Solution: Exercise 8 (Part 1)

Techniques of Integration: Substitution rule/Integration by parts/Trigonometric integrals/Partial fractions

1. Evaluate the integrals.

   (a) $\int \sin^3(x)\cos^2(x)\ dx$
   Solution: Let $u = \cos(x)$. Then $du = -\sin(x)dx$ and

$$\begin{aligned}\int \sin^3(x)\cos^2(x)\ dx &= \int \sin^2(x)\cos^2(x)\sin(x)\ dx = \int (1-\cos^2(x))\cos^2(x)\sin(x)\ dx \\ &= \int (1-u^2)u^2(-du) = \int -u^2+u^4 du = -\frac{u^3}{3}+\frac{u^5}{5}+C \\ &= -\frac{\cos^3(x)}{3}+\frac{\cos^5(x)}{5}+C.\end{aligned}$$

   (b) $\int_0^{\pi/2} \cos^5(x)\ dx$
   Solution: Let $u = \sin(x)$. Then $du = \cos(x)dx$ and
   when $x = 0 \Rightarrow u = 0$ and $x = \pi/2 \Rightarrow u = 1$,

$$\begin{aligned}\int_0^{\pi/2} \cos^5(x)\ dx &= \int_0^{\pi/2} [\cos^2(x)]^2\cos(x)\ dx = \int_0^{\pi/2} [1-\sin^2(x)]^2\cos(x)\ dx \\ &= \int_0^1 [1-u^2]^2 du = \int_0^1 1-2u^2+u^4 du = \left[u-\frac{2}{3}u^3+\frac{u^5}{5}\right]_0^1 \\ &= \left[1-\frac{2}{3}+\frac{1}{5}\right]-0 = \frac{8}{15}.\end{aligned}$$

2. Evaluate the integrals (integration by parts).

   (a) $\int x\ln(x)\ dx$
   Solution: Let $u = \ln(x)$, $dv = xdx \Rightarrow du = \frac{1}{x}dx$, $v = \frac{1}{2}x^2$.
   From integration by parts: $\int udv = uv - \int vdu$,

$$\int x\ln(x)\ dx = \ln(x)\frac{1}{2}x^2 - \int \frac{1}{2}x^2\frac{1}{x}dx = \frac{1}{2}x^2\ln(x) - \frac{1}{2}\int xdx = \frac{1}{2}x^2\ln(x) - \frac{x^2}{4}dx + C.$$

   (b) $\int xe^{2x}\ dx$
   Solution: Let $u = x$, $dv = e^{2x}dx \Rightarrow du = dx$, $v = \frac{1}{2}e^{2x}$.
   From integration by parts: $\int udv = uv - \int vdu$,

$$\int xe^{2x}\ dx = \frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x}dx = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C.$$

   (c) $\int \sin^{-1}(x)\ dx$

Solution: Let $u = \sin^{-1}(x)$, $dv = dx \Rightarrow du = \frac{1}{\sqrt{1-x^2}}dx$, $v = x$.
From integration by parts: $\int udv = uv - \int vdu$,

$$\begin{aligned}\int \sin^{-1}(x)\ dx &= x\sin^{-1}(x) - \int x\frac{1}{\sqrt{1-x^2}}dx \\ &= x\sin^{-1}(x) - \int (1-x^2)^{-1/2}\left(-\frac{1}{2}\right)d(1-x^2) \\ &= x\sin^{-1}(x) - \left(-\frac{1}{2}\right)\frac{(1-x^2)^{1/2}}{1/2} + C = x\sin^{-1}(x) + \sqrt{1-x^2} + C.\end{aligned}$$

(d) $\int x^2\cos(3x)\ dx$

Solution: Let $u = x^2$, $dv = \cos(3x)\ dx \Rightarrow du = 2xdx$, $v = \frac{1}{3}\sin(3x)$.
From integration by parts: $\int udv = uv - \int vdu$,

$$\int x^2\cos(3x) = \frac{1}{3}x^2\sin(3x) - \frac{2}{3}\underbrace{\int x\sin(3x)dx}_{(*)}$$

and from (*), we will apply again integration by parts.
Let $U = x$ and $dV = \sin(3x)dx \Rightarrow dU = dx$ and $V = -\frac{1}{3}\cos(3x)$

$$\int x\sin(3x)dx = -\frac{x}{3}\cos(3x) + \int\frac{1}{3}\cos(3x)dx = -\frac{x}{3}\cos(3x) + \frac{1}{9}\sin(3x) + \hat{C}.$$

By substituting (*) and letting $C = -\frac{2}{3}\hat{C}$, we have

$$\begin{aligned}\int x^2\cos(3x) &= \frac{1}{3}x^2\sin(3x) - \frac{2}{3}\left[-\frac{x}{3}\cos(3x) + \frac{1}{9}\sin(3x) + \hat{C}\right] \\ &= \frac{1}{3}x^2\sin(3x) + \frac{2}{3}\cos(3x) - \frac{2}{27}\sin(3x) + C.\end{aligned}$$

(e) $\int \sin(\ln(x))\ dx$

Solution: Let $u = \sin(\ln(x))$, $dv = dx \Rightarrow du = \cos(\ln(x))\frac{1}{x}dx$, $v = x$.
From integration by parts: $\int udv = uv - \int vdu$,

$$\int \sin(\ln(x))\ dx = x\sin(\ln(x)) - \int x\cos(\ln(x))\frac{1}{x}dx = x\sin(\ln(x)) - \underbrace{\int\cos(\ln(x))dx}_{(*)}$$

and from (*), we will apply again integration by parts.
Let $U = \cos(\ln(x))$ and $dV = dx \Rightarrow dU = -\sin(\ln(x))\frac{1}{x}dx$ and $V = x$

$$\int\cos(\ln(x))dx = x\cos(\ln(x)) + \int x\sin(\ln(x))\frac{1}{x}dx = x\cos(\ln(x)) + \int\sin(\ln(x))dx$$

By substituting (*) and letting $C = \frac{1}{2}\hat{C}$, we have

$$\begin{aligned}
\int \sin(\ln(x))\ dx &= x\sin(\ln(x)) - \left[x\cos(\ln(x)) + \int \sin(\ln(x))dx\right] \\
2\int \sin(\ln(x))\ dx &= x\sin(\ln(x)) - x\cos(\ln(x)) + \hat{C} \\
\int \sin(\ln(x))\ dx &= \frac{1}{2}x\sin(\ln(x)) - \frac{1}{2}x\cos(\ln(x)) + C.
\end{aligned}$$

(f) $\int e^{2t}\sin(3t)\ dt$

Solution:

Let $u = \sin(3t)$, $dv = e^{2t}dt \Rightarrow du = 3\cos(3t)dt$, $v = \frac{1}{2}e^{2t}$.

From integration by parts: $\int udv = uv - \int vdu$,

$$\int e^{2t}\sin(3t)dt = \frac{1}{2}e^{2t}\sin(3t) - \frac{3}{2}\underbrace{\int e^{2t}\cos(3t)dt}_{(*)}$$

and from (*), we will apply again integration by parts.

Let $U = \cos(3t)$ and $dV = e^{2t}dt \Rightarrow dU = -3\sin(3t)dt$ and $V = \frac{1}{2}e^{2t}$

$$\int e^{2t}\cos(3t)dt = \frac{1}{2}e^{2t}\cos(3t) + \frac{3}{2}\underbrace{\int e^{2t}\sin(3t)dt}_{(*)}$$

By substituting (*),

$$\begin{aligned}
\int e^{2t}\sin(3t)dt &= \frac{1}{2}e^{2t}\sin(3t) - \frac{3}{2}\left[\frac{1}{2}e^{2t}\cos(3t) + \frac{3}{2}\int e^{2t}\sin(3t)dt\right] \\
(1+\frac{9}{4})\int e^{2t}\sin(3t)dt &= \frac{1}{2}e^{2t}\sin(3t) - \frac{3}{4}e^{2t}\cos(3t) \\
\int e^{2t}\sin(3t)dt &= \frac{4}{13}\left[\frac{1}{2}e^{2t}\sin(3t) - \frac{3}{4}e^{2t}\cos(3t)\right] + C,
\end{aligned}$$

(g) $\int_0^{\pi/2} x\cos(2x)\ dx$

Solution: Let $u = x$, $dv = cos(2x)dx \Rightarrow du = dx$, $v = \frac{1}{2}\sin(2x)$.

From integration by parts: $\int udv = uv - \int vdu$,

$$\int_0^{\pi/2} x\cos(2x)\ dx = \left[\frac{x}{2}\sin(2x)\right]_0^{\pi/2} - \int_0^{\pi/2}\frac{1}{2}\sin(2x)\ dx = 0 + \frac{1}{4}\left[\cos(2x)\right]_0^{\pi/2} = \frac{1}{4}(-1-1) = -\frac{1}{2}.$$

(h) $\int_0^1 (x^2+1)e^{-x}\ dx$

Solution: Let $u = x^2+1$, $dv = e^{-x}dx \Rightarrow du = 2xdx$, $v = -e^{-x}$.

From integration by parts: $\int udv = uv - \int vdu$,

$$\int_0^1 (x^2+1)e^{-x}\ dx = \left[e^{-x}(x^2+1)\right]_0^1 + \int_0^1 2xe^{-x}\ dx = -2e^{-1} + 1 + 2\int_0^1 xe^{-x}\ dx.$$

Then $\int_0^1 xe^{-x}\ dx$ can be computed by integration by parts again: Let $U = x, dV = e^{-x}dx \Rightarrow dU = dx, V = -e^{-x}$,

$$\int_0^1 xe^{-x}\ dx = \left[-xe^{-x}\right]_0^1 + \int_0^1 e^{-x}dx = -2e^{-1} + 1.$$

That is,

$$\int_0^1 (x^2+1)e^{-x}\ dx = -2e^{-1} + 1 + 2(-2e^{-1}+1) = -6e^{-1} + 3.$$

(i) $\int_1^4 \ln(\sqrt{x})\ dx$
Solution:

$$\int_1^4 \ln(\sqrt{x})\ dx = \int_1^4 \frac{1}{2}\ln(x)\ dx = \frac{1}{2}\int_1^4 \ln(x)\ dx = \frac{1}{2}[x\ln(x) - x]_1^4 = 2\ln(4) - \frac{3}{2}.$$

We have used integration by parts: $\int udv = uv - \int vdu$, Let $u = \ln(x)$, $dv = dx \Rightarrow du = \frac{1}{x}dx$, $v = x$.

$$\int \ln(x)\ dx = x\ln(x) - \int x\frac{1}{x}\ dx = x\ln(x) - x + C.$$