Chapter 3: Linear Space (Vector Spaces): Part 3

1 Dimension

In the prior subsection we defined the basis of a vector space, and we saw that a space can have many different bases. We can find something about the bases that is uniquely associated with the space. This section shows that any two bases for a space have the same number of elements. So, with each space we can associate a number, the number of vectors in any of its bases.

Theorem 1.1. Let V be an n-dimensional vector space, and let $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\}$ be any basis.

(a) If a set in V has more than n vectors, then it is linearly dependent.

(b) If a set in V has fewer than n vectors, then it does not span V.

Theorem 1.2. In any finite-dimensional vector space, all of the bases have the **same** number of elements.

Remark: A vector space is finite-dimensional if it has a basis with only finitely many vectors.

Definition 1.1. The **dimension** of a finite-dimensional vector space V, denoted by $\dim(V)$, is defined to be the number of vectors in any of its bases for V.
The zero vector space is defined to have dimension zero.

Example 1.1. Dimension:

- Any basis for $\mathbb{R}^n$ has n vectors since the standard basis $\{\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_n\}$ has n vectors. Thus, this definition generalizes the most familiar use of term, that $\mathbb{R}^n$ is n-dimensional.
- The space P_n of polynomials of degree at most n has dimension $n+1$. E.g. one of its bases is $\{1, x, x^2, ..., x^n\}$.
- A trivial space is zero-dimensional since its basis is empty.

Example 1.2. (Exercise) No linearly independent set can have a size greater than the dimension of the enclosing space.

Poof Thm1.1

Example 1.3. Consider a linear system $\mathbf{Ax} = \mathbf{0}$ that has a general solution:

$$x_1 = -3r - 4s - 2t, \quad x_2 = r, \quad x_3 = -2s, \quad x_4 = s, \quad x_5 = t, \quad x_6 = 0,$$

where r, s, t are real numbers. Find a basis of the solution space for this linear system.

Theorem 1.3. Let S be a nonempty set of vectors in a vector space V.

(a) If S is a linearly independent set, and if v is a vector in V that is outside of $\text{span}(S)$, then the set $S \cup \{\mathbf{v}\}$ that results by inserting v into S is still linearly independent.

(b) If $\mathbf{v}$ is a vector in S that is expressible as a linear combination of other vectors in S, and if $S - \{\mathbf{v}\}$ denotes the set obtained by removing $\mathbf{v}$ from S, then S and $S - \{\mathbf{v}\}$ span the same space; that is,

$$\text{span}(S) = \text{span}(S - \{\mathbf{v}\})$$

Note: The above implies that
(a) Any linearly independent set can be expanded to make a basis.
(b) Any spanning set can be shrunk to a basis.

Theorem 1.4. Let $\mathbf{V}$ be an n-dimensional vector space, and let S be a set in V with exactly n vectors. Then,
(a) S is a basis for V **if and only if** S spans V.
(b) S is a basis for V **if and only if** S is linearly independent.

Example 1.4. Explain why the vectors

$$\mathbf{v}_1 = (2, 0, -1), \quad \mathbf{v}_2 = (4, 0, 7), \quad \mathbf{v}_3 = (-1, 1, 4)$$

form a basis for $\mathbb{R}^3$.

If W is a subspace of a finite-dimensional vector space V, then:

(a) W is finite-dimensional.

(b) $\dim(W) \le \dim(V)$.

(c) $W = V$ if and only if $\dim(W) = \dim(V)$.

Proof
(a) Exercise
(b) & (c)

2 Change of Basis

2.1 Coordinate Maps

Recall: If $S = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\}$ is a basis for a finite-dimensional vector space V, and if

$$(\mathbf{v})_S = (c_1, c_2, ..., c_n)$$

is the **coordinate vector** of $\mathbf{v}$ relative to S, i.e. $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$
then the mapping

$$\mathbf{v} \to (\mathbf{v})_S$$

creates a connection (a one-to-one correspondence) between vectors in the general vector space V and vectors in the Euclidean vector space $\mathbb{R}^n$. We call the above map as the **coordinate map** relative to S from V to $\mathbb{R}^n$.

Note: We can write coordinate vectors in the matrix form: $[v]_S = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$

Example 2.1. Suppose $B = \{\mathbf{u}_1, \mathbf{u}_2\}$ and $B = \{\mathbf{u}'_1, \mathbf{u}'_2\}$ be two bases for $\mathbb{R}^2$ and suppose $[u'_1]_B = \begin{bmatrix} a \\ b \end{bmatrix}$, $\quad [u'_1]_B = \begin{bmatrix} c \\ d \end{bmatrix}$. Let $\mathbf{v}$ be any vector in V, and let $[\mathbf{v}]_{B'} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$.
(a) Find $[\mathbf{v}]_B$.
(b) If we want to write $[\mathbf{v}]_B$ in the form $[\mathbf{v}]_B = \mathbf{P}[\mathbf{v}]_{B'}$, determine the matrix $\mathbf{P}$.

Change-of-Basis Let $B = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_n\}$ and $B' = \{\mathbf{u}'_1, \mathbf{u}'_2, ..., \mathbf{u}'_n\}$ be bases of a vector space $\mathbf{V}$.
- If we change the basis from an old basis B to a new basis B', then for each vector $\mathbf{v}$ in V, the old coordinate vector $[\mathbf{v}]_B$ is related to the new coordinate vector $[\mathbf{v}]_{B'}$ by the equation

$$[\mathbf{v}]_B = \mathbf{P}[\mathbf{v}]_{B'}$$

where the columns of P are the coordinate vectors of the new basis vectors relative to the old basis; that is,

$$\mathbf{P} = \mathbf{P}_{B' \to B} = \left[\ [u'_1]_B \mid [u'_2]_B \mid \cdots \mid [u'_n]_B\ \right].$$

Note: The matrix $\mathbf{P} = \mathbf{P}_{B' \to B}$ is called **transition matrix from B' to B.**
- If we change the basis from an old basis B' to a new basis B, then for each vector $\mathbf{v}$ in V, the old coordinate vector $[\mathbf{v}]_{B'}$ is related to the new coordinate vector $[\mathbf{v}]_B$ by the equation

$$[\mathbf{v}]_{B'} = \mathbf{P}[\mathbf{v}]_B$$

where the columns of P are the coordinate vectors of the new basis vectors relative to the old basis; that is,

$$\mathbf{P} = \mathbf{P}_{B \to B'} = \left[\ [u_1]_{B'} \mid [u_2]_{B'} \mid \cdots \mid [u_n]_{B'}\ \right].$$

Note: The matrix $\mathbf{P} = \mathbf{P}_{B \to B'}$ is called **transition matrix from B to B'.**

Example 2.2. Consider the bases $B = \{\mathbf{u}_1, \mathbf{u}_2\}$ and $B = \{\mathbf{u}'_1, \mathbf{u}'_2\}$ for $\mathbb{R}^2$, where

$$\mathbf{u}_1 = (1, 0), \mathbf{u}_2 = (0, 1), \mathbf{u}'_1 = (1, 1), \mathbf{u}'_2 = (2, 1)$$

(a) Find the transition matrix $\mathbf{P}_{B' \to B}$ from B' to B.
(b) Find the transition matrix $\mathbf{P}_{B \to B'}$ from B to B'.
(c) If $[\mathbf{v}]_{B'} = [-3, 5]^T$, find $[\mathbf{v}]_B$.

(Cont')

Remark:
If B and B' are bases for a finite-dimensional vector space V, then

$$\mathbf{P}_{B\to B'}\mathbf{P}_{B'\to B} = \mathbf{P}_{B\to B} = \mathbf{I}$$

I.e., since the net effect of the two operations is to leave each coordinate vector unchanged, we are led to conclude that $\mathbf{P}_{B\to B}$ must be the identity matrix

Theorem 2.1. If $\mathbf{P}$ is the transition matrix from a basis B' to a basis B for a finite dimensional vector space V, then

- $\mathbf{P}$ is invertible and
- $\mathbf{P}^{-1}$ is the transition matrix from B to B'.

A Procedure for Computing $\mathbf{P}_{B\to B'}$

Step 1. Form the matrix $[B'|B]$.

Step 2. Use elementary row operations to reduce the matrix in Step 1 to reduced row echelon form.

Step 3. The resulting matrix will be $[\mathbf{I}|\mathbf{P}_{B\to B'}]$.

Step 4. Extract the matrix $\mathbf{P}_{B\to B'}$ from the right side of the matrix in Step 3.

$$[B' \mid B] \quad \Rightarrow \text{Elementary row operations} \Rightarrow \quad [\,\mathbf{I} \mid \mathbf{P}_{B\to B'}]$$

[New Basis | Old Basis] $\Rightarrow$ Elementary row operations $\Rightarrow$ [**I** | Transition from old to new]

Example 2.3. Consider the bases $B = \{\mathbf{u}_1, \mathbf{u}_2\}$ and $B = \{\mathbf{u}'_1, \mathbf{u}'_2\}$ for $\mathbb{R}^2$, where

$$\mathbf{u}_1 = (1,0), \mathbf{u}_2 = (0,1), \mathbf{u}'_1 = (1,1), \mathbf{u}'_2 = (2,1)$$

(a) Find the transition matrix $\mathbf{P}_{B'\to B}$ from B' by using the above procedure. to B.
(b) Find the transition matrix $\mathbf{P}_{B\to B'}$ from B to B'.

Theorem 2.2. Let $B = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_n\}$ be any basis for the vector space Rn and let $S = \{\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_n\}$ be the standard basis for $\mathbb{R}^n$. If the vectors in these bases are written in column form, then

$$\mathbf{P}_{B\to S} = [\mathbf{u}_1 \mid \mathbf{u}_2 \mid \cdots \mid \mathbf{u}_n].$$