Matrix Algebra (Quick Review)

1 The notation

- A matrix

$$A = [a_{ij}] = \left[\begin{array}{c} \\ \\ \\ \\ \\ \\ \\ \end{array} \right]_{M \times N}$$

where
i denotes
j denotes
M denotes
N denotes
For example:

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 6 & 1 & 3 \end{bmatrix}_{2\times 3}$$

- Column vector

$$X = \begin{bmatrix} 3 \\ 4 \\ 5 \\ 9 \end{bmatrix}_{4\times 1}$$

- Row vector

$$B = [\ 3 \quad 4 \quad 5 \quad 9\ ]_{1\times 4}$$

- Transposition

$$X = Y'$$

$$A = \begin{bmatrix} 4 & 5 \\ 3 & 1 \\ 5 & 0 \end{bmatrix}_{3\times 2}, \qquad A' = \begin{bmatrix} 4 & 3 & 5 \\ 5 & 1 & 0 \end{bmatrix}_{2\times 3}$$

- Diagonal Matrix

$$A = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Identity Matrix

$$I_{3\times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}, I_{4\times 4} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{4\times 4}$$

$\Rightarrow$ property $AI = A$ or $IA = A$

- Equal Matrices

$A = B$ when all elements are the same.

$$A = \begin{bmatrix} 3 & 4 & 5 \\ 0 & -1 & 2 \\ 5 & 1 & 3 \end{bmatrix}_{3\times 3}, \quad B = \begin{bmatrix} 3 & 4 & 5 \\ 0 & -1 & 2 \\ 5 & 1 & 3 \end{bmatrix}_{3\times 3}$$

2 Matrix Operation

- Addition - $A + B$ is possible when A and B are of the same dimension.

$$\begin{aligned} A &= \begin{bmatrix} 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 \end{bmatrix}_{2\times 4}, \quad B = \begin{bmatrix} 1 & 0 & -1 & 3 \\ -2 & 0 & 1 & 5 \end{bmatrix}_{2\times 4} \\ A + B &= \begin{bmatrix} 3 & 3 & 3 & 8 \\ 4 & 7 & 9 & 14 \end{bmatrix}_{2\times 4} \end{aligned}$$

- Subtraction

$$A - B = \begin{bmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \end{bmatrix}_{2\times 4}$$

- Scalar Multiplication

$$\lambda A = \lambda \left[a_{ij} \right]_{i\times j}$$

Suppose $\lambda = 2$

$$\begin{aligned} A &= \begin{bmatrix} -3 & 5 \\ 8 & 7 \end{bmatrix} \\ \lambda A &= \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \end{aligned}$$

- Matrix Multiplication

$A_{(M\times N)}B_{(O\times P)}$ is possible if an only if $N = O$ ($\leftarrow$ Letter O). The multiplication result would be of dimension ___________.
Let

$$A = \begin{bmatrix} 3 & 4 & 7 \\ 5 & 6 & 1 \end{bmatrix}_{2\times 3}, \qquad B = \begin{bmatrix} 2 & 1 \\ 3 & 5 \\ 6 & 2 \end{bmatrix}_{3\times 2}$$

$$C = A \times B = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}_{2\times 2}$$

$$C = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}_{2\times 2}$$

$$C_{ij} =$$

- Matrix Differentiation

Rule1 - If $a' = [\, a_1 \quad a_2 \quad a_3 \quad \ldots \quad \ldots \quad a_n \,]$ and $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \ldots \\ x_n \end{bmatrix}$

Then

- Rule2 - If $X'AX = [\, x_1 \quad x_2 \quad x_3 \quad \ldots \quad \ldots \quad x_n \,]_{1\times n} \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}_{n\times n} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \ldots \\ x_n \end{bmatrix}_{n\times 1}$

Then

3 OLS Estimation using Matrix Notation

Let

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ .. \\ .. \\ .. \\ .. \\ \\ y_n \end{bmatrix}_{n\times 1} = \begin{bmatrix} 1 & x_{11} & x_{21} & ... & x_{k1} \\ 1 & x_{12} & & & .. \\ 1 & x_{13} & & & .. \\ .. & .. & & & .. \\ \\ \\ \\ \\ 1 & x_{1n} & & & x_{kn} \end{bmatrix}_{n\times(k+1)} \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \\ \hat{\beta}_2 \\ ... \\ \hat{\beta}_k \end{bmatrix}_{(k+1)\times 1} + \begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \hat{u}_3 \\ .. \\ .. \\ .. \\ .. \\ .. \\ \hat{u}_n \end{bmatrix}_{n\times 1}$$

Example: Let the estimated model be

$$y = \beta_0 + \beta_1 x_1 + u$$

Let
$\hat{\beta} = \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{bmatrix}_{2\times 1}$

$X'X =$

4 Variance-Covariance Matrix of $(\hat{\beta})$

$$
\begin{aligned}
\text{From Variance-Covariance Matrix of } (\hat{\beta}) &= E\{[\hat{\beta} - E(\hat{\beta})][\hat{\beta} - E(\hat{\beta})]'\} \\
&= E\{[\hat{\beta} - \beta][\hat{\beta} - \beta]'\} \\
\text{from } \hat{\beta} &= (X'X)^{-1}X'Y
\end{aligned}
$$

Multiple Regression Analysis (Inference)

Objectives

1. Students know how to test hypotheses about the parameter (β)

2. Students can test for the validity of the proposed population model.

Econometrics Analysis - The Steps

1 Sampling Distribution of the OLS estimators ($\hat{\beta}_{OLS}$)

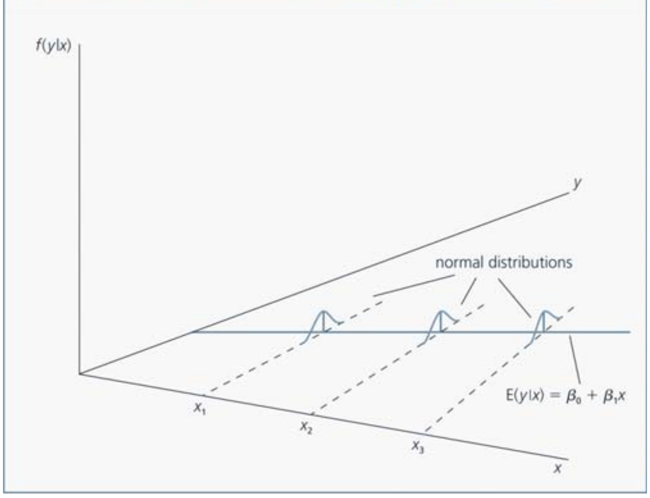
To be able to test hypotheses about the parameter ($\hat{\beta}$), we need an assumption about the distribution of u.

Assumption MLR 6 - Normality

The population error u is independent of the explanatory variables $X_1, X_2, ...X_k$ and is **normally distributed** with zero mean and variable σ^2.

FIGURE 4.1

The homoskedastic normal distribution with a single explanatory variable.



- By normality of the error term (u), we have nomality of $\hat{\beta}$

Or, using the standardization $\frac{Z-\mu}{\sigma} \sim$ normal (0,1), we have

2 Testing Hypotheses about an individual regression coefficient "the t-test"

- But we do not have $sd.(\hat{\beta}_j)$, we can only calculate the estimator of it :

- for degree of freedom > 30, $t \approx z$.We can use the $Z - score$ table to find the critical value(s).

2.1 Testing Against Two-Sided Alternatives

Consider the wage equation :

$$\log(wage) = \beta_0 + \beta_1 educ + \beta_2 \exp er + \beta_3 tenure + u.$$

Suppose we want to test whether experience has a partial effect on wage:

$$\begin{aligned} H_0 &: \beta_2 = 0 \quad \text{(experience has no partial effect)} \\ H_a &: \beta_2 \neq 0. \text{ (experience has a partial effect)} \end{aligned}$$

** Note that

Suppose $n = 34$, degree of freedom $= 34 - 3 - 1 = 30$.
From the table t-table, the 5% critical value for a 2-tailed test with 30 d.f. is __________

- If we change the significance level to 10%, then the 2-tailed critical value becomes __________

- If we cannot reject $H_0 : \beta_2 = 0$ (at a given significance level – e.g. 5%), we say that X_2 is statistically significant at the 5% level.

- In other words X_2 has a partial effect on the expected value of Y.

```
. regress log_wage educ exper tenure

      Source |       SS       df       MS              Number of obs =     526
-------------+------------------------------           F(  3,   522) =   80.39
       Model |  46.8741776     3  15.6247259           Prob > F      =  0.0000
    Residual |  101.455574   522  .194359337           R-squared     =  0.3160
-------------+------------------------------           Adj R-squared =  0.3121
       Total |  148.329751   525   .28253286           Root MSE      =  .44086

------------------------------------------------------------------------------
    log_wage |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
        educ |    .092029   .0073299    12.56   0.000     .0776292    .1064288
       exper |   .0041211   .0017233     2.39   0.017     .0007357    .0075065
      tenure |   .0220672   .0030936     7.13   0.000     .0159897    .0281448
       _cons |   .2843595   .1041904     2.73   0.007     .0796756    .4890435
------------------------------------------------------------------------------
```

2.2 Testing Against One-Sided Alternatives

Suppose the rule for rejecting H_0 becomes

$$H_a : \beta_j > 0.$$

We could have the H_0 as

$$\begin{aligned} H_0 &: \beta_j \leq 0 \qquad \text{or} \\ H_0 &: \beta_j = 0 \end{aligned}$$

depending on the context. For $H_0 : \beta_j = 0$, it implies that we are ruling out population values of β_j less than zero. For $H_0 : \beta_j \leq 0$,it implies that we are not ruling out population values of β_j less than zero. To given an example of a test against the one-sided alternative hypothesis, consider the wage equation again :

$$\log(wage) = \beta_0 + \beta_1 educ + \beta_2 \exp er + \beta_3 tenure + u$$

If we want to test whether experience has a positive partial effect on wage

$$\begin{aligned} H_0 &: \beta_2 = 0 \\ H_a &: \beta_2 > 0. \end{aligned}$$

We implicitly rule out the possibility that $\beta_2 < 0$ here.

3 Testing other hypotheses about β_j

- Most common H_0 is $H_0 : \beta_j = 0$.

- However, we can test other types of hypotheses. For example, $H_0 : \beta_j = a_j$ where a_j is a constant number.

If we want to test the H_0 using a 5% significance level, then we reject H_0 if

$$t_{wife\_income} > \_\_\_\_\_\_\_\_ \quad or$$
$$t_{wife\_income} < \_\_\_\_\_\_\_\_$$

4 Testing Hypotheses about a Single Linear Combination of the Parameter

Consider

$$\log(wage) = \beta_0 + \beta_1 jc + \beta_2 univ + \beta_3 \exp er + u$$

where $jc =$ number of years attending a two-year college
$univ =$ number of years at a four-year college
$\exp er =$ months in the workforce.
We want to test whether $\beta_1 = \beta_2$.

another possible hypothesis test (one-tailed alternative)

5 Computing p-Values for t-Tests

- What is the significance level given the computed t-statistics?

- p-value : $P(|T| > |t|)$

Example 1: $H_0 : \beta_j \geq 0, \quad H_a : \beta_j < 0, \quad$ d.f.$= 140$.

suppose the calculated $t_{\hat{\beta}_j} = -2.75$

- From the z-table, the value -2.75 corresponds to area = _____________.
- Thus, p-value = ___________.
- Would we reject H_0 if we use the significance level = 5%?

Example 2: $H_0 : \beta_j = a_j, \quad H_a : \beta_j \neq a_j, \quad$ d.f.$= 18$.

suppose the calculated $t_{\hat{\beta}_j} = -2.18$

- From the t-table, the value -2.18 corresponds to area = _____________.
- Thus, p-value = ___________.
- Would we reject H_0 if we use the significance level = 5%?

6 Confidence Intervals (CI)

- Confidence Intervales for the POPULATION PARAMETER (β_j)
- A 95% CI of β_j is given by

Example 1: **95% CI**

Example 2: **99% CI**