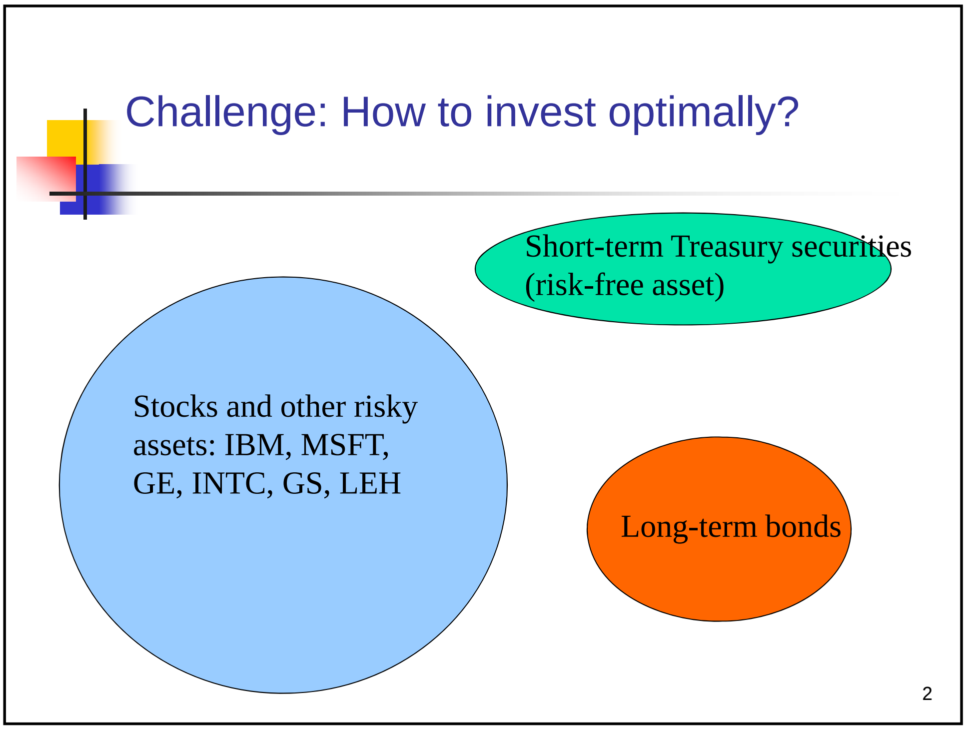
BACHELOR of ECONOMICS
International Program

Thammasat University | Thailand

FN 312 Investments



Challenge: How to invest optimally?

Short-term Treasury securities (risk-free asset)

Stocks and other risky assets: IBM, MSFT, GE, INTC, GS, LEH

Long-term bonds

Outline

- The interest rate
- Return
- Variance risk
- Investors preference
- Risk-return trade off

Real and Nominal Rates of Interest

- Interest rate
- US Treasury use as the risk-free rate
- Nominal interest rate
  - Growth rate of your money
- Real interest rate
  - Growth rate of your purchasing power
- If *R* is the nominal rate and *r* the real rate and *i* is the inflation rate:

$$R+1=(1+r)(1+i)$$

$$R \sim r+i$$

Factors Influencing Interest rates

- Supply
  - Households
- Demand
  - Businesses
- Government’s Net Supply and/or Demand
  - Federal Reserve Actions

Figure 5.1 Determination of the Equilibrium Real Rate of Interest

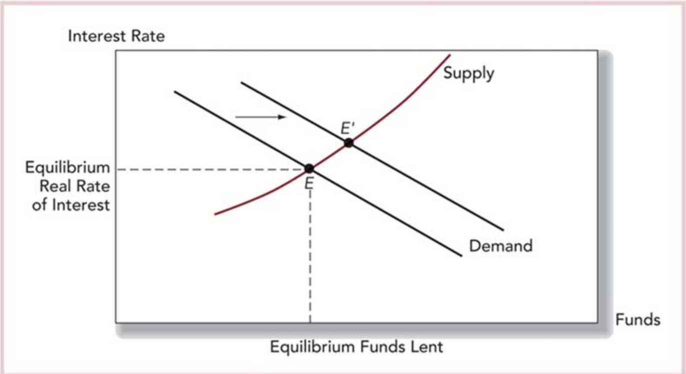
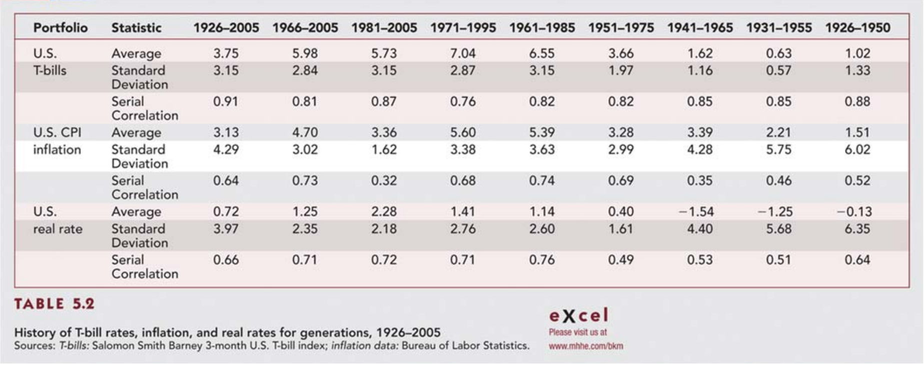


FIGURE 5.1 Determination of the equilibrium real rate of interest

Table 5.2 History of T-bill Rates, Inflation and Real Rates for Generations, 1926-2005



Portfolio	Statistic	1926–2005	1966–2005	1981–2005	1971–1995	1961–1985	1951–1975	1941–1965	1931–1955	1926–1950
U.S. T-bills	Average	3.75	5.98	5.73	7.04	6.55	3.66	1.62	0.63	1.02
	Standard Deviation	3.15	2.84	3.15	2.87	3.15	1.97	1.16	0.57	1.33
	Serial Correlation	0.91	0.81	0.87	0.76	0.82	0.82	0.85	0.85	0.88
U.S. CPI inflation	Average	3.13	4.70	3.36	5.60	5.39	3.28	3.39	2.21	1.51
	Standard Deviation	4.29	3.02	1.62	3.38	3.63	2.99	4.28	5.75	6.02
	Serial Correlation	0.64	0.73	0.32	0.68	0.74	0.69	0.35	0.46	0.52
U.S. real rate	Average	0.72	1.25	2.28	1.41	1.14	0.40	−1.54	−1.25	−0.13
	Standard Deviation	3.97	2.35	2.18	2.76	2.60	1.61	4.40	5.68	6.35
	Serial Correlation	0.66	0.71	0.72	0.71	0.76	0.49	0.53	0.51	0.64

TABLE 5.2

History of T-bill rates, inflation, and real rates for generations, 1926–2005

Sources: *T-bills:* Salomon Smith Barney 3-month U.S. T-bill index; *inflation data:* Bureau of Labor Statistics.

eXcel

Please visit us at

www.mhhe.com/bkm

Figure 5.2 Interest Rates and Inflation, 1926-2005

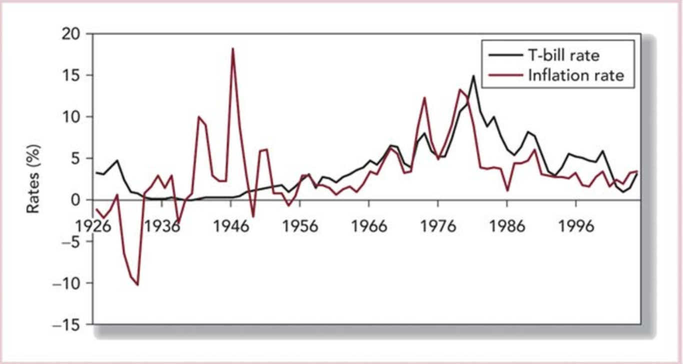


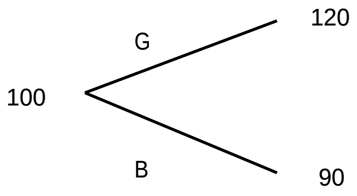
FIGURE 5.2 Interest and inflation rates, 1926–2005

Bills and Inflation, 1926-2005

- Entire post-1926 history of annual rates:
  - www.mhhe.com/bkm
- Average real rate of return on T-bills for the entire period was 0.72 percent
- Real rates are larger in late periods

Risky Prospects

The stock price of ABC is currently $100. Assume the stock price is either going to increase to $120 or decrease to $90 over the next year. In both cases the firm will pay a dividend of $5.



120

G

100

B

90

Rates of return

Rates of Return: Single Period

$$HPR = \frac{P_1 - P_0 + D_1}{P_0}$$

HPR = Holding Period Return

P_0 = Beginning price

P_1 = Ending price

D_1 = Dividend during period one

Rates of Return

- The rate of return in the G state is:

$$r_G = \frac{120 + 5 - 100}{100} = 25\%$$

- The rate of return in the B state is:

$$r_B = \frac{90 + 5 - 100}{100} = -5\%$$

Variance of Rate of Return

- The variance is defined as the expectation of the squared deviations from the expectation. In our case:

$$Var(\tilde{r}) = \sigma_r^2 = 0.4(0.25 - 0.07)^2 + 0.6(-0.05 - 0.07)^2 = 0.0216$$

- In general, the variance is given by:

$$Var(\tilde{r}) = \sigma_r^2 = E(\tilde{r} - E(\overline{r}))^2 = \sum_{j=1}^{n} p_j (r_j - \overline{r})^2$$

- The standard deviation is defined as the squared root of the variance. In our example: $\sigma_r = \sqrt{0.0216} = 14.7\%.$

Variance – An Alternative Method

- The variance can also be calculated as follows:

$$Var(\tilde{r}) = \sigma_r^2 = E(\tilde{r}^2) - E(\overline{r})^2 = \sum_{j=1}^{n} p_j {r_j}^2 - \left(\sum_{j=1}^{n} p_j {r_j}^2 \right)^2$$

- In our example:

$$Var(\overline{r}) = 0.4 \cdot 0.25^2 + 0.6 \cdot (-0.05)^2 - 0.07^2 = 0.0216$$

Expected Return

Expected returns

$$E(r) = \sum_{s} p(s) r(s)$$

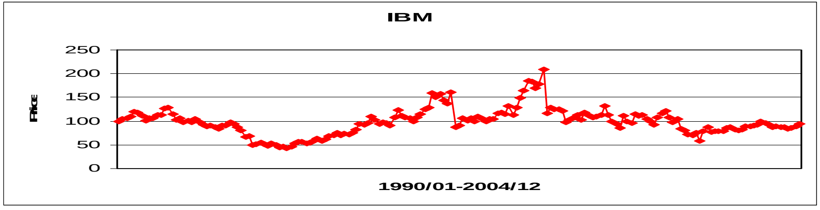
$p(s)$ = probability of a state
$r(s)$ = return if a state occurs
s = state

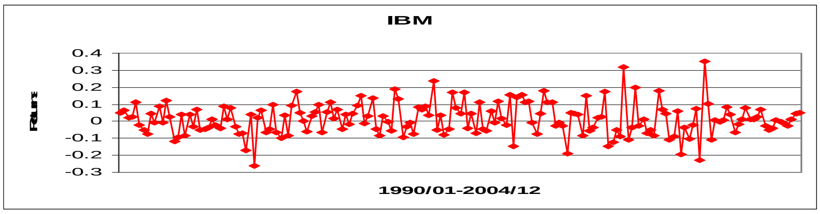
Scenario Returns: Example

State	Prob. of State	r in State
1	.1	-.05
2	.2	.05
3	.4	.15
4	.2	.25
5	.1	.35

$E(r)$ = (.1)(-.05) + (.2)(.05)… + (.1)(.35)
$E(r)$ = .15

Average past returns as expected return





Time Series Analysis of Past Rates of Return

- When using historical data to estimate expected return we assume that each data point has equal probability, therefore the probability = 1/n
- Expected Returns and the Arithmetic Average

$$E(r) = \sum_{s=1}^{n} p(s)r(s) = \frac{1}{n}\sum_{s=1}^{n} r(s)$$

Variance and Standard Deviation

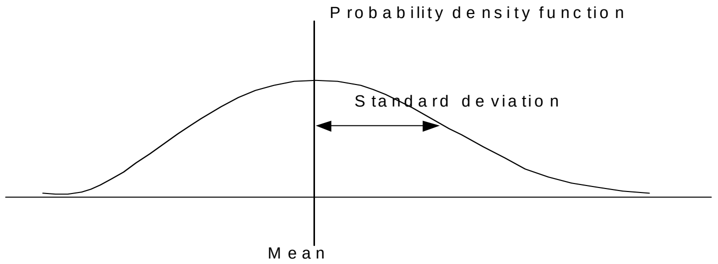
- Variance = expected value of squared deviations

$$\sigma^2 = \frac{1}{n}\sum_{s=1}^{n}\left[r(s)-\bar{r}\right]^2$$

- For sample data (not population data) when eliminating the bias because we use estimated expected return, Standard Deviation becomes:

$$\sigma = \sqrt{\frac{1}{n-1}}\sum_{j=1}^{n}\left[r(s)-\bar{r}\right]^2$$

Distribution of returns



Mean = $E[\tilde{r}_i] = 1/N\sum \tilde{r}_i$

Variance of returns = $\sigma^2[\tilde{r}_i] = 1/(N-1)\sum (\tilde{r}_i - E[\tilde{r}_i])^2$

Standard deviation = square root of variance

The Normal Distribution

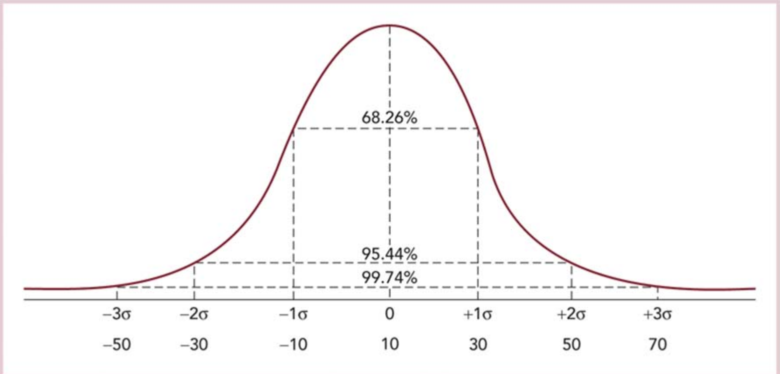


FIGURE 5.4 The normal distribution

Normal and Skewed Distributions (mean = 6% SD = 17%)

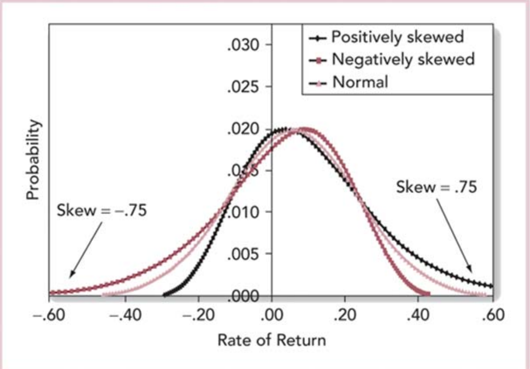


FIGURE 5.5A Normal and skewed distributions (mean = 6%, SD = 17%)

Normal and Fat-Tailed Distributions (mean = .1, SD =.2)

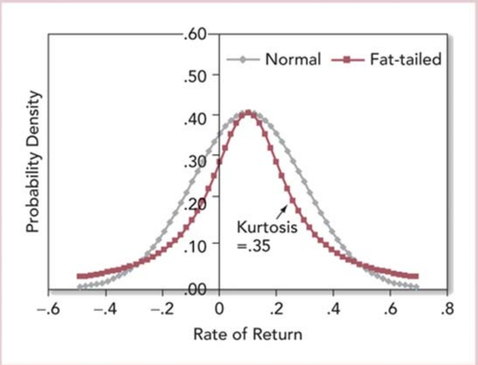


FIGURE 5.5B Normal and fat-tailed distributions (mean = .1, SD = .2)

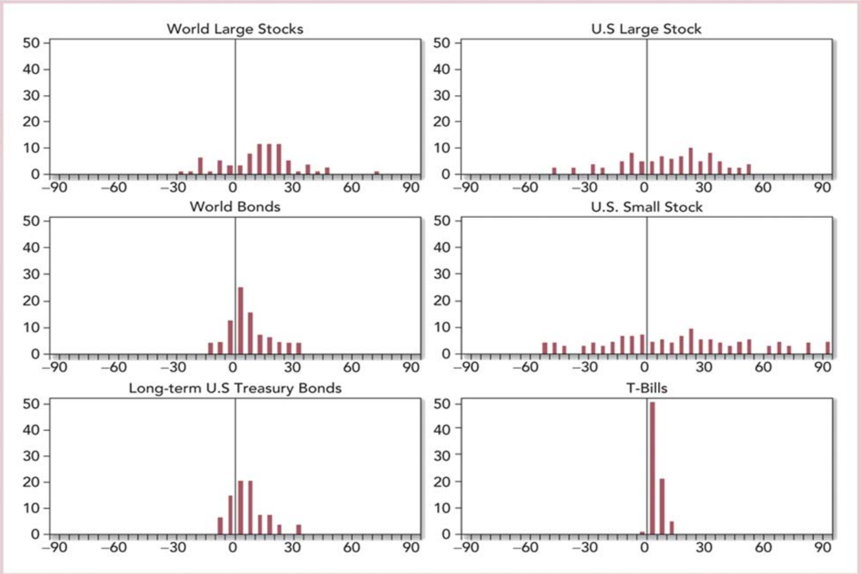
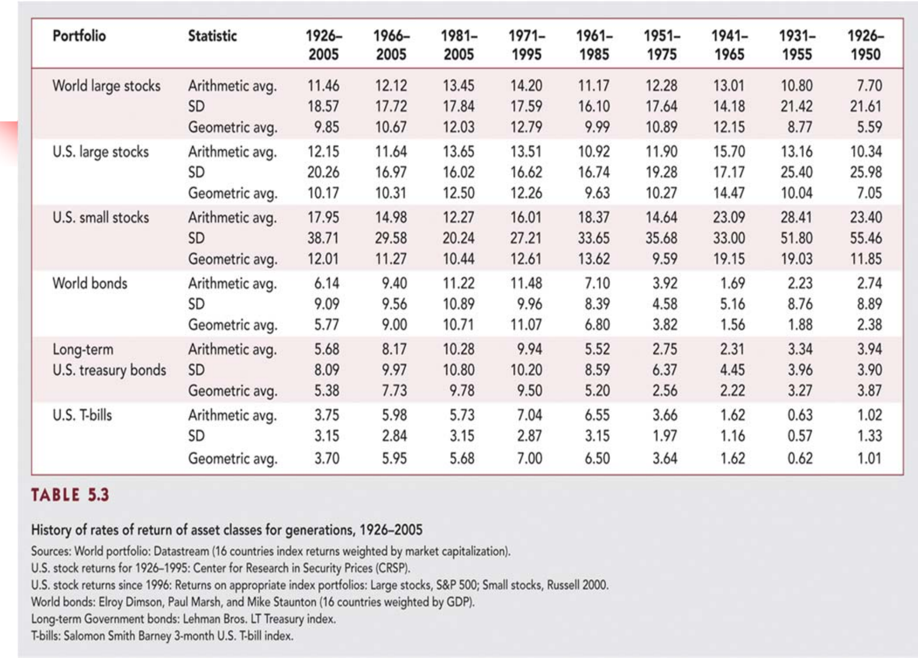


FIGURE 5.6 Frequency distributions of rates of return for 1926–2005

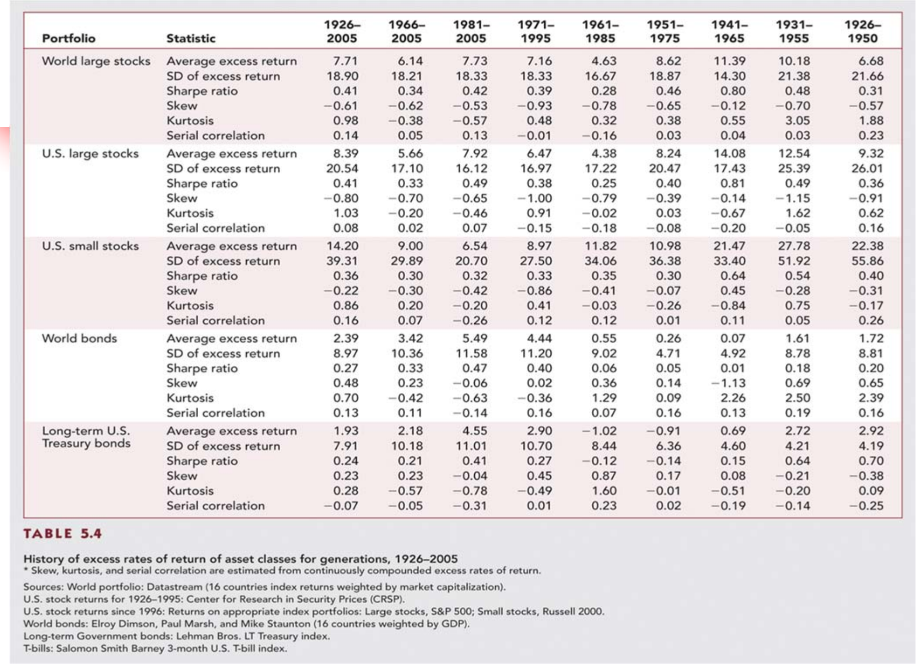


Portfolio	Statistic	1926–2005	1966–2005	1981–2005	1971–1995	1961–1985	1951–1975	1941–1965	1931–1955	1926–1950
World large stocks	Arithmetic avg.	11.46	12.12	13.45	14.20	11.17	12.28	13.01	10.80	7.70
	SD	18.57	17.72	17.84	17.59	16.10	17.64	14.18	21.42	21.61
	Geometric avg.	9.85	10.67	12.03	12.79	9.99	10.89	12.15	8.77	5.59
U.S. large stocks	Arithmetic avg.	12.15	11.64	13.65	13.51	10.92	11.90	15.70	13.16	10.34
	SD	20.26	16.97	16.02	16.62	16.74	19.28	17.17	25.40	25.98
	Geometric avg.	10.17	10.31	12.50	12.26	9.63	10.27	14.47	10.04	7.05
U.S. small stocks	Arithmetic avg.	17.95	14.98	12.27	16.01	18.37	14.64	23.09	28.41	23.40
	SD	38.71	29.58	20.24	27.21	33.65	35.68	33.00	51.80	55.46
	Geometric avg.	12.01	11.27	10.44	12.61	13.62	9.59	19.15	19.03	11.85
World bonds	Arithmetic avg.	6.14	9.40	11.22	11.48	7.10	3.92	1.69	2.23	2.74
	SD	9.09	9.56	10.89	9.96	8.39	4.58	5.16	8.76	8.89
	Geometric avg.	5.77	9.00	10.71	11.07	6.80	3.82	1.56	1.88	2.38
Long-term U.S. treasury bonds	Arithmetic avg.	5.68	8.17	10.28	9.94	5.52	2.75	2.31	3.34	3.94
	SD	8.09	9.97	10.80	10.20	8.59	6.37	4.45	3.96	3.90
	Geometric avg.	5.38	7.73	9.78	9.50	5.20	2.56	2.22	3.27	3.87
U.S. T-bills	Arithmetic avg.	3.75	5.98	5.73	7.04	6.55	3.66	1.62	0.63	1.02
	SD	3.15	2.84	3.15	2.87	3.15	1.97	1.16	0.57	1.33
	Geometric avg.	3.70	5.95	5.68	7.00	6.50	3.64	1.62	0.62	1.01

TABLE 5.3

History of rates of return of asset classes for generations, 1926–2005

Sources: World portfolio: Datastream (16 countries index returns weighted by market capitalization).
U.S. stock returns for 1926–1995: Center for Research in Security Prices (CRSP).
U.S. stock returns since 1996: Returns on appropriate index portfolios: Large stocks, S&P 500; Small stocks, Russell 2000.
World bonds: Elroy Dimson, Paul Marsh, and Mike Staunton (16 countries weighted by GDP).
Long-term Government bonds: Lehman Bros. LT Treasury index.
T-bills: Salomon Smith Barney 3-month U.S. T-bill index.



Portfolio	Statistic	1926–2005	1966–2005	1981–2005	1971–1995	1961–1985	1951–1975	1941–1965	1931–1955	1926–1950
World large stocks	Average excess return	7.71	6.14	7.73	7.16	4.63	8.62	11.39	10.18	6.68
	SD of excess return	18.90	18.21	18.33	18.33	16.67	18.87	14.30	21.38	21.66
	Sharpe ratio	0.41	0.34	0.42	0.39	0.28	0.46	0.80	0.48	0.31
	Skew	−0.61	−0.62	−0.53	−0.93	−0.78	−0.65	−0.12	−0.70	−0.57
	Kurtosis	0.98	−0.38	−0.57	0.48	0.32	0.38	0.55	3.05	1.88
	Serial correlation	0.14	0.05	0.13	−0.01	−0.16	0.03	0.04	0.03	0.23
U.S. large stocks	Average excess return	8.39	5.66	7.92	6.47	4.38	8.24	14.08	12.54	9.32
	SD of excess return	20.54	17.10	16.12	16.97	17.22	20.47	17.43	25.39	26.01
	Sharpe ratio	0.41	0.33	0.49	0.38	0.25	0.40	0.81	0.49	0.36
	Skew	−0.80	−0.70	−0.65	−1.00	−0.79	−0.39	−0.14	−1.15	−0.91
	Kurtosis	1.03	−0.20	−0.46	0.91	−0.02	0.03	−0.67	1.62	0.62
	Serial correlation	0.08	0.02	0.07	−0.15	−0.18	−0.08	−0.20	−0.05	0.16
U.S. small stocks	Average excess return	14.20	9.00	6.54	8.97	11.82	10.98	21.47	27.78	22.38
	SD of excess return	39.31	29.89	20.70	27.50	34.06	36.38	33.40	51.92	55.86
	Sharpe ratio	0.36	0.30	0.32	0.33	0.35	0.30	0.64	0.54	0.40
	Skew	−0.22	−0.30	−0.42	−0.86	−0.41	−0.07	0.45	−0.28	−0.31
	Kurtosis	0.86	0.20	−0.20	0.41	−0.03	−0.26	−0.84	0.75	−0.17
	Serial correlation	0.16	0.07	−0.26	0.12	0.12	0.01	0.11	0.05	0.26
World bonds	Average excess return	2.39	3.42	5.49	4.44	0.55	0.26	0.07	1.61	1.72
	SD of excess return	8.97	10.36	11.58	11.20	9.02	4.71	4.92	8.78	8.81
	Sharpe ratio	0.27	0.33	0.47	0.40	0.06	0.05	0.01	0.18	0.20
	Skew	0.48	0.23	−0.06	0.02	0.36	0.14	−1.13	0.69	0.65
	Kurtosis	0.70	−0.42	−0.63	−0.36	1.29	0.09	2.26	2.50	2.39
	Serial correlation	0.13	0.11	−0.14	0.16	0.07	0.16	0.13	0.19	0.16
Long-term U.S. Treasury bonds	Average excess return	1.93	2.18	4.55	2.90	−1.02	−0.91	0.69	2.72	2.92
	SD of excess return	7.91	10.18	11.01	10.70	8.44	6.36	4.60	4.21	4.19
	Sharpe ratio	0.24	0.21	0.41	0.27	−0.12	−0.14	0.15	0.64	0.70
	Skew	0.23	0.23	−0.04	0.45	0.87	0.17	0.08	−0.21	−0.38
	Kurtosis	0.28	−0.57	−0.78	−0.49	1.60	−0.01	−0.51	−0.20	0.09
	Serial correlation	−0.07	−0.05	−0.31	0.01	0.23	0.02	−0.19	−0.14	−0.25

TABLE 5.4

History of excess rates of return of asset classes for generations, 1926–2005
* Skew, kurtosis, and serial correlation are estimated from continuously compounded excess rates of return.

Sources: World portfolio: Datastream (16 countries index returns weighted by market capitalization).
U.S. stock returns for 1926–1995: Center for Research in Security Prices (CRSP).
U.S. stock returns since 1996: Returns on appropriate index portfolios: Large stocks, S&P 500; Small stocks, Russell 2000.
World bonds: Elroy Dimson, Paul Marsh, and Mike Staunton (16 countries weighted by GDP).
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T-bills: Salomon Smith Barney 3-month U.S. T-bill index.

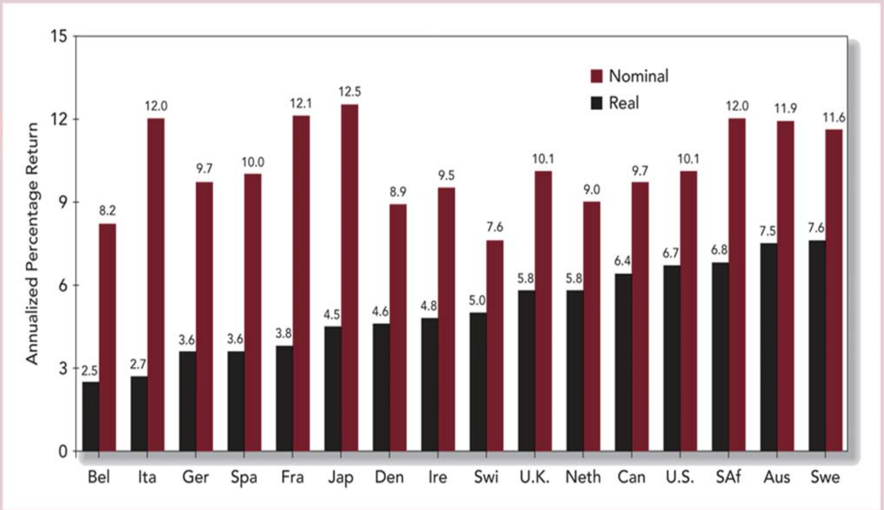


FIGURE 5.7 Nominal and real equity returns around the world, 1900–2000

Source: Elroy Dimson, Paul Marsh, and Mike Staunton, *Triumph of the Optimists: 101 Years of Global Investment Returns* (Princeton University Press, 2002), p. 50. Reprinted by permission of the Princeton University Press.

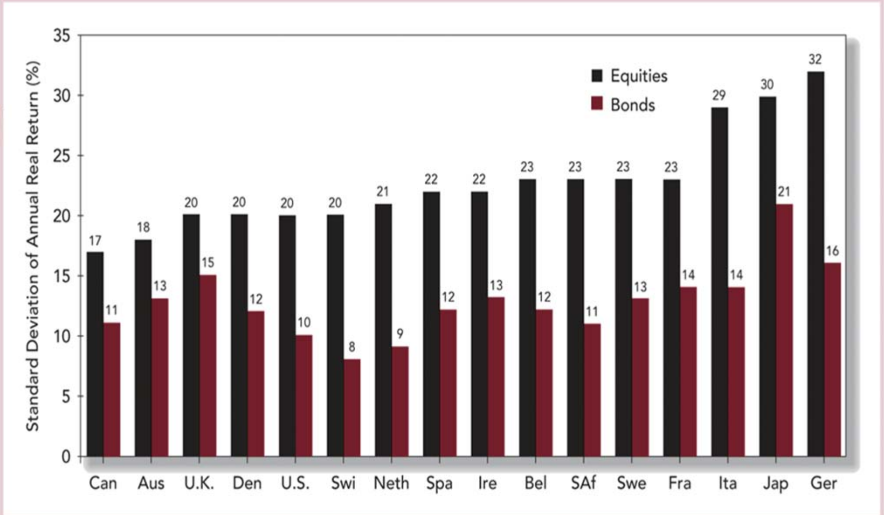


FIGURE 5.8 Standard deviations of real equity and bond returns around the world, 1900–2000

Source: Elroy Dimson, Paul Marsh, and Mike Staunton, *Triumph of the Optimists: 101 Years of Global Investment Returns* (Princeton University Press, 2002), p. 61. Reprinted by permission of the Princeton University Press.

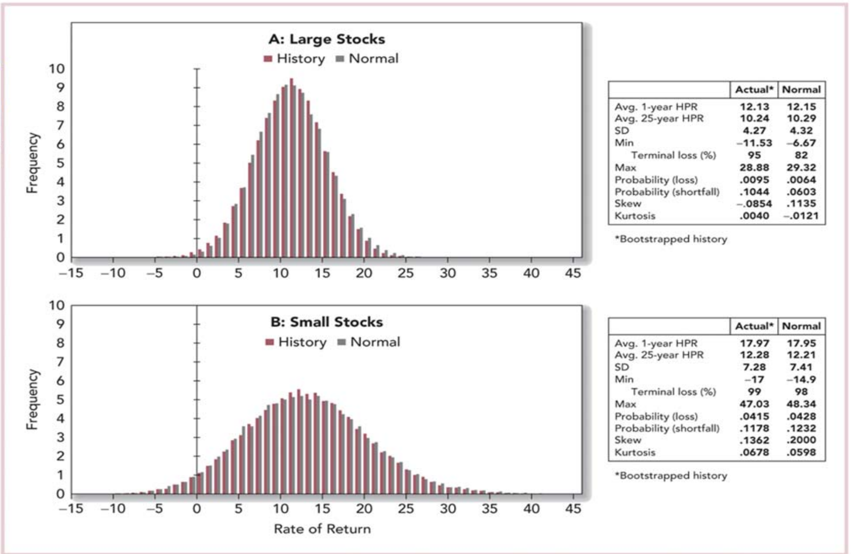
Growth rate of investment

$$TV_n = (1+r_1)(1+r_2)x\ldots x = (1+r_n)$$

TV = Terminal Value of the Investment of one dollar

g= geometric average rate of return

$$g = TV^{1/n} - 1$$



	Actual*	Normal
Avg. 1-year HPR	12.13	12.15
Avg. 25-year HPR	10.24	10.29
SD	4.27	4.32
Min	−11.53	−6.67
Terminal loss (%)	95	82
Max	28.88	29.32
Probability (loss)	.0095	.0064
Probability (shortfall)	.1044	.0603
Skew	−.0854	.1135
Kurtosis	.0040	−.0121

*Bootstrapped history

	Actual*	Normal
Avg. 1-year HPR	17.97	17.95
Avg. 25-year HPR	12.28	12.21
SD	7.28	7.41
Min	−17	−14.9
Terminal loss (%)	99	98
Max	47.03	48.34
Probability (loss)	.0415	.0428
Probability (shortfall)	.1178	.1232
Skew	.1362	.2000
Kurtosis	.0678	.0598

*Bootstrapped history

FIGURE 5.10 Annually compounded, 25-year HPRs from bootstrapped history and a normal distribution (50,000 observations)

Table 5.5 Risk Measures for Non-Normal Distributions

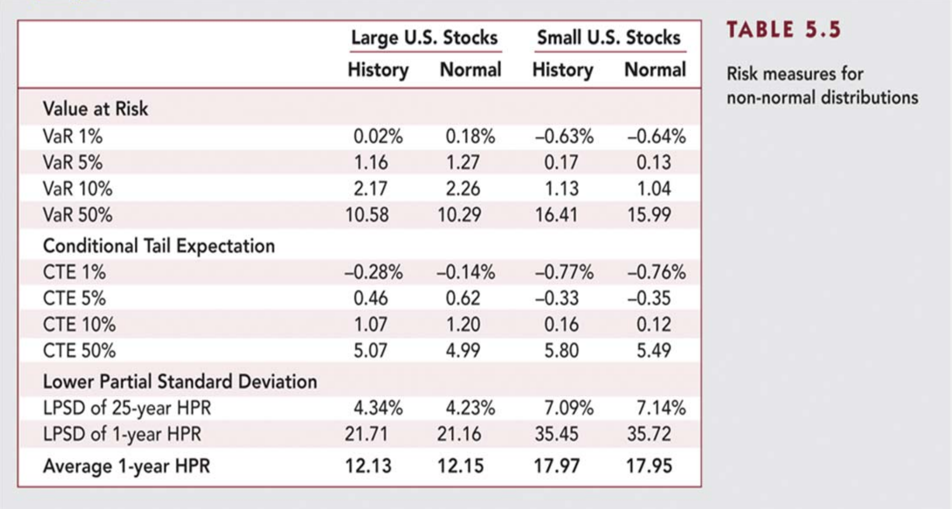


TABLE 5.5

Risk measures for non-normal distributions

	Large U.S. Stocks		Small U.S. Stocks	
	History	Normal	History	Normal
Value at Risk				
VaR 1%	0.02%	0.18%	–0.63%	–0.64%
VaR 5%	1.16	1.27	0.17	0.13
VaR 10%	2.17	2.26	1.13	1.04
VaR 50%	10.58	10.29	16.41	15.99
Conditional Tail Expectation				
CTE 1%	–0.28%	–0.14%	–0.77%	–0.76%
CTE 5%	0.46	0.62	–0.33	–0.35
CTE 10%	1.07	1.20	0.16	0.12
CTE 50%	5.07	4.99	5.80	5.49
Lower Partial Standard Deviation				
LPSD of 25-year HPR	4.34%	4.23%	7.09%	7.14%
LPSD of 1-year HPR	21.71	21.16	35.45	35.72
Average 1-year HPR	**12.13**	**12.15**	**17.97**	**17.95**

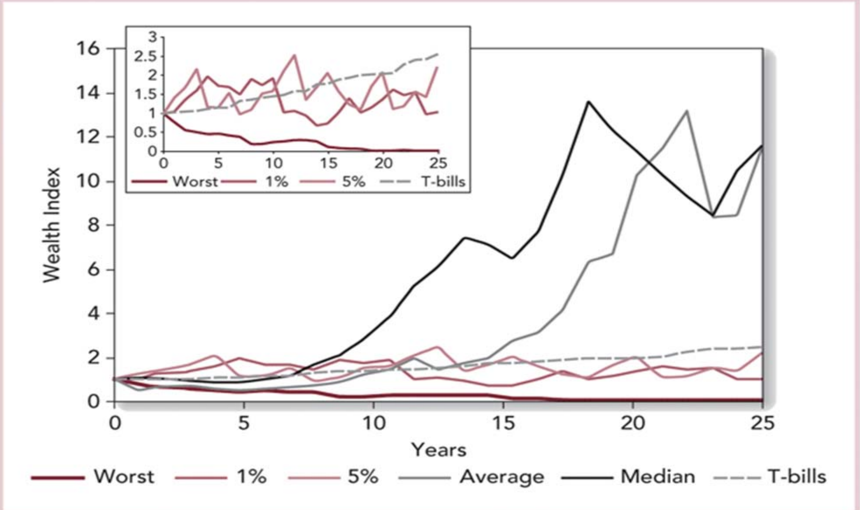


FIGURE 5.12 Wealth indexes of selected outcomes of large stock portfolios and the average T-bill portfolio. Inset: Focus on worst, 1%, 5% outcome versus bills.

Covariance

- The covariance of two random rates of return is

$$Cov(\tilde{r}_1,\tilde{r}_2)=\sigma_{12}=E\left(\left(\tilde{r}_1-E(r_1)\right)\left(\tilde{r}_2-E(r_2)\right)\right)=\sum_{j=1}^{n}p_j\left(r_{1j}-\bar{r}_1\right)\left(r_{2j}-\bar{r}_2\right)$$

- An alternative formula

$$Cov(\tilde{r}_1,\tilde{r}_2)=\sigma_{12}=E(\tilde{r}_1\tilde{r}_2)-E(\tilde{r}_1)E(\tilde{r}_2)$$

- Question: What is $Cov(\tilde{r},\tilde{r})$?

Correlation Coefficient

- The correlation coefficient is

$$\rho_{12}=\frac{Cov(r_1,r_2)}{\sigma_1\sigma_2}$$

- It satisfies

$$-1\le\rho_{12}\le1$$

- What is the meaning of $\rho=0,1,-1$?

Return and standard deviation of a portfolio

Given two uncertain stock returns, X and Y:

$$E[\tilde{r}_X] \quad \sigma[\tilde{r}_X] \quad E[\tilde{r}_Y] \quad \sigma[\tilde{r}_Y] \quad \rho_{X,Y}$$

A portfolio, P, that invests W_1 in X and W_2 in Y:

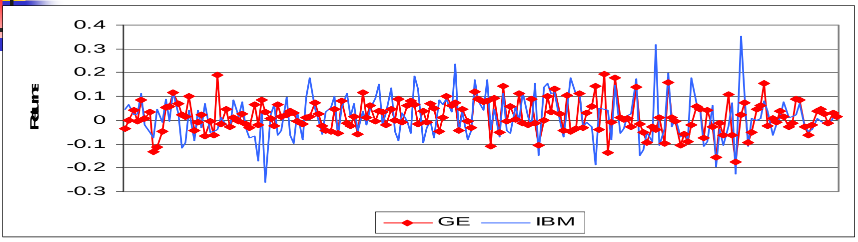
Mean

$$E[\tilde{r}_p] = w_1 E[\tilde{r}_X] + w_2 E[\tilde{r}_Y]$$

Standard deviation (STD)

$$\sigma[\tilde{r}_p] = \sigma_p = \sqrt{w_1^2\sigma_X^2 + w_2^2\sigma_Y^2 + 2w_1w_2\rho_{X,Y}\sigma_X\sigma_Y}$$

Invest 50% in each stock; correlation= 0.34



GE: mean = 0.0146, standard deviation = 0.0657

IBM: mean = 0.01345 , standard deviation = 0.09233

Portfolio mean = 0.5x0.0146 + 0.5x0.01345 = 0.014 (monthly)

Portfolio STD = $\sqrt{w_1^2\sigma_X^2 + w_2^2\sigma_Y^2 + 2w_1w_2\rho_{X,Y}\sigma_X\sigma_Y}$ = 0.06512

Some Useful Formulas

$$E(\tilde{r}_1+\tilde{r}_2)=E(\tilde{r}_1)+E(\tilde{r}_2)$$

$$E(\alpha\tilde{r})=\alpha E(\tilde{r})$$

$$Var(\alpha\tilde{r})=\alpha^2 Var(\tilde{r})$$

$$Var(\tilde{r}_1+\tilde{r}_2)=Var(\tilde{r}_1)+Var(\tilde{r}_2)+2Cov(\tilde{r}_1\tilde{r}_2)=Var(\tilde{r}_1)+Var(\tilde{r}_2)+2\sigma_1\sigma_2\rho_{12}$$

$$Cov(\tilde{r}_1+\tilde{r}_2,\tilde{r}_3)=Cov(\tilde{r}_1,\tilde{r}_3)+Cov(\tilde{r}_2,\tilde{r}_3)$$

$$Cov(\alpha\tilde{r}_1,\tilde{r}_2)=\alpha Cov(\tilde{r}_1,\tilde{r}_2)$$

More Useful Stuff

If α is a constant then:

$$Var(\alpha)=0$$

$$Cov(\tilde{r},\alpha)=0$$

$$E(\alpha)=\alpha$$

Also:

$$Cov(\tilde{r}_1,\tilde{r}_2)=Cov(\tilde{r}_2,\tilde{r}_1)$$

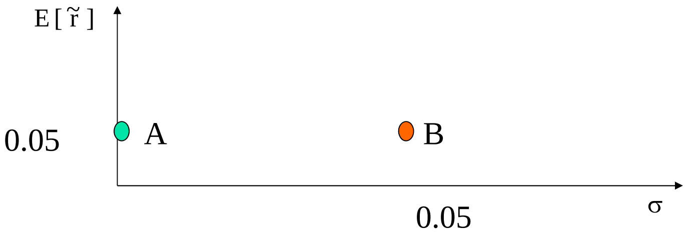
$$\rho_{12}=\rho_{21}$$

Investor’s preference

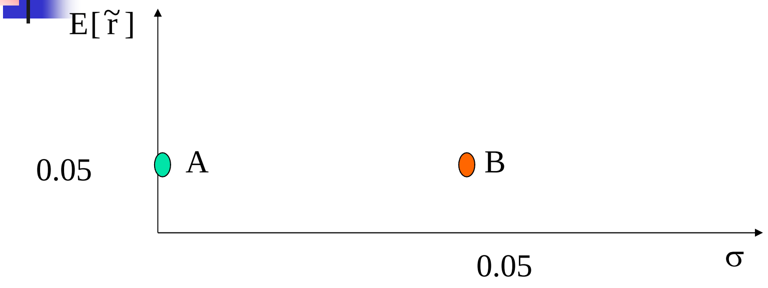
Choose between

Security A: certain investment that returns 5%

Security B: uncertain investment with expected return of 5% and STD = 5%



Investor’s preference



Risk averse investors prefer A; A is the dominant security

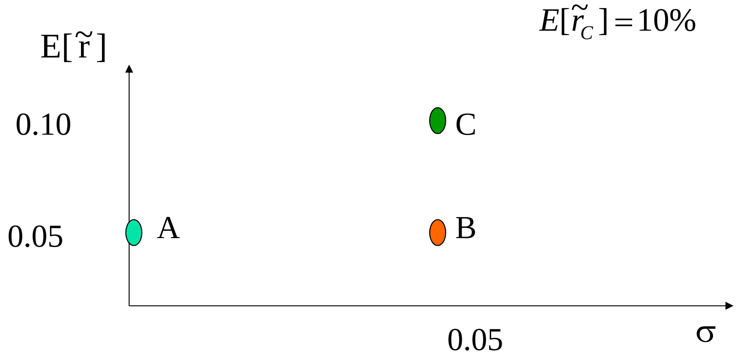
(1) $E[\tilde{r}_A] \geq E[\tilde{r}_B]$

(2) $\sigma_A \leq \sigma_B$, and strict inequality in at least one case

Risk and return trade off

- Consider another security C with $\sigma[\tilde{r}_C] = 5\%$

$E[\tilde{r}_C] = 10\%$



Risk and return trade off

- The risk premium is how much an investor is being compensated beyond the riskfree rate

= $E[\tilde{r}] - r_f$

- Risk premium of security C = 0.1-0.05=0.05 (5%)
- The Sharpe ratio is how much an investor is being compensated beyond the riskfree rate per one standard deviation of uncertainty
- Sharpe ratio of C = $\frac{E[\tilde{r}] - r_f}{\sigma} = \frac{0.05}{0.05}$ = 1

Risk and return trade off

- How much risk premium does an investor require in order to take an investment depends on his/her preference
- To compare investment opportunities an investor can assign a welfare, or utility score to competing investments
- A utility number is a cardinal number
- What should be part of the utility function?

Risk and return trade off

- What investors like?
- What investors dislike?
- Utility function?

  U =

Risk and return trade off

- What investors like? $E[\tilde{r}]$
- What investors dislike? σ^2
- Utility function?

$$U = E[\tilde{r}] \qquad \sigma^2$$

Risk and return trade off

- One common utility measure used in economics is:

$$U = E[\tilde{r}] - 0.5 A \sigma^2$$

Parameter measuring sensitivity to risk

- What A>0 means?

Risk and return trade off

- To compare investment opportunities investors can assign a utility score to competing investments
- The higher utility measure, the preferred investment

Concept check

What should be the value of A (the coefficient of risk aversion) if investors are risk neutral, i.e., do not care about risk? What sign should A have if investors are risk loving, i.e., like risk.

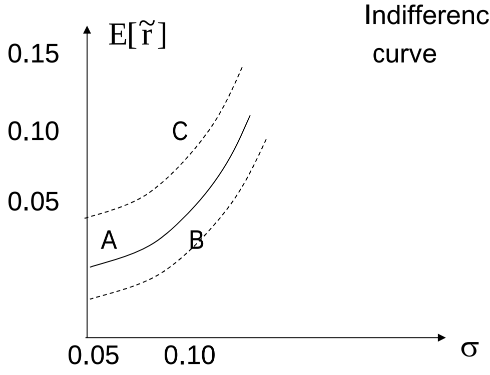
G) Risk neutral A=0, risk loving A<0

Y) Risk neutral A=1, risk loving A<0

R) Risk Neutral A= -1, risk loving A=0

Risk-return tradeoff

Let A=10



Indifference curve

$E[\tilde{r}]$	σ	U
0.05	0	0.05
0.066	0.04	0.05
0.075	0.05	0.05
0.15	0.10	0.05

$$U = E[\tilde{r}] - 0.5A\sigma^2$$

Risk-return tradeoff: example

A stock portfolio has an expected return of 20% and a standard deviation of 20%. T-bills yield 7% for sure. Which investment is preferred if $A = 4$?

$U = E[r] - .5A\sigma^2$

Risk-return tradeoff: example

A stock portfolio has an expected return of 20% and a standard deviation of 20%. T-bills yield 7% for sure. Which investment is preferred if *A* = 8?

$U = E[r] - .5A\sigma^2$

Risk-return tradeoff

- The level of risk aversion determines whether an investor chooses to invest in risky assets and how much he/she chooses to invest in risky assets.
- What determines the risk aversion level of an investor?

Risk-return tradeoff

- The level of risk aversion determines whether an investor chooses to invest in risky assets and how much he/she chooses to invest in risky assets.
- What determines the risk aversion level of an investor?
  - Current wealth
  - Future income
  - Personality
  - Investment horizon

Risk-return tradeoff

- How is risk aversion level determined in practice?

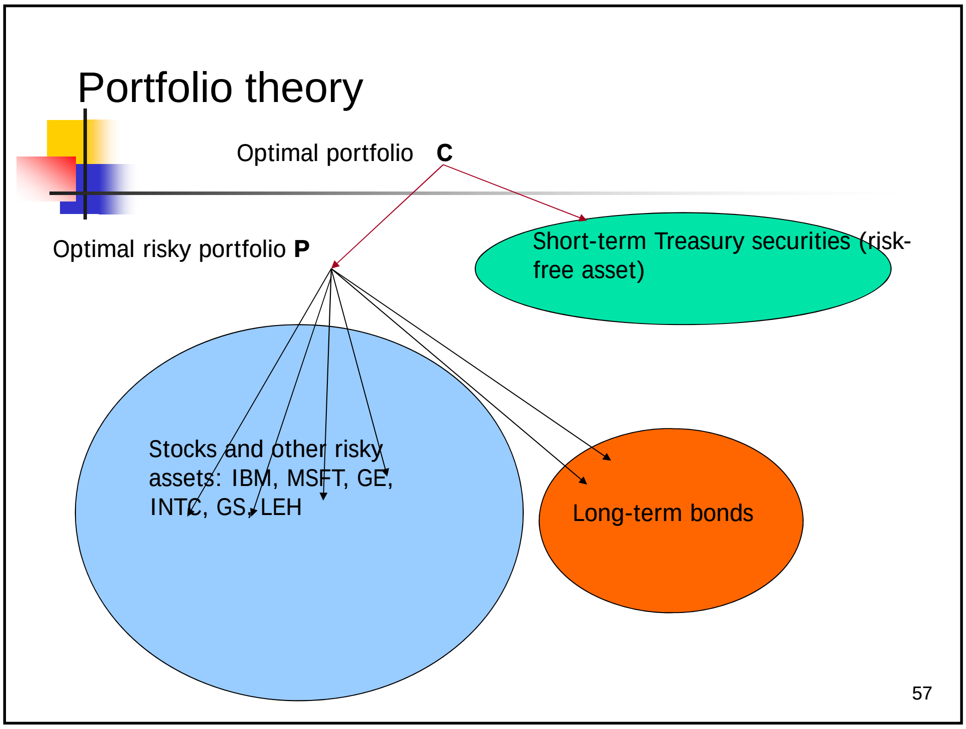
  One way is to use survey methods to asses the relative level of risk aversion of an investor compared to that of an average investor which is approximately 3.0 (See how to compute this number in class notes for asset allocation).

Historical Data of Common Stocks

	S&P 500 Stocks		One-month Bills		Excess Return	
	E[r]	σ	r_f	σ	$E[r] - r_f$	σ
1926-1942	7.2	29.6	1.0	1.7	6.2	29.9
1943-1960	17.4	18.0	1.4	1.0	16.0	18.3
1961-1979	8.6	17.0	5.2	2.0	3.4	17.7
1980-1999	18.6	13.1	7.0	3.0	11.6	13.8
1926-1999	13.1	20.2	3.8	3.3	**9.3**	**20.6**

Historical facts (1929-1999)

- Rate of return of the S&P500 = 13.1 %.
- One month T-bill rate 3.8 %
- Risk premium of the S&P500 is 9.3% per year
- Volatility of the market = 20.6%
- Sharpe ratio = 9.3/20.6 = 0.45



Portfolio theory

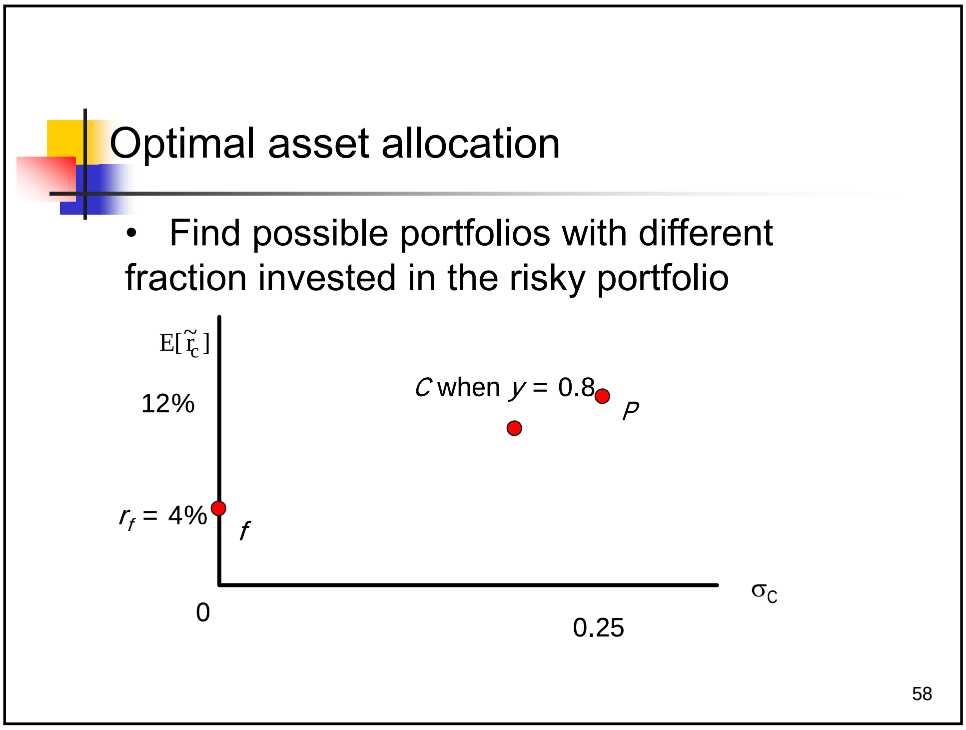
Optimal portfolio **C**

Optimal risky portfolio **P**

Short-term Treasury securities (risk-free asset)

Stocks and other risky assets: IBM, MSFT, GE, INTC, GS, LEH

Long-term bonds



Optimal asset allocation

- Find possible portfolios with different fraction invested in the risky portfolio

Optimal asset allocation

What happens if we choose to put 80% in P and 20% in the riskfree rate. What will be the expected return and standard deviation of this new portfolio C?

$$E(\tilde{r}_c) = yE(\tilde{r}_P) + (1-y)r_f \quad =$$

0.8*0.12+0.2*0.04=0.104

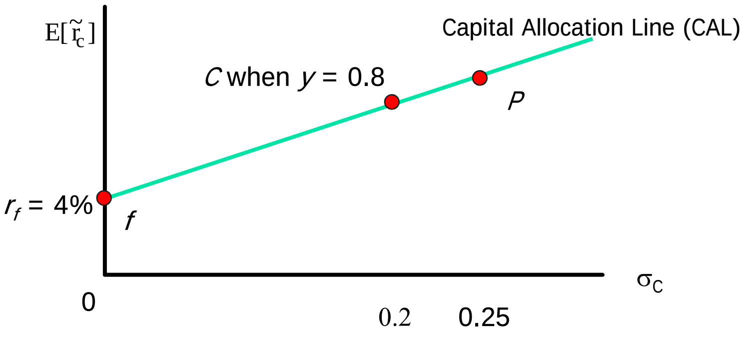
$$\sigma_C^2 = y^2\sigma_P^2 + (1-y)^2\sigma_f^2 + 2y(1-y)\sigma_P\sigma_f\rho_{P,f}$$

$$\sigma_C^2 = y^2\sigma_P^2$$

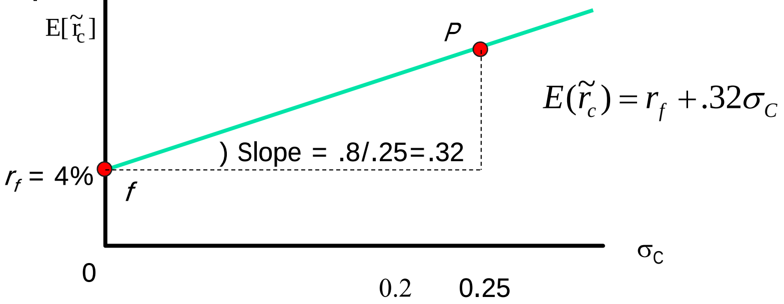
$$\sigma_C = y\sigma_P \qquad = 0.25*0.8=0.2$$

Optimal asset allocation

y	$E[\tilde{r}_c]$	$\sigma[\tilde{r}_c]$
0	0.04	0.0
0.2	0.056	0.05
0.8	0.104	0.20
1.0	0.120	0.25



Optimal asset allocation



- Slope of the CAL equals reward-to-variability ratio or the Sharpe ratio

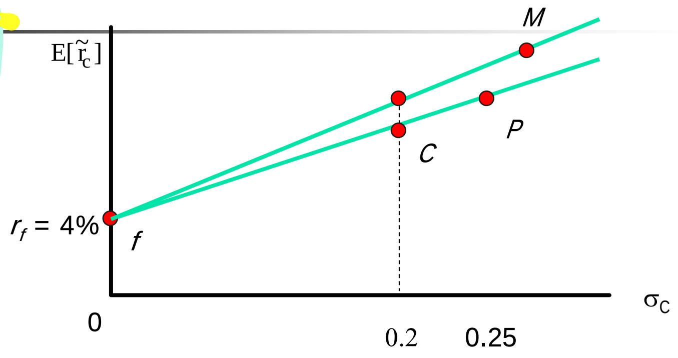
$$\frac{E[\tilde{r}_p] - r_f}{\sigma_p} = \frac{0.08}{.25} = 0.32$$

Optimal asset allocation

The capital allocation line:

- The CAL represents the investment opportunity set of combinations of the riskfree asset and a risky asset
- The slope of the CAL equals the Sharpe ratio
- When the borrowing and lending rate is the same, all combinations of P and the riskfree asset have the same slope

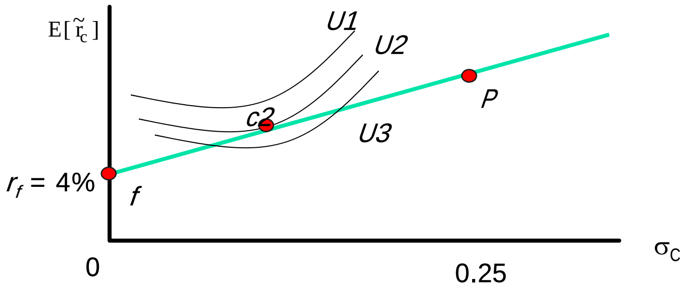
Concept check



- Do you prefer the combination of the riskfree asset and M or P?
  - G) P
  - Y) M
  - R) no difference

Optimal asset allocation

To choose the optimal y, we maximize utility given all the possible choices of the complete portfolio



Theoretical derivation of y* (skip)

$$\underset{y}{Max}\, U = E[\tilde{r}_C] - .5A\sigma_C^2$$

$$E[\tilde{r}] = r_f + y[E[\tilde{r}_P] - r_f]$$

$$\sigma^2 = y^2 \sigma^2{}_P$$

$$\underset{y}{Max}\, U = r_f + y(E[\tilde{r}_P] - r_f) - .5Ay^2\sigma_P^2$$

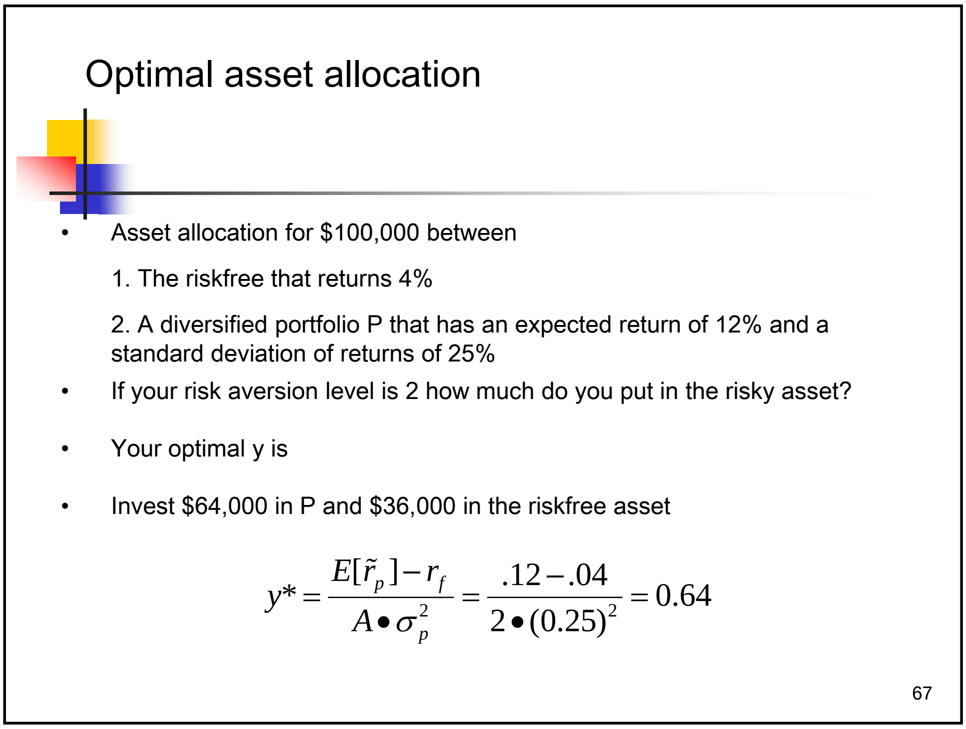
$$\frac{dU(y)}{dy} = 0 \rightarrow [E[\tilde{r}_P] - r_f] - Ay^*\sigma_P^2 = 0$$

$$\frac{d^2u}{dy^2} < 0 \rightarrow \max imum$$

$$y^* = \frac{E[\tilde{r}_P] - r_f}{A\sigma_P^2}$$

Optimal asset allocation

- Let y* denote the optimal y
- y* is at the point where the tangent of the utility indifferent curve equals the slope of the CAL
- $y^* = \frac{E[r] - r_f}{A\sigma^2}$
- y* is negatively related to risk aversion, but positively related to the risk premium



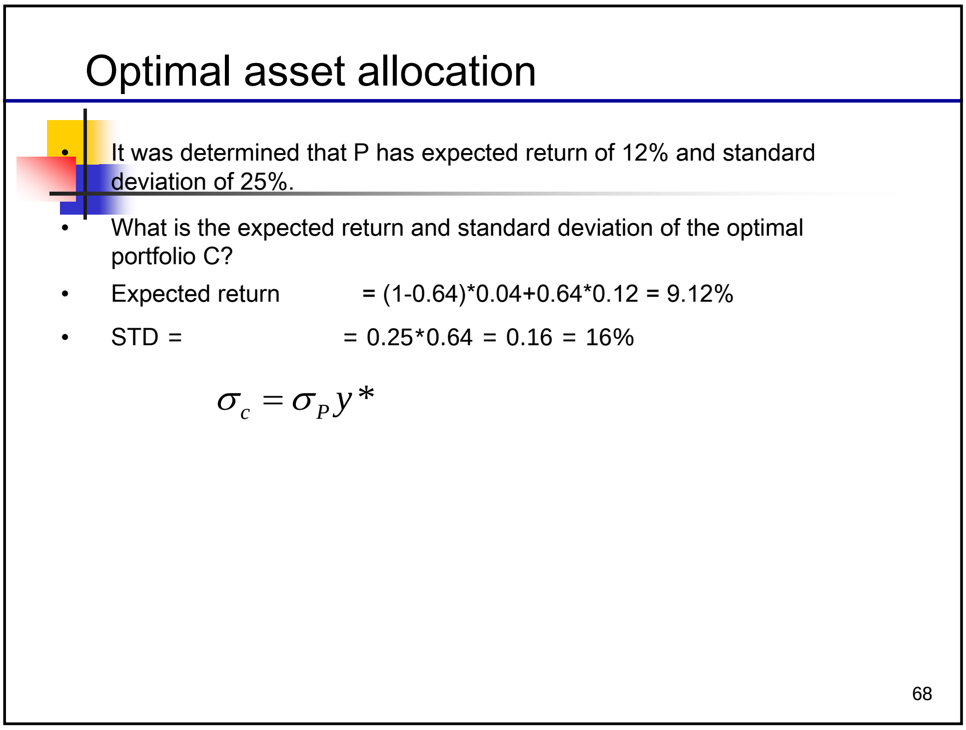
Optimal asset allocation

- Asset allocation for $100,000 between

  1. The riskfree that returns 4%

  2. A diversified portfolio P that has an expected return of 12% and a standard deviation of returns of 25%
- If your risk aversion level is 2 how much do you put in the risky asset?
- Your optimal y is
- Invest $64,000 in P and $36,000 in the riskfree asset

$$y^* = \frac{E[\tilde{r}_p] - r_f}{A \bullet \sigma_p^2} = \frac{.12 - .04}{2 \bullet (0.25)^2} = 0.64$$

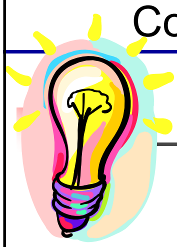


Optimal asset allocation

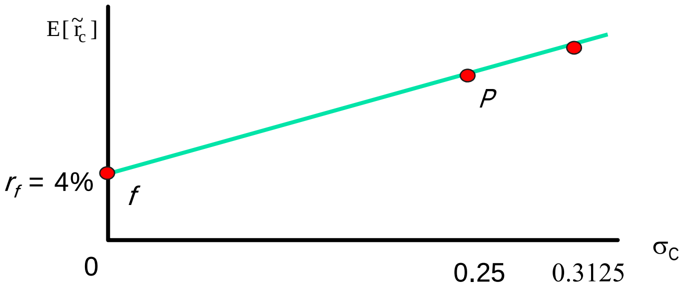
- It was determined that P has expected return of 12% and standard deviation of 25%.
- What is the expected return and standard deviation of the optimal portfolio C?
- Expected return = (1-0.64)*0.04+0.64*0.12 = 9.12%
- STD = = 0.25*0.64 = 0.16 = 16%

$$\sigma_c = \sigma_P y^*$$

Concept check

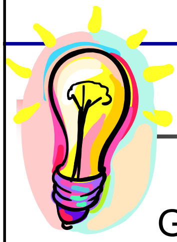


y	$E[\tilde{r}_c]$	$\sigma[\tilde{r}_c]$
0.8	0.104	0.20
1.0	0.120	0.25
1.25	0.140	0.3125



What happens when y >1? For example y=1.25

Concept check



G) You made a mistake in the calculations

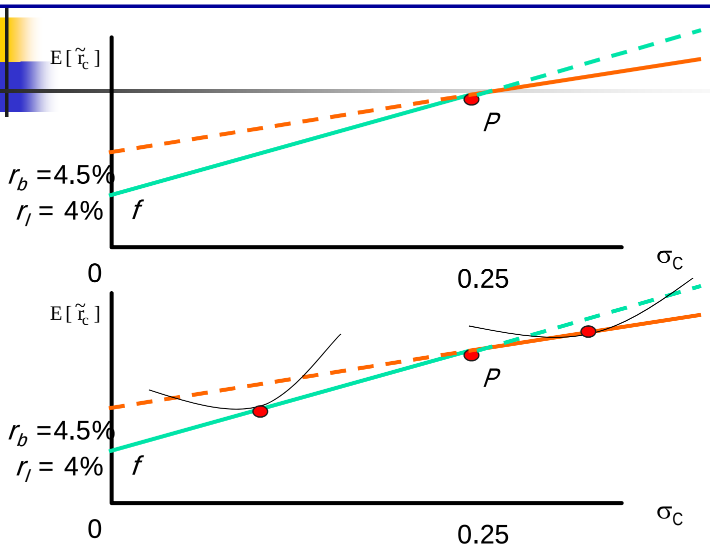
Y) The optimal investment is to invest 25% of your portfolio in the riskfree asset and 125% in the risky asset

R) The optimal investment is to borrow 25% of the original investment. Invest 125% in the risky asset

Practice question

- Suppose you are deciding to invest $500,000 between a portfolio of risky assets P that has an expected return of 14% and a standard deviation of 20% and the riskfree asset, which returns 4%. You have determined that your risk aversion level is around 4. How much money should you invest in P and in the riskfree asset?
- What is the expected return and standard deviation of the new portfolio?

Different borrowing and lending rates



Summary

- Investor’s preference
  - Most investors prefer securities with high return and lower risk
- Risk and return trade off
  - An investor may choose a higher risk security if he/she is adequately compensated
  - Utility is a welfare score used to formalized preference
  - Investors with higher risk aversion levels choose investments with lower risk, all else equal