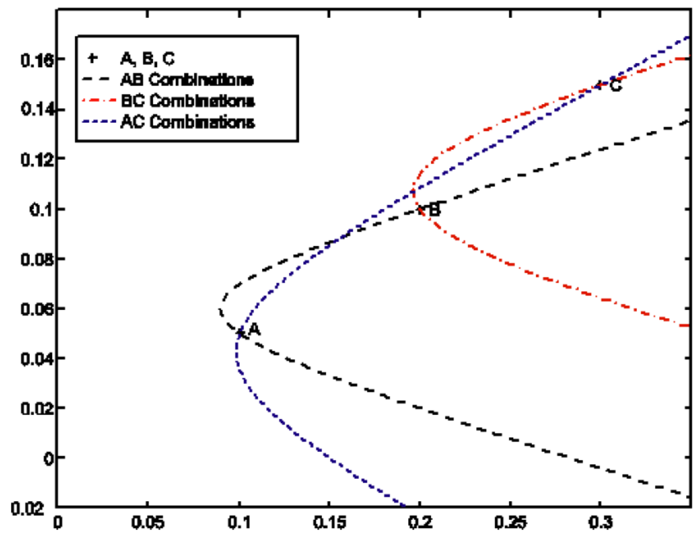
- Adding the risk free asset generates a number of CAL choices
- The feasible set of portfolios becomes more attractive
- We can identify an optimal risky portfolio which dominates all other risky portfolios (irrespective of risk preferences)
- The optimal (tangency/MVE) portfolio has the highest Sharpe ratio among all feasible portfolios (highest slope)

Optimal Portfolio with Many Risky Assets

Asset	$E(r)$	σ
A	5%	10%
B	10%	20%
C	15%	30%

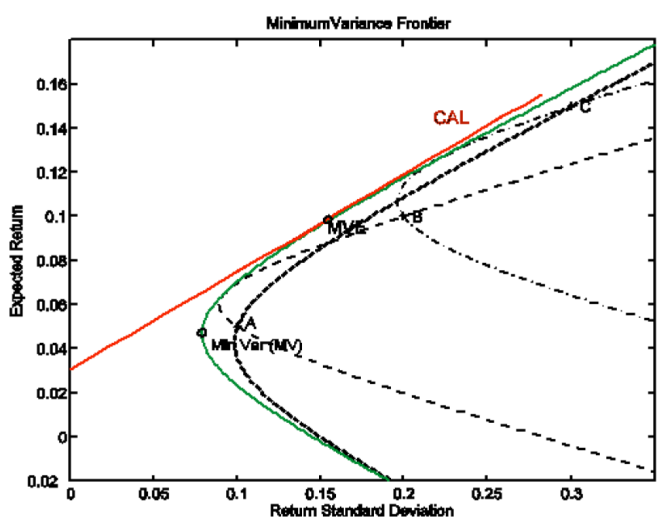
Correlations			
Assets	A	B	C
A	1.0	0.0	0.5
B	0.0	1.0	0.5
C	0.5	0.5	1.0

- What does the minimum variance frontier look like now?

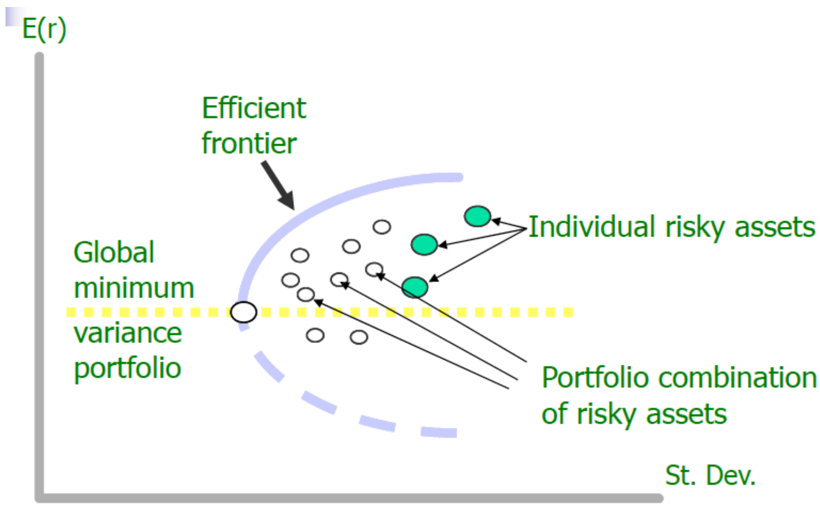


- We get the possible combinations. Can we do better?

Portfolios of Portfolios



The Case of N Risky Assets



The 'Efficient' Set of Risky Portfolios

- Among the set of all possible portfolios constructed from risky assets, the efficient portfolios give the minimum risk for given expected return or the maximum expected return for given risk
- Harry Markowitz developed a mathematical algorithm based on quadratic programming to determine the set of efficient portfolios

Optimal Portfolio Selection

1) Estimate expected return, variances-covariances matrix
2) The weights are determined by quadratic programming optimization with constraints based on either:
   – For given expected return, find the minimum variance
   – For given variance, find the maximum expected return

Note: You can do this by using Solver in Excel

$$\underset{w_i}{\text{Min}} \ \sigma_p^2 = \sum_{i=1}^{n} w_i^2 \sigma_i^2 + 2 \sum_{i=1}^{n} \sum_{j>i}^{n} w_i w_j \sigma_{ij}$$

$$\text{s.t.} \quad \sum_{i=1}^{n} w_i = 1$$

$$\sum_{i=1}^{n} w_i \mu_i = \bar{\mu}_p$$

$$w_i \geq 0$$

$$|w_i| < k$$

4) Trace out the minimum variance frontier by repeating the optimization for all levels of expected return

5) Graph the efficient frontier

6) With a risk-free asset, find CAL with highest Sharpe ratio, with the point of tangency with efficient frontier being the optimal risky portfolio

Note: We can also maximize the Sharpe ratio by introducing the risk free rate asset from the outset and find the portfolio with the steepest CAL

- no constraint on expected return or variance just the the portfolio weights sum to one
- won't get entire minimum-variance frontier

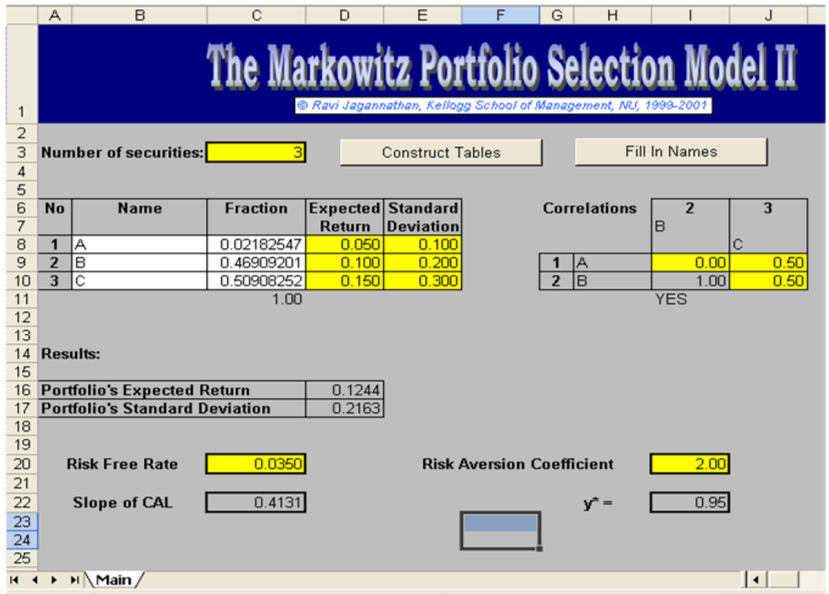
Example

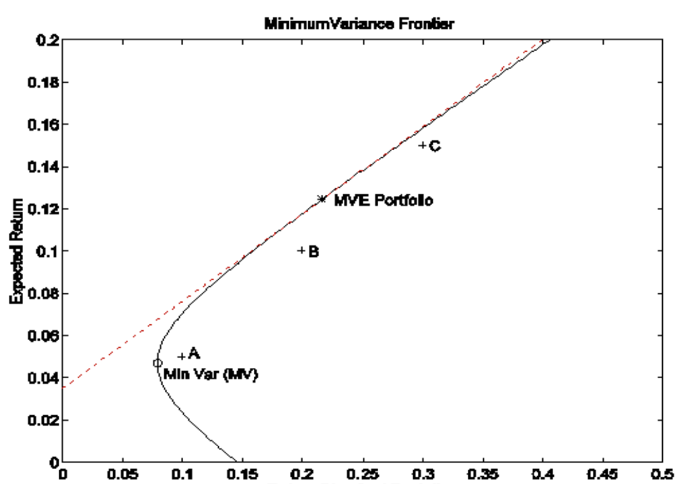
- Consider three risky assets, A, B, C with risk free return at 3.5%

Asset	$E(r)$	σ
A	5%	10%
B	10%	20%
C	15%	30%

Correlations			
Assets	A	B	C
A	1.0	0.0	0.5
B	0.0	1.0	0.5
C	0.5	0.5	1.0

- Refer to Markowitz.xls and tutorial on moodle





$$\mathbf{w}_{MVE} = \begin{bmatrix} 0.0218 \\ 0.4619 \\ 0.5091 \end{bmatrix} \qquad \begin{aligned} \bar{r}_{MVE} &= 0.1244 \\ \sigma_{MVE} &= 0.2163 \\ SR_{MVE} &= 0.4131 \end{aligned}$$

The Investor's Optimal Portfolio

- Lies on the expanded efficient portfolio locus
- The position depends on the investor's attitude towards risk ie. degree of risk aversion which influences the shape of the indifference curve

Example: Optimal Risky Portfolio

- Investing $100,000, risk-free = 0.5%
- Stock 1 has expected return 8%, stdev 12%
- Stock 2 has expected return 13%, stdev 20%
- The correlation between the two stock returns is zero.

Optimal Complete Portfolio

- Investing $100,000, A=8, risk-free = 0.5%

Two Fund Separation

- Key result in modern portfolio theory
- An investment problem can be broken into two parts
  - Find the optimal portfolio of risky securities
  - Find the best combination of the risk free asset and this optimal risky portfolio
- Separation property – the portfolio choice problem is separated into 2 independent tasks.
  - The 1st task (finding the optimal risky portfolio) is purely technical and is the same for all clients.
  - The 2nd task (capital allocation) depends on risk preference

Passive Strategy is Efficient

- Basic message: your risk/return tradeoff is improved by holding many assets with less than perfect correlation
- The optimal risky portfolio is the same for every investor, and is the market portfolio
- No need for stock selection
- Investors need only to adjust the mix of risk-free asset and the market portfolio based on risk aversion