

Mean Variance Optimization

R2, Chapter 5,6

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Recap

- Single Asset Risk vs. Portfolio Risk
- ρ is between -1 and 1
- σ^2 is a function of Σ and X
 - which X will make a minimum σ^2 ?



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Agenda

- Combination Of Risky Assets
- Efficient Frontier
- Calculation Technique
 - Minimum Variance Portfolio
 - Efficient Portfolio
 - Efficient Frontier



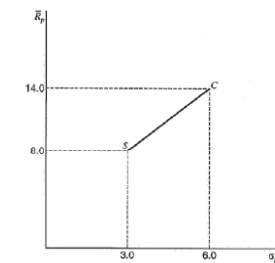
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Consider 2 Risky Assets Portfolio

When $\rho=1$

$E(R_C)=8, E(R_S)=14$

- The relationship between risk and return is basically a linear line
- No benefits from diversification at all



C=Coronel Motor; S=Separated Edison

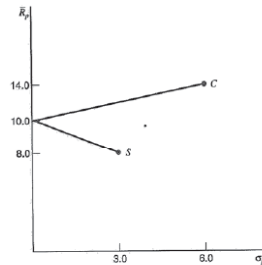


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Consider 2 Risky Assets Portfolio

When $\rho = -1$

- It is possible to find X which makes $\sigma_p = 0$

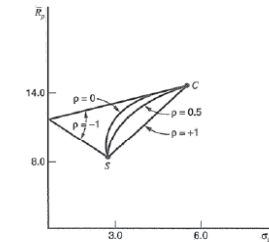


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Consider 2 Risky Assets Portfolio

When $\rho = 0, 0.5$

- Even though risk is not virtually eliminated, it is possible to find X which yields **lowest σ_p**
 - Thus called “Minimum Variance Portfolio”



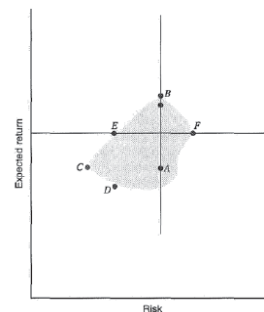
ρ closer to -1, σ_p closer to zero

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The Shape of Efficient Frontier

Risk-Return Space

- In theory, there are infinite number of possible combination of risky assets
- In practice, we can eliminate much of them and consider only some parts
- Efficient Rules**
 - Higher return, given same risk
 - Lower risk, given same return



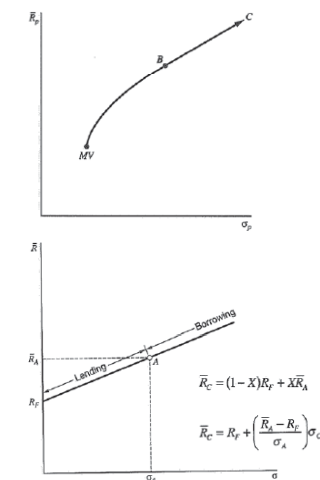
C is prefer to A, E is prefer to D and F, B is prefer to A
Hence, C-E-B are the only efficient portfolio

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The Shape of Efficient Frontier

Efficient Frontier

- Starting from minimum variance portfolio
 - Short Sale Not Allow, **MVB**
 - Short Sale Allow, **MVC**
- If a riskless asset is included in a risky portfolio, the efficient frontier become straight line and is called **“Capital Market Line”**



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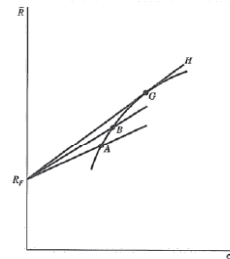
The Shape of Efficient Frontier

Notice That..

- $R_F B$ is superior to $R_F A$
- $R_F G$ is superior to $R_F B$
- **Why?**
- Risk Averse Investor
 - Move along $R_F - G$
- Risk Lover Investor
 - Move along $G - H$

Combination of a riskless asset and various risky portfolios

Separation Theorem, R2 – Page 87

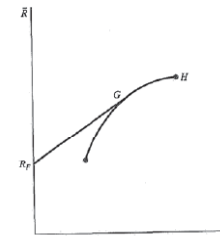


All investors will hold a combination of riskless asset (either lending or borrowing) and a portfolio G

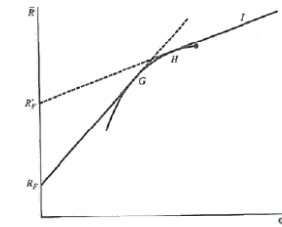
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The Shape of Efficient Frontier

Lending but not borrowing at riskless rate



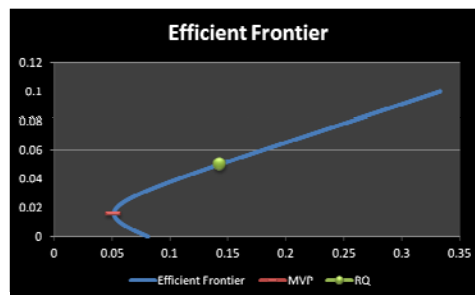
Different lending and borrowing rate



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Calculation Techniques

- Minimum Variance Portfolio
- Efficient Portfolio at given Expected Returns
- Efficient Frontier With Only Risky Assets



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Minimum Variance Portfolio

- Consider 3 Risky Asset Portfolio

$$\min_{x_A, x_B, x_C} \sigma_p^2 = x_A^2 \sigma_A^2 + x_B^2 \sigma_B^2 + x_C^2 \sigma_C^2 + 2x_A x_B \sigma_{AB} + 2x_A x_C \sigma_{AC} + 2x_B x_C \sigma_{BC}$$

$$\text{st. } x_A + x_B + x_C = 1$$

$$L(x_A, x_B, x_C, \lambda) = x_A^2 \sigma_A^2 + x_B^2 \sigma_B^2 + x_C^2 \sigma_C^2 + 2x_A x_B \sigma_{AB} + 2x_A x_C \sigma_{AC} + 2x_B x_C \sigma_{BC} + \lambda(x_A + x_B + x_C - 1)$$

$$\frac{\partial L}{\partial x_A} = 2x_A \sigma_A^2 + 2x_B \sigma_{AB} + 2x_C \sigma_{AC} + \lambda = 0$$

$$\frac{\partial L}{\partial x_B} = 2x_A \sigma_{AB} + 2x_B \sigma_B^2 + 2x_C \sigma_{BC} + \lambda = 0$$

$$\frac{\partial L}{\partial x_C} = 2x_A \sigma_{AC} + 2x_B \sigma_{BC} + 2x_C \sigma_C^2 + \lambda = 0$$

$$\frac{\partial L}{\partial \lambda} = x_A + x_B + x_C - 1 = 0$$

Solve 4 equations, 4 Unknowns to get x_A, x_B, x_C, λ

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Minimum Variance Portfolio

- Consider 3 Risky Asset Portfolio – **Matrix**

$$\min_{\mathbf{X}} \sigma_p^2 = \mathbf{X}' \Sigma \mathbf{X} \text{ st. } \mathbf{X}' \mathbf{1} = 1$$

$$\frac{\partial L}{\partial \mathbf{X}} = \begin{pmatrix} 2\sigma_A^2 & 2\sigma_{AB} & 2\sigma_{AC} & 1 \\ 2\sigma_{AB} & 2\sigma_B^2 & 2\sigma_{BC} & 1 \\ 2\sigma_{AC} & 2\sigma_{BC} & 2\sigma_C^2 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_A \\ x_B \\ x_C \\ \lambda \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{X} \\ \lambda \end{pmatrix} = \begin{pmatrix} 2\Sigma & \mathbf{1}' \\ \mathbf{1}' & 0 \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{0} \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x_A \\ x_B \\ x_C \\ \lambda \end{pmatrix} = \begin{pmatrix} 2\sigma_A^2 & 2\sigma_{AB} & 2\sigma_{AC} & 1 \\ 2\sigma_{AB} & 2\sigma_B^2 & 2\sigma_{BC} & 1 \\ 2\sigma_{AC} & 2\sigma_{BC} & 2\sigma_C^2 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$



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Exercise

- Minimum Variance Portfolio



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Efficient Portfolio

- Consider 3 Risky Asset Portfolio – **Matrix**

$$\min_{\mathbf{X}} \sigma_p^2 = \mathbf{X}' \Sigma \mathbf{X} \text{ st. } \mathbf{X}' \mathbf{R} = \mu_p \text{ and } \mathbf{X}' \mathbf{1} = 1$$

$$\frac{\partial L}{\partial \mathbf{X}} = \begin{pmatrix} 2\Sigma & \mathbf{R} & \mathbf{1} \\ \mathbf{R}' & 0 & 0 \\ \mathbf{1}' & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{X} \\ \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mu_p \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{X} \\ \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 2\Sigma & \mathbf{R} & \mathbf{1} \\ \mathbf{R}' & 0 & 0 \\ \mathbf{1}' & 0 & 0 \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{0} \\ \mu_p \\ 1 \end{pmatrix}$$



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Exercise

- Efficient Portfolio
- Efficient Frontier



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Conclusion

- ρ, Σ and X plays important role in risk reduction, if $\rho = -1$, the payoff from diversification is highest; if $\rho = 1$, there is no benefits gained from diversification at all
- Given that $\rho \neq 1$, it is possible to find MVP
- Efficient Frontier is the upper side of the parabola and starts from MVP



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Group Homework

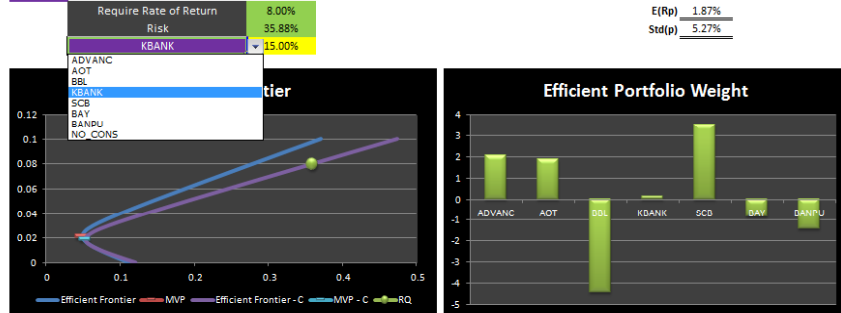
- Implement a financial model of a selectable 7-Asset portfolio
- The portfolio must show **investments proportion, risks and returns** of an **MVP** and **Efficient Portfolio** at any given required rate of return
- Add 1 additional constraint of **investment proportion** e.g. invest exactly 15% in a selected asset



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Group Homework

Stock	Name	Beta	Current PE	ROE	ROA	ROC	WebURL	MVP	EP
ADVANC	ADVANCED INFO SERVICE PCL	0.6334205	20.49978	63.85362	27.52739	38.94442	ADVANCED.INFO.SERVICE.PCL	0.851021	2.068057
AOT	AIRPORTS OF THAILAND PCL	0.9770615	23.03183	6.748394	3.452664	4.730938	AIRPORTS.OF.THAILAND.PCL	-0.33166	1.896781
BBL	BANGKOK BANK PUBLIC CO LTD	1.122179	11.07654	12.40294	1.402461	6.545659	BANGKOK.BANK.PUBLIC.CO.LTD	-0.38315	-4.41173
KBANK	KASIKORN BANK PCL	1.146595	13.87803	18.64102	1.617641	8.689319	KASIKORN.BANK.PCL	0.15	0.15
SCB	SIAM COMMERCIAL BANK PUB CO	1.153906	14.63829	19.3603	1.87959	7.744585	SIAM.COMMERCIAL.BANK.PUB.CO	0.407365	3.537149
BAY	BANK OF AYUDHYA PCL	1.174226	15.78722	10.7969	1.161088	3.466481	BANK.OF.AYUDHYA.PCL	0.121674	-0.807
BANPU	BANPU PUBLIC CO LTD	1.111976	7.975561	17.79078	6.115622	12.26795	BANPU.PUBLIC.CO.LTD	0.184757	-1.43326



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Title: [FN451]Homework 2

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