5. A person has utility function

$$u(x, y) = 100xy + x + 2y$$

Suppose that the price per unit of x is \$2, and that the price per unit of y is \$4. The person receives \$1000 that all has to be spent on the two commodities x and y. Solve the utility maximization problem.

8. A firm produces and sells two commodities. By selling x tons of the first commodity the firm gets a price per ton given by $p = 96 - 4x$. By selling y tons of the other commodity the price per ton is given by $q = 84 - 2y$. The total cost of producing and selling x tons of the first commodity and y tons of the second is given by $C(x, y) = 2x^2 + 2xy + y^2$.

   (a) Show that the firm's profit function is $P(x, y) = -6x^2 - 3y^2 - 2xy + 96x + 84y$.

   (b) Compute the first-order partial derivatives of P, and find its only stationary point.

   (c) Suppose that the firm's production activity causes so much pollution that the authorities limit its output to 11 tons in total. Solve the firm's maximization problem in this case. Verify that the production restrictions do reduce the maximum possible value of $P(x, y)$.

SM 9. Consider the utility maximization problem

$$\max x^a + y \quad \text{subject to} \quad px + y = m$$

where all constants are positive, $a \in (0, 1)$.

   (a) Find the demand functions, $x^*(p, m)$ and $y^*(p, m)$.

   (b) Find the partial derivatives of the demand functions w.r.t. p and m, and check their signs.

   (c) How does the optimal expenditure on the x good vary with p? (Check the elasticity of $px^*(p, m)$ w.r.t. p.)

   (d) Put $a = 1/2$. What are the demand functions in this case? Denote the maximal utility as a function of p and m by $U^*(p, m)$, the value function, also called the indirect utility function. Verify that $\partial U^*/\partial p = -x^*(p, m)$.

SM 10. Consider the problem

$$\max U(x, y) = 100 - e^{-x} - e^{-y} \quad \text{subject to} \quad px + qy = m$$

   (a) Write down the first-order conditions for the problem and solve them for x, y, and λ as functions of p, q, and m. What assumptions are needed for x and y to be nonnegative?

   (b) Verify that x and y are homogeneous of degree 0 as functions of p, q, and m.

3. A consumer's demands x, y, z for three goods are chosen to maximize the utility function

$$U(x, y, z) = x + \sqrt{y} - 1/z \quad (x \geq 0,\ y > 0,\ z > 0)$$

subject to the budget constraint $px + qy + rz = m$, where $p, q, r > 0$ and $m \geq \sqrt{pr} + p^2/4q$.

(a) Write out the first-order conditions for a constrained maximum.

(b) Find the utility-maximizing demands for all three goods as functions of the four variables (p, q, r, m).

(c) Show that the maximized utility is given by the indirect utility function

$$U^*(p, q, r, m) = \frac{m}{p} + \frac{p}{4q} - 2\sqrt{\frac{r}{p}}$$

(d) Find $\partial U^*/\partial m$ and comment on your answer.