

# EE 411 (1/2018)

## Microeconomic Analysis

### Lecture 1: Consumer Theory

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Concepts: Preferences, Utility, UMP, Indirect and Expenditure function, Demand function and its properties



# applications

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- If you introduce a new product, how much should it charge?
- To answer, we need to know more about consumer preferences and their behaviors.
- Able to explain how consumers allocate incomes to the purchase of different goods and services.



# Part 1: mathematical approach


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From Jehle and Reny, chapter 1.  
Consumer Theory



# 1.1 Notations

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- $\mathbf{x} = (x_1, x_2, \dots, x_n)$ ; a consumption bundle/plan
- $x_i \in P$  represents the number of good  $i$
- $\mathbf{x} \in X$ , a consumption set;  $X =$  
- Assumption 1.1: Properties of  $X$ .  
 $X$  is closed, convex, can be zero.
- $B$  is a feasible set  $\subset X$
- Behavior Assumption: consumer chooses choice that is most preferred

# 1.2 Preferences and Utility of consumers



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## ■ 1.2.1. Preferences Relations over $X$

1. We start with axioms of consumer choice to make sure his choice can be made and consistent.
2. A binary relation notation:
  - If  $x^1 \succsim x^2$ , we say that “ $x^1$  is at least a good as  $x^2$ .”.
  - Consumers compare two plans at a time and then make a decision.



## 1.2.1. Preferences Relations

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### 2. Two Axioms for preference relation

- A1: Completeness implies consumers can compare his choices.

Either  $x^1 \succsim x^2$ , or  $x^2 \succsim x^1$

- A2: Transitivity implies consistent choices.
  - If  $x^1 \succsim x^2$  and  $x^2 \succsim x^3$  then  $x^1 \succsim x^3$

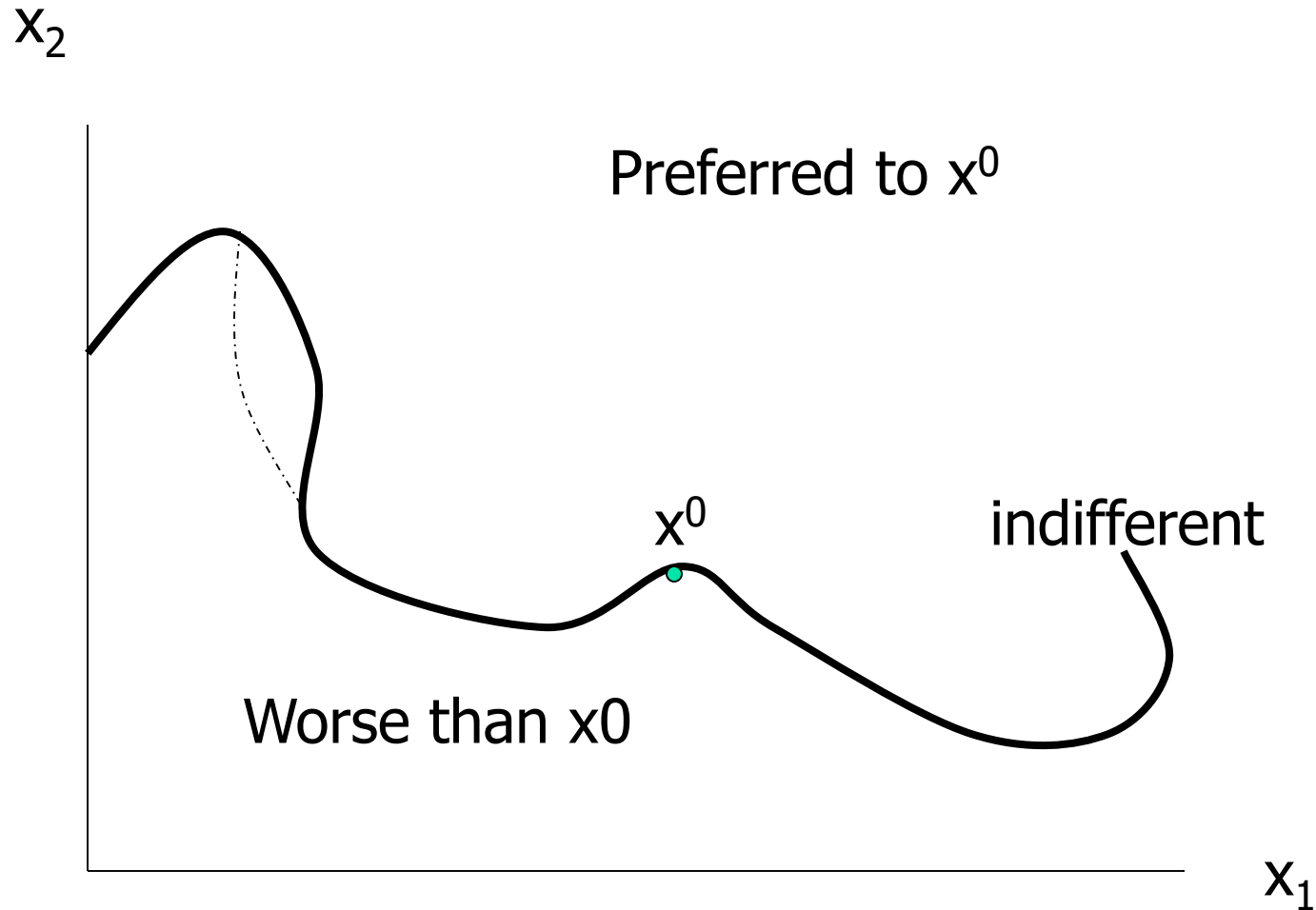


## 1.2.1. Preferences Relations

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- Strict preference relation use  $\succ$
- Indifference relation use  $\sim$
- With all these preference relations, we know that, we can rank any two plans:  
 $x^1 \succ x^2$  or  $x^2 \succ x^1$  or  $x^1 \sim x^2$
- Now, let think of two-goods case, pick a point  $\mathbf{x}^0$ , and we can compare it to all other points by graphing these sets of preferences satisfying all axioms.

We make hypothetical sets in  $X$ .



All points must be in three categories. Gap and thickness are possible at the moment.



## 1.2.1. Preferences Relations

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- A3: Continuity implies no hole or gaps in indifference curve, and no sudden reverse in preference. (for purely math aspects)
- Next, we make further assumptions on tastes.
- A4': Local nonsatiation implies unlimited wants or no thick indifference curve; For a given  $x^0$ , there will have at least one point that consumer prefers to  $x^0$ . Or, a preferred plan nearby always exists, not necessary more is better yet. You should feel different if deviating from  $x^0$  by epsilon.



## 1.2.1. Preferences Relations

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- A4: Monotonicity implies more is always preferred to less, or indifference cannot bend upward;



## 1.2.1. Preferences Relations

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- If  $\mathbf{x}^0 \geq \mathbf{x}^1$ , then  $\mathbf{x}^0 \succsim \mathbf{x}^1$ .

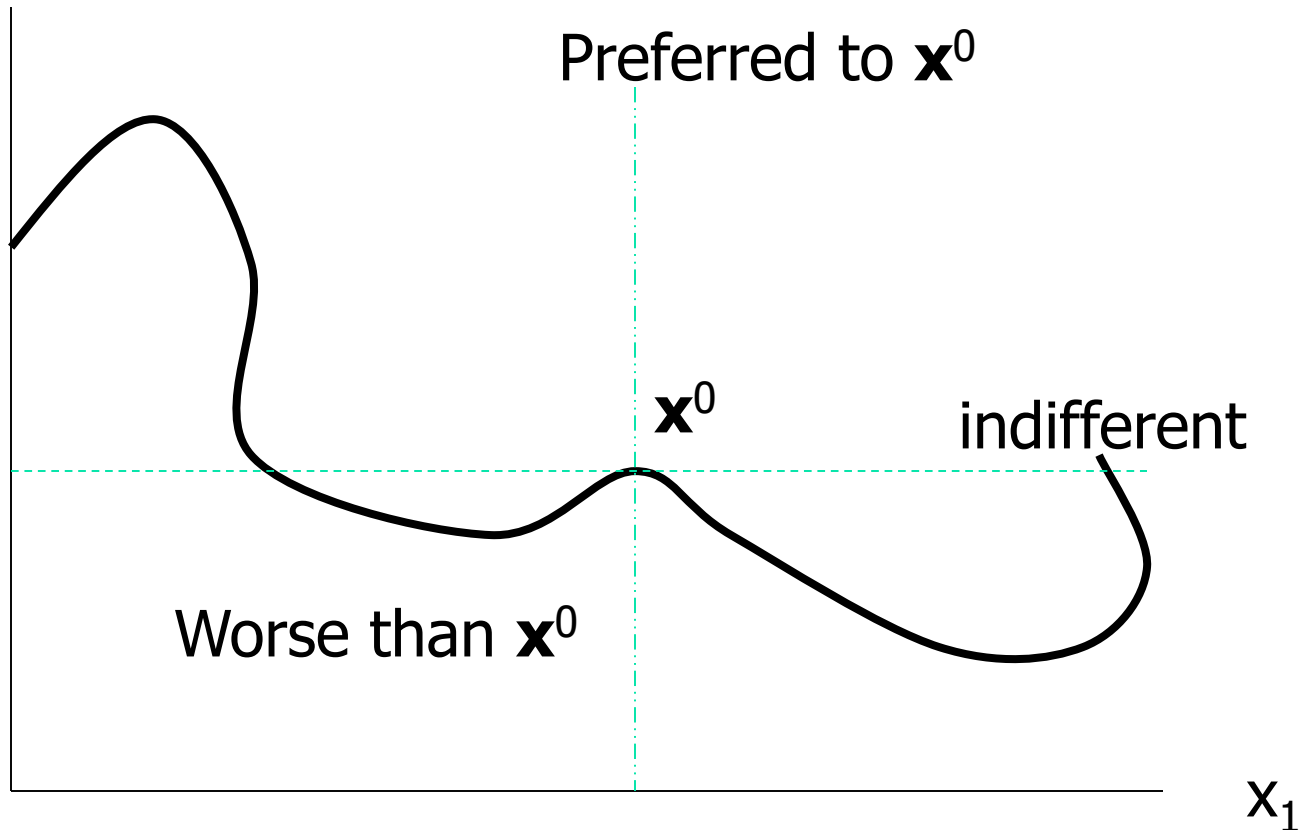
“If  $\mathbf{x}^0$  contains at least as much of goods as  $\mathbf{x}^1$ , then  $\mathbf{x}^0$  is at least as good as  $\mathbf{x}^1$ .”

- If  $\mathbf{x}^0 \gg \mathbf{x}^1$ , then  $\mathbf{x}^0 \succ \mathbf{x}^1$ .

“ If  $\mathbf{x}^0$  contains strictly more of every good, then  $\mathbf{x}^0$  is strictly preferred to  $\mathbf{x}^1$ ”.

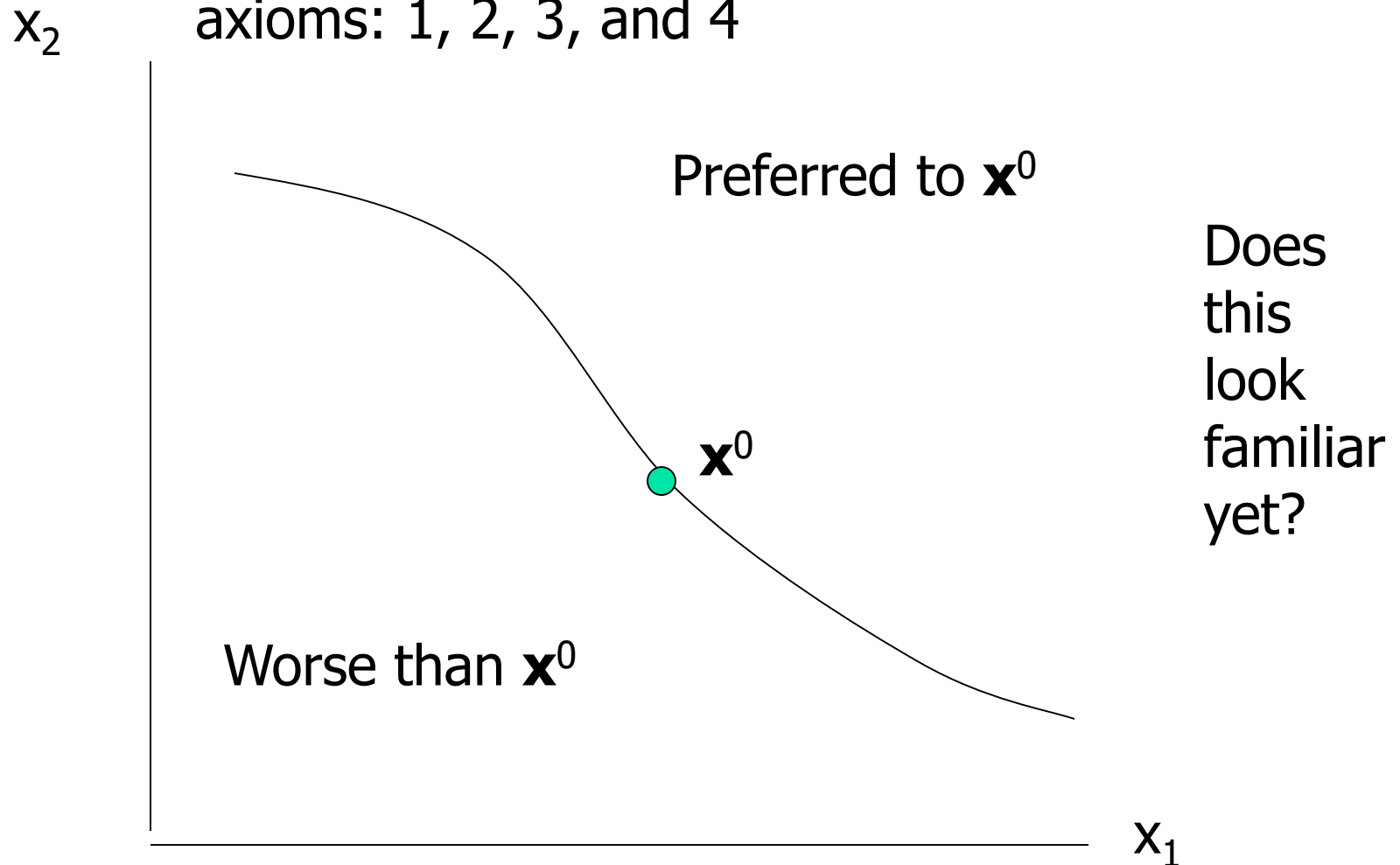
Hypothetical preferences satisfying  
axioms: 1, 2, 3, and 4'

$x_2$



Indifferent to  $\mathbf{x}^0$  must not be NE and SW quadrants.

Hypothetical preferences satisfying  
axioms: 1, 2, 3, and 4



Axioms 1-4 tells possible Indifferent curve.



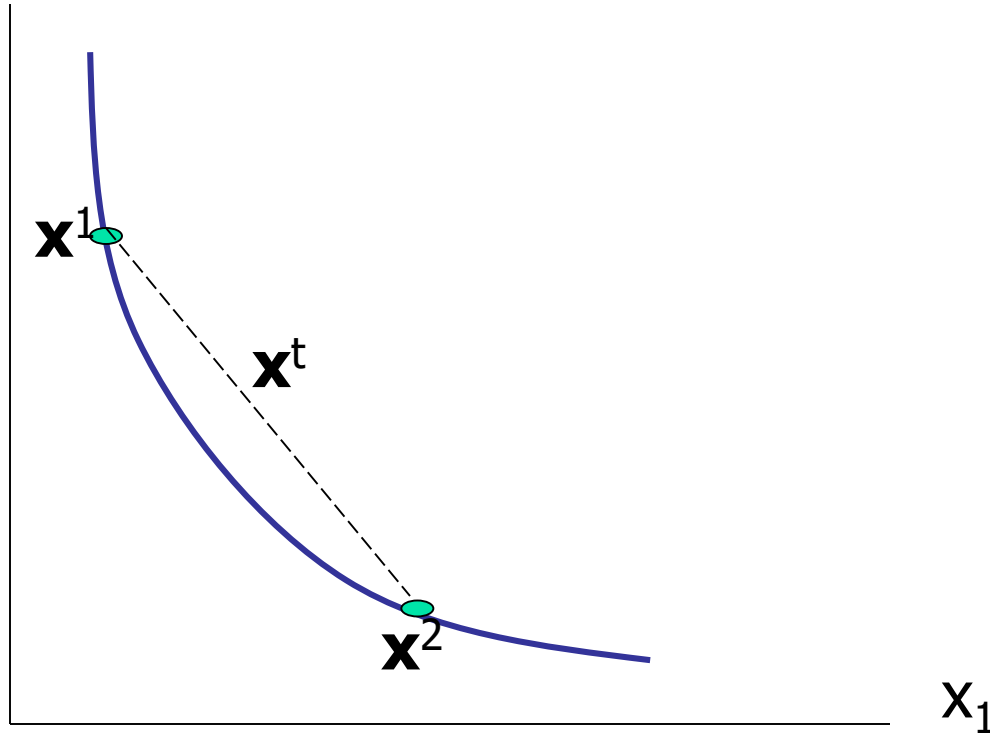
## 1.2.1. Preferences Relations

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- A5': Convexity implies balanced bundles is preferred to extreme ones.
- If  $\mathbf{x}^1 \sim \mathbf{x}^2$ , then  $t\mathbf{x}^1 + (1-t)\mathbf{x}^2 \succsim \mathbf{x}^1$ , for all  $t \in [0,1]$ .
- A5: Strong convexity: replace above with  $\succ$
- We rule out the portion that is concave to origin.

Satisfy Axioms 1-5: Diminishing MRS: rate of giving up  $x_2$  in exchange of a unit of  $x_1$

$x_2$





# 1.2.1. Preferences Relations: summary

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- A1, A2 tells consumer can make consistent comparisons among choices.
- A3 guarantees nice topology and existence of the mutually exclusive sets.
- A4 and A5 characterizes consumer's tastes over choices. Tastes displays nonsatiation (weak or strong) and favor balanced consumption.



## 1.2.2 Utility Function

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- A way to summarize or represent information in consumer's preference relation.
- With helps of calculus, it is convenient to work with a utility function instead.



## 1.2.2 Utility Function

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- Def 1.5.

A real valued function  $u : \mathbb{R}^n_+ \rightarrow \mathbb{R}$  is called a utility function representing the preference relation, if for all  $\mathbf{x}^0, \mathbf{x}^1$ ,  
 $u(\mathbf{x}^0) \geq u(\mathbf{x}^1) \Leftrightarrow \mathbf{x}^0 \succsim \mathbf{x}^1$ .

- So, it assigns higher numbers to preferred bundles.



## 1.2.2 Utility Function

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- It can be shown that any binary relation that is complete, transitive, and continuous can be represented by a continuous real-valued utility function. (See Debreu 1954)
- This existence of the utility function does not depend on assumptions about consumer tastes (convexity and Monotonicity).



## 1.2.2 Utility Function

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- Theorem 1.1. Existence of utility function
- (skip) we claim that at least one such function exists and represents given preferences.
- Theorem 1.2. Utility function is invariant to positive monotonic transformation. If  $u$  represents your preferences, then so  $v=u+5$  or  $w=u^2$ , since these functions will rank bundles the same way as  $u$  does.



## 1.2.2 Utility Function

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*For example :for  $a, b > 0$*

*$u = x_1^a x_2^b$  known as Cobb - Douglas function*

*let  $v = u^{1/a+b}$*

$$v = x_1^{a/a+b} x_2^{b/a+b}$$

$$= x_1^a x_2^b$$

$$= x_1^a x_2^{1-a}$$



## 1.2.2 Utility Function

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- Theorem 1.3. If preference relation is represented by  $u$ , then we have
  - 1.  $u(\mathbf{x})$  is **strictly increasing** iff preference relation is **strictly monotonic**.
  - 2.  $u(\mathbf{x})$  is **quasiconcave** iff preference relation is **convex**.
  - 3.  $u(\mathbf{x})$  is strictly **quasiconcave** iff preference relation is **strictly convex**.



## 1.2.2 Utility Function

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- Continuity does not rule out kink.
- To use calculus, we assume differentiability of  $u$ , which implies IC are smooth and continuous.
- Slope of IC is MRS
- $u(x_1, x_2) = \text{constant}$
- $u_1 dx_1 + u_2 dx_2 = 0$ , so MRS or  $dx_2/dx_1$
- $MRS_{12} = -u_1 / u_2$  this is a function to be evaluated



## 1.2.2 Utility Function

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- For any quasiconcave utility function, Hessian matrix of  $u$  will satisfy

$$\mathbf{y}^T \mathbf{H}(\mathbf{x}) \mathbf{y} \leq 0 \text{ for all } \mathbf{y} \text{ such that}$$

$$\square \mathbf{u}(\mathbf{x}) \cdot \mathbf{y} = 0.$$

- If the inequality is strict, this says that moving from  $\mathbf{x}$  in a direction  $\mathbf{y}$  that is tangent to the indifference surface through  $\mathbf{x}$  reduces utility.



## 1.2.2 Utility Function

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- MRS tells us the rate at which  $x_1$  can be substituted for  $x_2$  without changing utility level.
- By strict convexity, MRS is always strictly diminishing.
- By strictly monotonicity and differentiability of  $u$ , MU is always strictly positive,  $\partial u / \partial x_i > 0$



## 1.3 Consumer's Problem

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- We set up the problem by assuming that a utility function is continuous, strictly increasing, and strictly quasiconcave.
- Our domain set is the positive orthants.
- We assume that commodities market prices are strictly positive ( $\mathbf{p} \succ \mathbf{0}$ ) and income is fixed ( $y \geq 0$ ).



## 1.3 Consumer's Problem

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- Assumption 1.2 Consumer Preference:  
Preference relation satisfies A1-5 (complete, transitive, continuous, strictly monotonic and strictly convex), it can be represented by a real-valued utility function that is continuous, strictly increasing, and strictly quasiconcave.
- We have a nice IC map for a 2-good case: non-intersect, strictly convex to origin, increasingly NE.



## 1.3 Consumer's Problem

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- Budget Set,  $B = \{ (x_1, x_2) / p_1x_1 + p_2x_2 \leq y \}$ .
- UMP: Maximize  $u(x)$  subject to  $B$ .
- Given that (a)  $u$  is continuous, and (b) budget set is nonempty, closed and bounded, thereby compact, this guarantees existence of extreme values of  $u$  over budget set.
- Since  $u$  is strictly quasiconcave, the maximizer,  $x^*$ , is unique.



## 1.3 Consumer's Problem

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- For Existence of extreme values (Weierstrass Theorem, see p. 432 in JR).
- Note a real-valued function map vectors in  $\mathbb{R}^n$  into numbers in real line  $\mathbb{R}$ .
- This W theorem specifies sufficient conditions under which the existence of a max or min of a function is assured.



## 1.3 Consumer's Problem

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- Weierstrass Theorem (see p. 432 in JR).

Let  $f: S \rightarrow \mathbb{R}$  be a continuous real-valued mapping where  $S$  is a nonempty compact subset of  $\mathbb{R}^n$ . Then there exists a vector  $\mathbf{x}^* \in S$  and  $\mathbf{x}^{**} \in S$  such that

$$f(\mathbf{x}^*) \leq f(\mathbf{x}) \leq f(\mathbf{x}^{**}) \text{ for all } \mathbf{x} \in S.$$

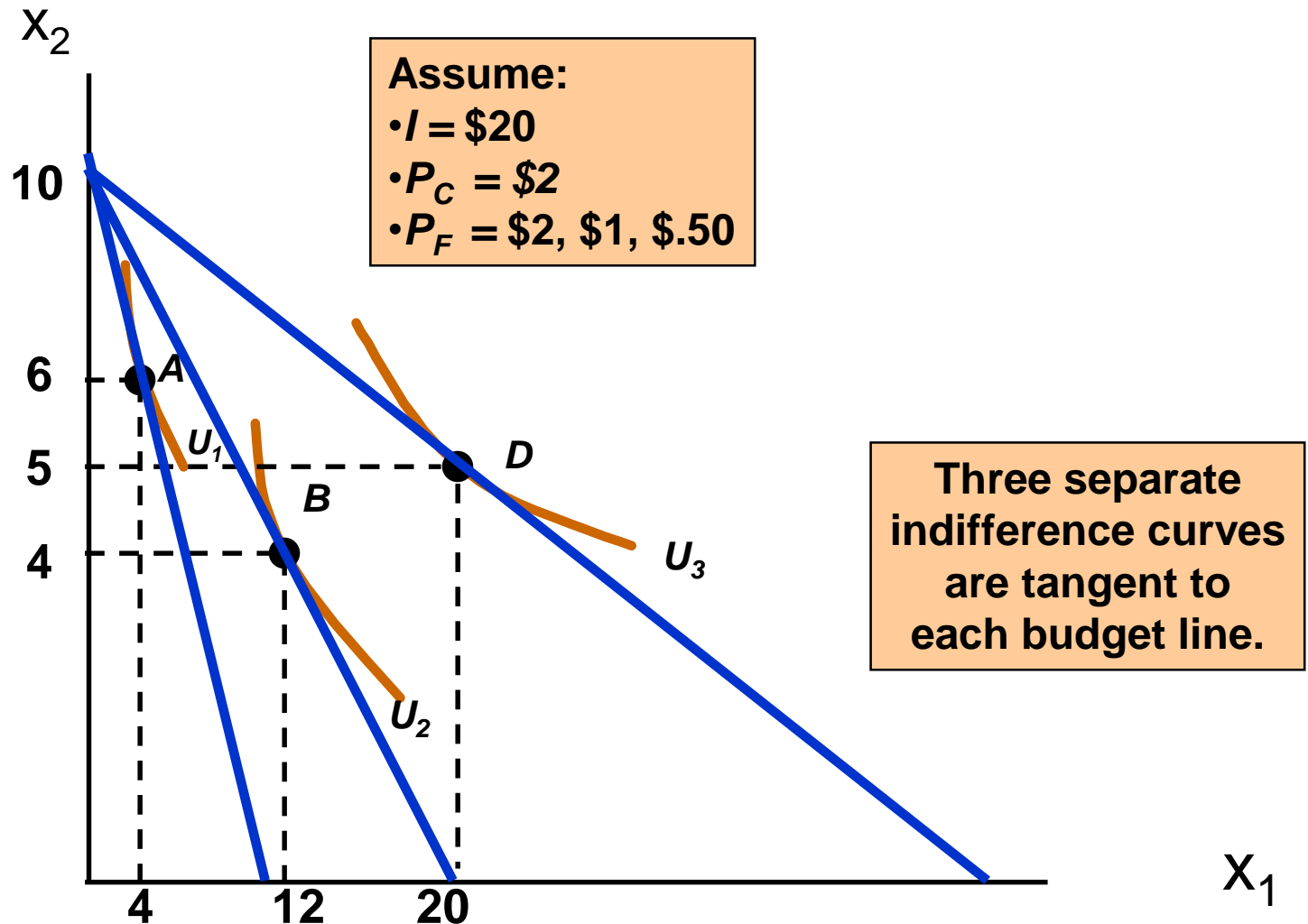


## 1.3 Consumer's Problem

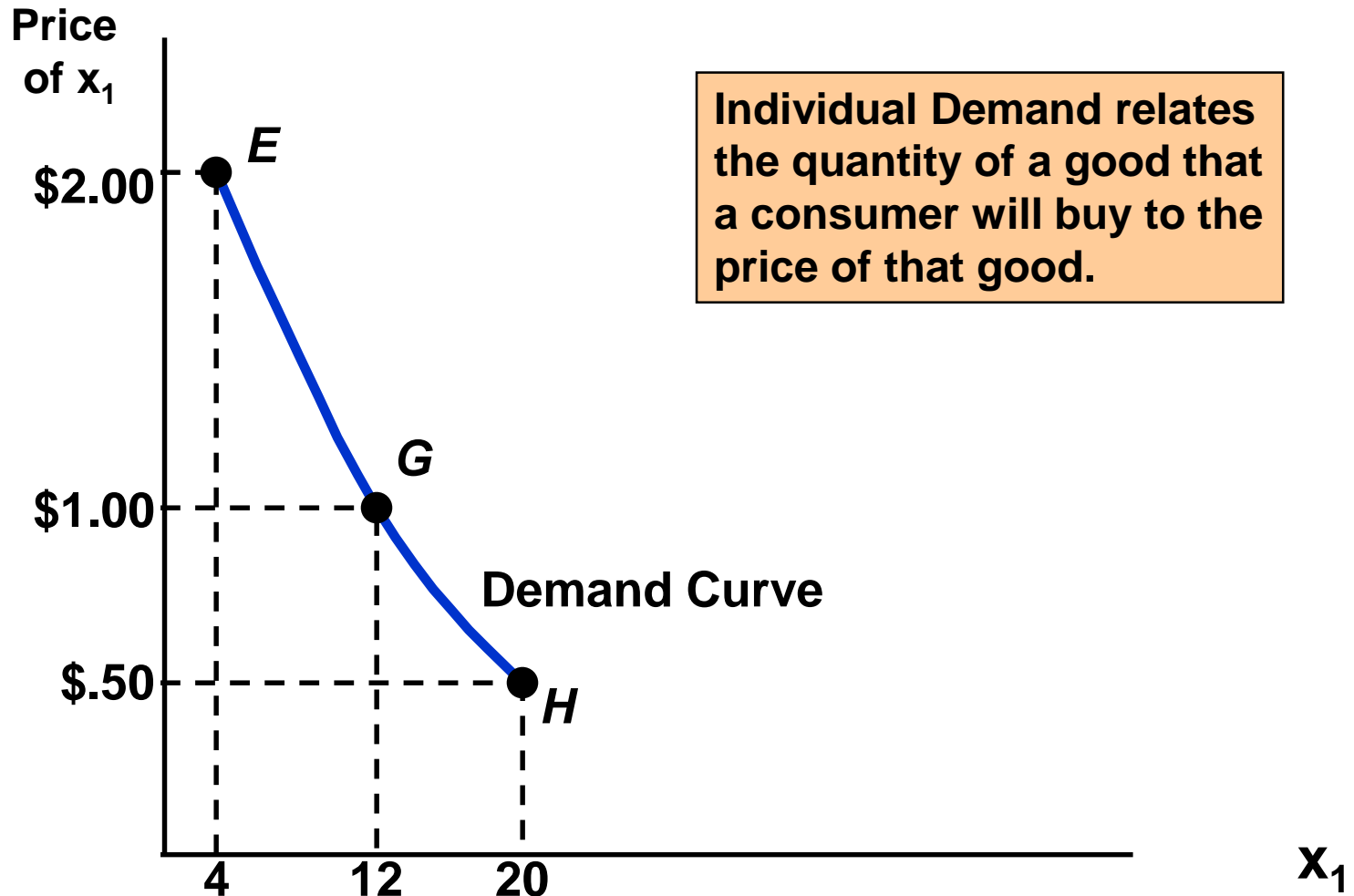
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- Solutions to the U-max problem are *Marshallian demand* functions,
- $x_1(p_1, p_2, y)$  and  $x_2(p_1, p_2, y)$  or in vector  $\mathbf{x}^* = \mathbf{x}(\mathbf{p}, y)$
- We can draw an individual demand curve by allowing for price changes, assuming income fixed.
- Let suppose the price of  $x_1$  changes.

# Demand for $x_1$



# Demand for $x_1$





## 1.3 Consumer's Problem

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- If we add differentiability of  $u$ , we can solve this problem by Lagrange technique.
- Let rewrite constraint as  $y - \mathbf{p}\mathbf{x} \geq 0$  then form
- $L(\mathbf{x}, \lambda) = u(\mathbf{x}) + \lambda [ y - \mathbf{p}\mathbf{x} ]$
- Assuming that solution  $\mathbf{x}^*$  is strictly positive, we can apply Kuhn-Tucker conditions as

$$\frac{\partial L}{\partial x_i} = 0,$$

$$\frac{\partial L}{\partial \lambda} \geq 0, \quad \lambda^* \geq 0 \text{ and } \lambda^* \frac{\partial L}{\partial \lambda} = 0$$



## 1.3 Consumer's Problem

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- If we argue that budget constraint is binding, FOCs become

$$\frac{\partial L}{\partial x_i} = 0, \frac{\partial L}{\partial \lambda} = 0$$

- We solve 3 equations  $L_1 = L_2 = L_\lambda = 0$  to find 3 unknowns: optimum  $x_1$  and  $x_2$  and  $\lambda$ .
- At optimum, MRS = price ratio.
- Note these are just necessary conditions for a local optimum. However, for this particular problem these necessary conditions are also sufficient for a global optimum.



## 1.3 Consumer's Problem

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- Theorem 1.4 Sufficiency of consumer's FOC.  
Suppose  $u$  is continuous and quasiconcave on  $\mathbb{R}^n_+$  and  $(\mathbf{p}, y) \gg \mathbf{0}$ . If  $u$  is differentiable at  $\mathbf{x}^*$  and  $(\mathbf{x}^*, \lambda^*) \gg \mathbf{0}$  solves FOCs, then  $\mathbf{x}^*$  solves UMP.

Proof: skip (see p. 24)



## 1.3 Consumer's Problem

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- Ex1.  $u(x_1, x_2) = x_1^\alpha x_2^\beta$
- Answer:
- $x_1^* = [\alpha / (\alpha + \beta)] y / p_1$
- $x_2^* = [\beta / (\alpha + \beta)] y / p_2$ .
- Check if  $(\alpha + \beta) = 1$ , then  $\alpha = p_1 x_1^* / y$  or budget share of good 1



## 1.3 Consumer's Problem

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- Ex2.  $u(x_1, x_2) = (x_1^\rho + x_2^\rho)^{1/\rho}$  where  $\rho < 1$  and  $\neq 0$ , called CES or constant elasticity of substitution.
- Answer: if let  $r = \rho/(\rho-1)$
- $x_1^* = [p_1^{r-1}] y / [p_1^r + p_2^r]$
- $x_2^* = [p_2^{r-1}] y / [p_1^r + p_2^r]$
- $U(x_1^*, x_2^*) = [p_1^r + p_2^r]^{-1/r} y$
- See textbook for details



## 1.3 Consumer's Problem

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- Ex3.  $u(x_1, x_2) = a \ln x_1 + (1-a) \ln x_2$
- Answer:
- $x_1^* = ay / p_1$
- $x_2^* = (1-a)y / p_2$
- $U(x_1^*, x_2^*) = \ln y - a \ln p_1 - (1-a) \ln p_2$



## 1.3 Consumer's Problem

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- Ex4.  $u(x_1, x_2) = \min \{ax_1, bx_2\}$
- Answer:
- We cannot use the tangency properties here since  $u$  is not differentiable
- We know that at optimum  $ax_1 = bx_2$  or  $x_1 = bx_2/a$ , and we use all income.
- $p_1 bx_2/a + p_2 x_2 = y$
- $x_2^* = ay / [bp_1 + ap_2]$
- $x_1^* = by / [bp_1 + ap_2]$



## 1.3 Consumer's Problem

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- Ex5.  $u(x_1, x_2) = ax_1 + bx_2$
- Answer: We know that goods are perfectly substitute, so this consumer will buy one good if it is cheaper.
- When we have a corner solution, MRS=price ratio condition is not hold.
- Since MRS =  $a/b$  and constant
- If  $MRS > (p_1 / p_2)$ ,  $x_1^* = y / p_1$  and  $x_2^* = 0$
- If  $MRS = (p_1 / p_2)$ ,  $p_1 x_1^* + p_2 x_2^* = y$ .
- Or we will have interior multiple solutions for  $mrs=price$  ratio.



## 1.3 Consumer's Problem

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- When  $MRS > (p_1 / p_2)$  ,  $x_1^* = y / p_1$  and  $x_2^* = 0$
- When  $MRS > \text{Price ratio}$ , this means consumers are willing to give up lots of good 2 for a unit of good 1. Since his rate of substitution for good 1 is higher than the relative prices of good 1. He can do better by trading at the market prices and has money left to spend.



## 1.3 Consumer's Problem

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- The relative price of good 1 in terms of good 2 measures the units of good 2 foregone per one unit of good 1 acquired. For example, in absolute terms,

$$\text{MRS} = \frac{dx_2}{dx_1} = \frac{u_1}{u_2} = \frac{2}{1} > \frac{1}{1} = \frac{p_1}{p_2} = \frac{\text{units of } x_2}{\text{units of } x_1}.$$



## 1.3 Consumer's Problem

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- MRS shows that to consume an additional unit of  $x_1$ , he is willing to give up 2 units of  $x_2$ , but the price of an additional unit of  $x_1$  in terms of  $x_2$  is only 1, so he will be better off to purchase good 1 at the market price.



## 1.3 Consumer's Problem

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- Later we'll focus on the demand curve that is differentiable (not just continuous)
- Theorem 1.5 Differentiable Demand
- Let  $\mathbf{x}^* \gg \mathbf{0}$ ,  $\mathbf{p} \gg \mathbf{0}$ ,  $y > 0$ , if  $u$  is twice continuously differentiable (C2) on  $\mathbb{R}^n_+$ ,  $u_i(\mathbf{x}^*) > 0$  for all  $i$ , Bordered H has a nonzero det, then  $\mathbf{x}^*(\mathbf{p}, y)$  is differentiable.