EE320 (1/2013)

INTRODUCTORY MATHEMATICAL ECONOMICS

OPTIMIZATION UNDER EQUALITY CONSTRAINTS

Topics

- Effects of a constraint
- Finding the stationary (optimal) values
  - Elimination method
  - Lagrange multiplier
  - *n*-variable and multi-constraint cases
- Second-order conditions for constrained optimization
- Economic applications
- Extensions: n-variable and multiconstraint cases

Effects of a Constraint

- In the previous topic, we studied optimization problems, where the choice variables can be chosen freely.
- Example: A profit-maximizing firm's objective is:

$$\pi = PQ - [C(Q_1) + C(Q_2)]$$

where $Q = Q_1 + Q_2$.

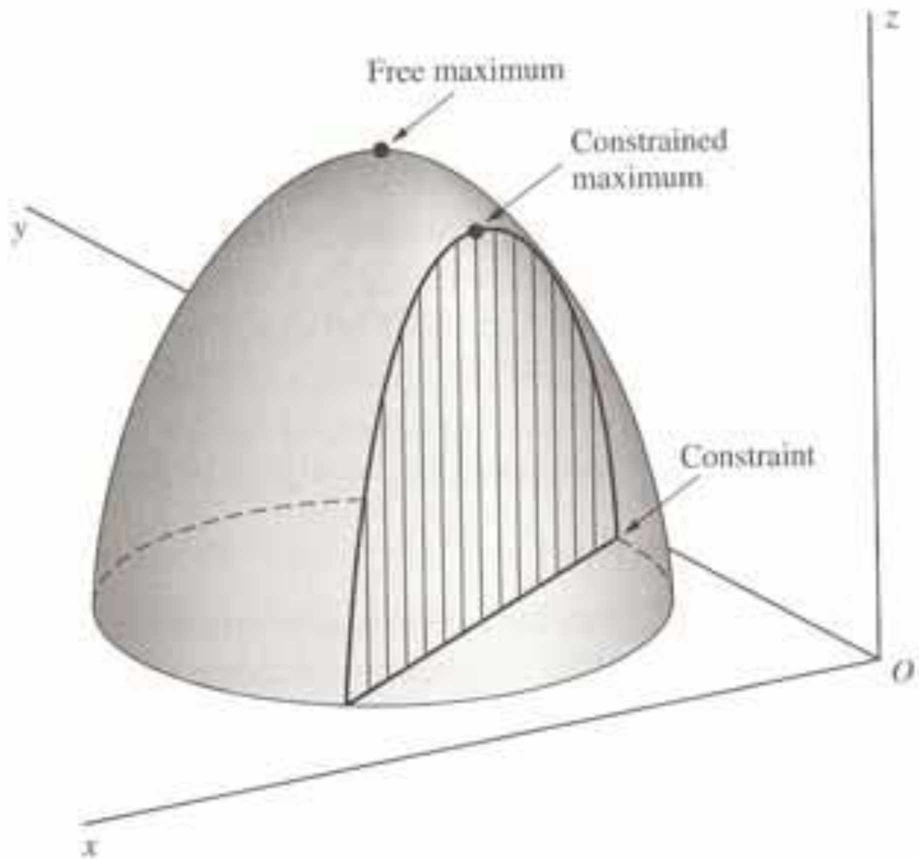
  - Q_1 and Q_2 can be chosen freely in order to maximize profit.

    ➔ Here, Q_1^* and Q_2^* are *free optimal values*.

  - Question: If there is a *production constraint*, e.g. $Q_1 + Q_2 = 900$, then how would the firm change its behavior?
  - ➔ Q_1^{**} and Q_2^{**} will now become the *constrained optimal values*.

Effects of a Constraint

- **Figure:** "Free extremum" vs. "Constrained extremum"



FINDING THE STATIONARY VALUES

Two Choice Variables with One Equality Constraint

- Use *elimination* (or *substitution*) method

Example: $$\max_{x_1,x_2} U = x_1 x_2 + 2x_1$$

subject to $$4x_1 + 2x_2 = 60$$

Two Choice Variables with One Constraint

➢Use *Lagrange-multiplier* method

Consider the following maximization problem.

$$\max_{x,y} f(x,y)$$

subject to $g(x,y)=c$.

Define the Lagrangian function by

$$L(x,y,\lambda)=f(x,y)+\lambda[c-g(x,y)]$$

F.O.N.C.

$$L_x(x,y)=f_x(x,y)-\lambda g_x(x,y)=0$$

$$L_y(x,y)=f_y(x,y)-\lambda g_y(x,y)=0$$

$$L_\lambda(x,y)=c-g(x,y)=0$$

Example 1: 2 choice variables & 1 constraint

- Find the optimal values for the following function.

$$U = x_1 x_2 + 2x_1$$

subject to $4x_1 + 2x_2 = 60$

Step 1- Set up the Lagrangian function:

$$L(x_1, x_2, \lambda) = x_1 x_2 + 2x_1 + \lambda[60 - 4x_1 - 2x_2]$$

Step 2 - Find FONC:

Example 2: 2 choice variables & 1 constraint

- Find the optimal values for the following function:

$$f(x_1, x_2) = x_1^2 + x_2^2$$

subject to $g(x_1, x_2) = x_1 + 4x_2 = 2$

Step 1- Set up the Lagrangian function:

$$L(x_1, x_2, \lambda) = x_1^2 + x_2^2 + \lambda[2 - x_1 - 4x_2]$$

Step 2 - Find FONC:

Interpretation of Lagrange Multiplier

- Consider the problem

$$\max_{x,y} (\min) f(x,y)$$
$$\text{subject to } g(x,y) = c$$

- ➢ Solutions to this problem: *x*(c), y*(c),* and *f*(c).*
- ➢ *f*(c)* is called optimal value function.

- We can show that $\dfrac{df^*(c)}{dc} = \lambda(c)$

- ➢ The Lagrange multiplier λ is the rate at which the optimal value of the objective function changes with respect to changes in the constraint constant, c.
- ➢ In economics, λ is called a "shadow price".

Interpretation of Lagrange Multiplier

- In economic applications, *c* denotes the available stock of some resources, *f(x, y)* denotes utility or profit.
- *λ(c)dc* is the change in utility or profit that can be obtained from *dc* units more of the resource.
- Thus, *λ* is called a shadow price of the resource.
- Examples:
  - For cost minimization problem, $\lambda = \frac{dC}{dQ_0} = MC$
  - For utility maximization problem, $\lambda = \frac{dU}{dY_0}$ = MU of income
  - For profit maximization problem, $\lambda = \frac{d\pi}{dQ_0}$
  - For output maximization problem, $\lambda = \frac{dQ}{dC_0}$

SECOND-ORDER CONDITIONS FOR CONSTRAINED OPTIMIZATION

Second-Order Total Differential

- From the constraint $g(x, y) = c$, $dg = g_x dx + g_y dy = 0$.

  ➔ $dy = -(g_x / g_y)dx$

- From the objective function $z = f(x, y)$, $dz = f_x dx + f_y dy$.

  $d^2z =$

➢ $$d^2z = \left(f_{xx} - \frac{f_y}{g_y} g_{xx} \right) dx^2 + 2\left(f_{xy} - \frac{f_y}{g_y} g_{xy} \right) dxdy + \left(f_{yy} - \frac{f_y}{g_y} g_{yy} \right) dy^2$$

Second-Order Total Differential

- Recall the FONC for $L(x, y, \lambda) = f(x, y) + \lambda[c - g(x, y)]$

$$L_x(x, y) = f_x(x, y) - \lambda g_x(x, y) = 0$$

$$L_y(x, y) = f_y(x, y) - \lambda g_y(x, y) = 0$$

- By partially differentiating the FONC, the second derivatives are:

$$L_{xx}$$

$$L_{xy}$$

$$L_{yy}$$

➢ Thus, d^2z can be rewritten as:

$$d^2z = L_{xx}dx^2 + L_{xy}dxdy$$
$$+ L_{yx}dydx + L_{yy}dy^2$$

Second-Order Conditions

- The second-order sufficient conditions are:
  - For *minimum* of z: *d^2z is positive definite*, subject to $dg = 0$
  - For *maximum* of z: *d^2z is negative definite*, subject to $dg = 0$

  where $dg = g_x dx + g_y dy = 0$.

- One can show the sign definiteness of d^2z can be determined from the following criterion:

$$d^2z \text{ is } \begin{Bmatrix} positive\_definite \\ negative\_definite \end{Bmatrix} \text{ subject to } g_x dx + g_y dy = 0$$

$$\text{iff } \begin{vmatrix} 0 & g_x & g_y \\ g_x & L_{xx} & L_{xy} \\ g_y & L_{yx} & L_{yy} \end{vmatrix} \begin{cases} < 0 \\ > 0 \end{cases}$$

Second-Order Conditions

- The symmetric determinant $|\overline{H}| = \begin{vmatrix} 0 & g_x & g_y \\ g_x & L_{xx} & L_{xy} \\ g_y & L_{yx} & L_{yy} \end{vmatrix}$ is called the *Bordered Hessian*.

- Summary:

For a stationary value of $L(x,y) = f(x,y) + \lambda[c - g(x,y)]$

The second-order sufficient conditions are:

For *maximum* of z, the *bordered Hessian is positive* ($|\overline{H}| > 0$);

For *minimum* of z, the *bordered Hessian is negative* ($|\overline{H}| < 0$).

Example

- Determine whether the extremum of the objective function

$$U = x_1 x_2 + 2x_1 \quad \text{subject to} \quad 4x_1 + 2x_2 = 60$$

gives a minimum or a maximum.

Example

- Determine whether the extremum of the objective function

$$f\left(x_1, x_2\right) = x_1^2 + x_2^2 \quad \text{subject to} \quad g(x_1, x_2) = x_1 + 4x_2 = 2$$

gives a minimum or a maximum.

ECONOMIC APPLICATIONS OF CONSTRAINED OPTIMIZATION

Utility Maximization Problem

- Objective function: $\max_{Q_1, Q_2} U = U(Q_1, Q_2)$
- Subject to: $Y_0 = p_1 Q_1 + p_2 Q_2$

➢ Lagrangian function:

Utility Maximization Problem

- Illustration of an Indifference Curve

Utility Maximization Problem

- Second-order sufficient condition

Utility Maximization Problem

- Example: Suppose that $U = 50A + 200B - 0.5A^2 - 2.5B^2$.
  Find A and B that maximize the utility given that $P_A = 10$, $P_B = 5$, and $Y = 490$.

Expenditure Minimization

- Objective function: $\underset{Q_1,Q_2}{Min} E = p_1 Q_1 + p_2 Q_2$
- Subject to: $\overline{U} = U(Q_1, Q_2)$

➢ Lagrangian function:

Expenditure Minimization

- Example: Suppose that $U = 50A + 200B - 0.5A^2 - 2.5B^2$, and the consumer would like to have a utility fixed at U_0. Find A and B that minimize the expenditure given that $P_A = 10$ and $P_B = 5$.

Profit Maximization Problem

- Objective function: $\max\limits_{Q_1,Q_2} \pi = p_1 Q_1 + p_2 Q_2 - c(Q_1, Q_2)$
- Subject to: $Q_0 = Q_1 + Q_2$

➢ Lagrangian function:

Profit Maximization Problem

- Example: Suppose that P_1 = 80, P_2 = 100, $TC = 100 + 0.1Q_1^2 + 0.2Q_2^2$

And $Q_1 + Q_2 = 325$. Find Q_1 and Q_2 that maximize the profit.

Output Maximization Problem

- Objective function: $\max_{K,L} Q = Q(K,L)$
- Subject to: $C_0 = wL + rK$

➢ Lagrangian function:

Output Maximization Problem

- Example: Suppose $Q = KL$, w = 6, r = 10, C_0 = 60. Find K and L that maximizes output.

Cost Minimization

- Objective function: $\underset{K,L}{Min}\, C = wL + rK$
- Subject to: $Q_0 = f(K, L)$

➢ Lagrangian function:

Cost Minimization

- Graph: Expansion path

Cost Minimization

- Example: Suppose $Q = KL$. A firm will produce 15 units of the good. Find K and L that minimize total cost given that w = 6, r =10.

EXTENSION: N-CHOICE VARIABLE AND MULTI-CONSTRAINT CASES

Optimization with Equality Constraints:

n-Variable and One Equality Constraint

- The objective function

$$z = f\left(x_1, x_2, \ldots, x_n\right)$$

Subject to the constraint $g\left(x_1, x_2, \ldots, x_n\right) = c$

The Lagrangian function:

$$L(x_1, \ldots, x_n, \lambda) = f(x_1, \ldots, x_n) + \lambda[c - g(x_1, \ldots, x_n)]$$

FONC:

$$L_1 = f_1(x_1, \ldots, x_n) - \lambda g_1(x_1, \ldots, x_n) = 0$$

$$\vdots$$

$$L_n = f_n(x_1, \ldots, x_n) - \lambda g_n(x_1, \ldots, x_n) = 0$$

$$L_\lambda(x_1, \ldots, x_n) = c - g(x_1, \ldots, x_n) = 0$$

Second-Order Conditions:
n-Variable and One Equality Constraint

- The objective function $z = f(x_1, x_2, \ldots, x_n)$

  Subject to $g(x_1, x_2, \ldots, x_n) = c$

➢ Bordered Hessian:

$$|\overline{H}| = \begin{vmatrix} 0 & g_1 & g_2 & \cdots & g_n \\ g_1 & L_{11} & L_{12} & \cdots & L_{1n} \\ g_2 & L_{21} & L_{22} & \cdots & L_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ g_n & L_{n1} & L_{n2} & \cdots & L_{nn} \end{vmatrix}$$

➢ Bordered leading principal minors can be defined as:

$$|\overline{H}_2| = \begin{vmatrix} 0 & g_1 & g_2 \\ g_1 & L_{11} & L_{12} \\ g_2 & L_{21} & L_{22} \end{vmatrix}; |\overline{H}_3| = \begin{vmatrix} 0 & g_1 & g_2 & g_3 \\ g_1 & L_{11} & L_{12} & L_{13} \\ g_2 & L_{21} & L_{22} & L_{23} \\ g_3 & L_{31} & L_{32} & L_{33} \end{vmatrix}; \ldots; |\overline{H}_n| = |\overline{H}|$$

Second-Order Conditions:

n-Variable and One Equality Constraint

- The conditions for positive and negative definiteness of d^2z are:

d^2z is $\begin{Bmatrix} positive\_definite \\ negative\_definite \end{Bmatrix}$ subject to $dg = 0$ iff $\begin{cases} |\bar{H}_2|, |\bar{H}_3|, \ldots, |\bar{H}_n| < 0 \\ |\bar{H}_2| > 0; |\bar{H}_3| < 0; |\bar{H}_4| > 0; etc. \end{cases}$

Condition	Maximum	Minimum
First-order necessary condition	$L_\lambda = L_1 = L_2 = \ldots = L_n = 0$	$L_\lambda = L_1 = L_2 = \ldots = L_n = 0$
Second-order necessary condition*	$\|\bar{H}_2\| > 0; \|\bar{H}_3\| < 0;$ $\|\bar{H}_4\| > 0; \ldots; (-1)^n \|\bar{H}_n\| > 0$	$\|\bar{H}_2\|, \|\bar{H}_3\|, \ldots, \|\bar{H}_n\| < 0$

Cost Minimization

- Objective function: $\underset{K,L,T}{Min} C = wL + iK + rT$
- Subject to: $Q_0 = f(K,L,T)$

➢ Lagrangian function:

Optimization with Equality Constraints: *n*-Variable and Two Equality Constraints

- The objective function

$$z = f(x_1, x_2, \ldots, x_n)$$

Subject to the constraints $g(x_1, x_2, \ldots, x_n) = c$

$$h(x_1, x_2, \ldots, x_n) = d$$

➢ **The Lagrangian function:**

$$L = f(x_1, x_2, \ldots, x_n) + \lambda_1[c - g(x_1, x_2, \ldots, x_n)] + \lambda_2[d - h(x_1, x_2, \ldots, x_n)]$$

➢ **FONC:**

$$L_i = f_i(x_1, \ldots, x_n) - \lambda_1 g_i(x_1, \ldots, x_n) - \lambda_2 h_i(x_1, \ldots, x_n) = 0$$ for $i = 1, \ldots, n$

$$L_{\lambda_1}(x_1, \ldots, x_n) = c - g(x_1, \ldots, x_n) = 0$$

$$L_{\lambda_2}(x_1, \ldots, x_n) = d - h(x_1, \ldots, x_n) = 0$$

Optimization with Equality Constraints: *n*-Variable and *k* Equality Constraints

- The objective function

$$z = f(x_1, x_2, \ldots, x_n)$$

Subject to the constraints

$$g_1(x_1, x_2, \ldots, x_n) = c_1$$
$$g_2(x_1, x_2, \ldots, x_n) = c_2$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$g_k(x_1, x_2, \ldots, x_n) = c_k$$

➢ **The Lagrangian function:**

$$L = f(x_1, x_2, \ldots, x_n) + \lambda_1\left[c_1 - g^1(x_1, x_2, \ldots, x_n)\right] + \lambda_2\left[c_2 - g^2(x_1, x_2, \ldots, x_n)\right]$$
$$+ \ldots + \lambda_k\left[c_k - g^k(x_1, x_2, \ldots, x_n)\right]$$

Optimization with Equality Constraints: *n*-Variable and *k* Equality Constraints

- **FONC:**

$$\frac{\partial L}{\partial x_1} = f_1(x_1,\ldots,x_n) - \lambda_1 g_1^1(x_1,\ldots,x_n) - \ldots - \lambda_k g_1^k(x_1,\ldots,x_n) = 0$$

$$\frac{\partial L}{\partial x_2} = f_2(x_1,\ldots,x_n) - \lambda_1 g_2^1(x_1,\ldots,x_n) - \ldots - \lambda_k g_2^k(x_1,\ldots,x_n) = 0$$

....

$$\frac{\partial L}{\partial x_n} = f_n(x_1,\ldots,x_n) - \lambda_1 g_n^1(x_1,\ldots,x_n) - \ldots - \lambda_k g_n^k(x_1,\ldots,x_n) = 0$$

$$\frac{\partial L}{\partial \lambda_1} = c_1 - g^1(x_1,\ldots,x_n) = 0$$

$$\frac{\partial L}{\partial \lambda_2} = c_2 - g^2(x_1,\ldots,x_n) = 0$$

....

$$\frac{\partial L}{\partial \lambda_k} = c_k - g^k(x_1,\ldots,x_n) = 0$$

Second-Order Conditions: *n*-Variable and *k* Equality Constraints

The Lagrangian condition is:

$$L = f(x_1, x_2, \ldots, x_n) + \sum_{j=1}^{k} \lambda_j \left[c_j - g^j(x_1, x_2, \ldots, x_n)\right]$$

- Bordered Hessian:

$$|\overline{H}| \equiv \left| \begin{array}{cccc:cccc} 0 & 0 & \cdots & 0 & g_1^1 & g_2^1 & \cdots & g_n^1 \\ 0 & 0 & \cdots & 0 & g_1^2 & g_2^2 & \cdots & g_n^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & g_1^k & g_2^k & \cdots & g_n^k \\ \hdashline g_1^1 & g_1^2 & \cdots & g_1^k & L_{11} & L_{12} & \cdots & L_{1n} \\ g_2^1 & g_2^2 & \cdots & g_2^k & L_{21} & L_{22} & \cdots & L_{1n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ g_n^1 & g_n^2 & \cdots & g_n^k & L_{n1} & L_{n2} & \cdots & L_{nn} \end{array} \right|$$

Second-Order Conditions: *n*-Variable and *k* Equality Constraints

- The second-order sufficient conditions are stated in terms of the signs of the following *(n-k)* bordered leading principal minors:

$$\left|\bar{H}_{k+1}\right|, \left|\bar{H}_{k+2}\right|, \ldots, \left|\bar{H}_{n}\right| \qquad \text{where } \left|\bar{H}_{n}\right| = \left|\bar{H}\right|$$

- SOSC:

1) For a *maximum of* z, the bordered leading principal minors alternate in sign, and the sign of $\left|\bar{H}_{k+1}\right| = (-1)^{k+1}$.
2) For a *minimum of* z, all the bordered leading principal minors take the same sign, namely that of $(-1)^{k}$.

Example

- The objective function

$$z = f(x_1, x_2, x_3)$$

Subject to $g(x_1, x_2, x_3) = c$

$$h(x_1, x_2, x_3) = d$$

- The Lagrangian function:

$$L = f(x_1, x_2, x_3) + \lambda_1[c - g(x_1, x_2, x_3)] + \lambda_2[d - h(x_1, x_2, x_3)]$$

- FONC: $L_1 = L_2 = L_3 = L_{\lambda_1} = L_{\lambda_2} = 0$
- SOSC: 1) Max: $|H_3| < 0$

  2) Min: $|H_3| > 0$