

6 Integration

6.1 Definition of Integral

For a function, $F(x)$ that its derivative is $f(x)$

Or mathematically $F'(x) = f(x)$

This function, $F(x)$, then can be called an '**anti-derivative**' of $f(x)$. For example,

- If $f(x) = nx^{n-1}$, then $F(x) = x^n$; as long as $n \neq 0$.
- If $f(x) = Ce^{Cx}$, then $F(x) = e^{Cx}$.
- If $f(x) = \frac{1}{x}$, then $F(x) = \ln x$.

It should be noted that a function does not have a **unique** anti-derivative.

For example, if $f(x) = 3x^2$

then $F(x) = x^3$ is one anti-derivative

or $F(x) = x^3 + 1$, $F(x) = x^3 + 8$, etc (Infinite number of anti-derivatives)

We may spot that any $F(x) = x^3 + C$ where C is a constant, can be anti-derivative of $f(x)$.

(You can try differentiate $F(x)$ and you will see that this is true because $\frac{d}{dx}(C) = 0$.)

Hence, for this function $f(x) = 3x^2$,

$F(x) = x^3 + C$ where C is a constant and is called **a constant of integration**.

This process of anti-differentiation is called **INTEGRATION**.

In other words, to integrate a function, $f(x)$ is to find $F(x)$ such that $F'(x) = f(x)$.

A symbol for integrating a function $f(x)$ is written as $\int f(x)dx$

$$\int f(x)dx = F(x) + C \quad \text{if and only if} \quad F'(x) = f(x)$$

where $\int$ is called the *integral sign*.

$f(x)$ is called the *integrand*.

C is called the *constant of integration*.

dx is part of the integral notation and indicates the variable involved.

REMEMBER that the integration is not unique; **DON'T FORGET THE CONSTANT OF INTEGRATION!**

6.2 Indefinite Integral

If we have to integrate a function, $f(x)$ and values of x are not given, we said we have to integrate without a limit. Hence, the integral will be a function of x .

From some of the differentiation rules, the following integration formulas can be derived:

1	$\int kdx = kx + C$, k is a constant
2	$\int x^n dx = \frac{x^{n+1}}{n+1} + C$, $n \neq -1$
3	$\int e^x dx = e^x + C$
4	$\int \frac{1}{x} dx = \ln x + C$
5	$\int kf(x)dx = k \int f(x)dx$, k is a constant
6	$\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$

NOTE: we can always check the integration result by differentiating it!

Ex.1 Find the following integrals.

(i) $\int 5dx$

[ANS: $5x + C$]

(ii) $\int x^4 dx$

[ANS: $\frac{x^5}{5} + C$]

$$(iii) \quad \int \frac{1}{\sqrt{x}} dx \quad [ANS: 2\sqrt{x} + C]$$

$$(iv) \quad \int (2\sqrt[5]{x^4} - 7x^3 + 10e^x - 1) dx \quad [ANS: \frac{10}{9}x^{\frac{9}{5}} - \frac{7}{4}x^4 + 10e^x - x + C]$$

$$(v) \quad \int \left(\frac{(x-1)(2x+7)}{3} \right) dx \quad [ANS: \frac{1}{3} \left[\frac{2}{3}x^3 + \frac{5}{2}x^2 - 7x \right] + C]$$

Ex.2: Suppose a firm has a marginal cost

$$MC = a + bQ + dQ^2$$

Find the equation of its total cost curve.

$$[Ans: C(Q) = aQ + \frac{bQ^2}{2} + \frac{dQ^3}{3} + e. \quad e \text{ is the constant of integration. It is the value of } C(Q)$$

when $Q = 0$, i.e. the fixed cost of production.]

If a value of $f(x_i)$ [initial boundary condition] is given,
the constant of integration (C) can be evaluated to obtain a specific equation.

$$\textbf{Ex.3:} \text{ If } f'(x) = 2x \text{ and } f(1) = 4, \text{ find } f(x). \quad [ANS: x^2 + 3]$$

Ex.4: Find the function $f(x)$ whose tangent has slope $\sqrt{x\sqrt{x\sqrt{x}}}$ for each value of x and whose graph passes through the point (1,1). [ANS:

$$\frac{8}{15}x^{\frac{15}{8}} + \frac{7}{15}]$$

Ex.5: For a firm, the marginal cost is $MC = 3Q + 4$. If the fixed cost are 40, find the total cost function $C(Q)$. [ANS: $C = \frac{3}{2}Q^2 + 4Q + 40$]

If there are n numbers of constants of integration, n numbers of initial boundary conditions $f(x_i)$ are required to evaluate all constants of integration.

Ex.6: If $y'' = x^2 + x + 1$, find $y(x)$ for [ANS: $\frac{x^4}{12} + \frac{x^3}{6} + \frac{x^2}{2} + C_1x + C_2$]

(i) when $y(0) = 0$ and $y(1) = \frac{7}{4}$ [ANS:

$$\frac{x^4}{12} + \frac{x^3}{6} + \frac{x^2}{2} + x]$$

(ii) when $y(0) = 0$ and $y'(0) = 0$ [ANS: $\frac{x^4}{12} + \frac{x^3}{6} + \frac{x^2}{2}$]

NOTE that if $y'(a)$ and $y'(b)$ are given, these are not enough because C_2 cannot be determined.

Ex.7: Given the marginal-revenue function $MR = 2000 - 20Q - 3Q^2$. Find the demand function. [ANS: $2000 - 10Q - Q^2$]

Hint: If there is no product sold ($Q = 0$) then the revenue is zero.

6.3 Power Rule for Integration and Integration by Substitution.

$$\frac{d}{dx} \left(\frac{[u(x)]^{n+1}}{n+1} \right) = [u(x)]^n \cdot u'(x)$$

$$\therefore \int [u(x)]^n \cdot u' dx = \frac{[u(x)]^{n+1}}{n+1} + C$$

But $u'(x) = \frac{du}{dx} \Rightarrow u'(x)dx = du$

$$\therefore \int [u(x)]^n du = \frac{u^{n+1}}{n+1} + C$$

Power Rule

$$\text{If } u \text{ is differentiable and a function of } x, \text{ then } \int u^n du = \frac{u^{n+1}}{n+1} + C \text{ if } n \neq -1.$$

Ex.8: Find $\int (x+5)^5 dx$.

Integration by substitution, try $u = x+5 \Rightarrow du = 1 \cdot dx$

$$\int (x+5)^5 dx = \int u^5 du = \frac{u^6}{6} + C$$

Substitute u gives $\int (x+5)^5 dx = \frac{(x+5)^6}{6} + C$

Note that you **MUST** give the answer explicitly in term of x **NOT** u .

Ex.9: Find $\int \sqrt[3]{6x} dx$. (**Hint:** $u=6x$) [ANS: $\frac{(6x)^{\frac{4}{3}}}{8} + C$]

Ex.10: Show that $\int x^2(3x^3 + 7)^3 dx = \frac{(3x^3 + 7)^4}{36} + C$

[**Hint:** $u = 3x^3 + 7$]

6.4 More Integration by Substitution

Let u be a function of x ,

$$\int e^u du = e^u + C$$

$$\int \frac{1}{u} du = \ln|u| + C$$

Ex.11: Show that $\int e^{ax} dx = \frac{1}{a} e^{ax} + C$.

[**Hint:** $u = ax$]

Ex.12: Show that $\int a^x dx = \frac{1}{\ln a} a^x + C$.

Ex.13: Show that $\int (2t+1)e^{t^2+t} dt = e^{t^2+t} + C$.

[**Hint:** $u = t^2 + t$]

Ex.14: Show that $\int \frac{7t}{5t^2 - 6} dt = \frac{7}{10} \ln|5t^2 - 6| + C$.

Fill in the table by specifying the substitution you may choose for each integral.

Integral	Substitution u
1. $\int (3x+4)^{\frac{5}{2}} dx$	
2. $\int \left(\frac{4}{3-x} \right) dx$	
3. $\int t e^{2-t^2} dt$	
4. $\int t(2+t^2)^3 dt$	
5. $\int \frac{3}{(2x-5)^4} dx$	

6. $\int x^2 e^{-x^3} dx$	
7. $\int \frac{e^t}{e^t + 1} dt$	
8. $\int \frac{t+3}{\sqrt[3]{t^2+6t+5}} dt$	

6.4.1 Integration Hints

- Try to arrange the integrand into a familiar integration forms (seen formulas). If you can't find the right form, try some more! There is no shortcut! **ONLY PRACTICE!**
- When integrating fraction forms, sometimes a preliminary long division is needed to get familiar integration forms.

Ex.15: Show that $\int \left(\frac{x+3}{x+6} \right) dx = x - 3 \ln|x+6| + C$.

Ex.16: Show that $\int \left(\frac{x^3 + x^2 - x - 3}{x^2 - 3} \right) dx = \frac{x^2}{2} + x + \ln|x^2 - 3| + C$.

Ex.17: Show that $\int \left(\frac{9x^2 + 5}{3x} \right) dx = \frac{3}{2}x^2 + \frac{5}{3} \ln|x| + C$.

Ex.18: Show that $\int \left(\frac{6x^2 - 11x + 5}{3x - 1} \right) dx = x^2 - 3x + \frac{2}{3} \ln|3x - 1| + C$.

Ex.19: Show that $\int 2x(x^2 + 3)^5 dx = \frac{(x^2 + 3)^6}{6} + C$.

Ex.20: Show that $\int x^3 e^{4x^4} dx = \frac{1}{16} e^{4x^4} + C$.

Ex.21: $y' = (3 - 2x)^2$, $y(0) = 1$, $y(x) = ?$.

[ANS: $-\frac{1}{6}(3 - 2x)^3 + \frac{11}{2}$]

Ex.22: $y'' = \frac{1}{x^2}$, $y'(-1) = 1$, $y(1) = 0$, $y(x) = ?$.

[ANS: $-\ln|x|$]

6.5 Summation (Revisions)

$$\sum_{i=0}^n i = 0 + 1 + 2 + \dots + n = \sum_{k=0}^n k$$

Σ = notation for summation called 'sigma'.

i, k = index of summation (a 'dummy' symbol)

$0, n$ = limits of summation

If $n = 5$, $\sum_{i=0}^5 i = 0 + 1 + 2 + 3 + 4 + 5$

Ex.23: Show that $\sum_{i=1}^5 3i^2 = 3 \sum_{i=0}^5 i^2$.

ANS:

$$\begin{aligned} \sum_{i=1}^5 3i^2 &= 3(1)^2 + 3(2)^2 + 3(3)^2 + 3(4)^2 + 3(5)^2 \\ &= 3(1^2 + 2^2 + 3^2 + 4^2 + 5^2) \\ &= 3 \sum_{i=0}^5 i^2 \end{aligned}$$

$$\boxed{\sum_{i=0}^n Cx_i = C \sum_{i=0}^n x_i}$$

Ex.24: If $\sum_{k=1}^n k = 1 + 2 + 3 + \dots + n$, show that $\boxed{\sum_{k=1}^n k = \frac{n(n+1)}{2}}$ is the general form in term of n .

ANS: Let $S = \sum_{k=1}^n k = 1 + 2 + 3 \dots + n$ (1)

$$\text{Reverse} \quad S = n + (n-1) + (n-2) \dots + 1 \quad (2)$$

$$(1)+(2) \quad 2S = (n+1) + (n-1+2) + (n-2+3) + \dots + (n+1)$$

$$2S = (n+1) + (n+1) + (n+1) + \dots + (n+1)$$

$$2S = n \cdot (n+1)$$

$$S = \frac{n \cdot (n+1)}{2} = \sum_{k=1}^n k$$

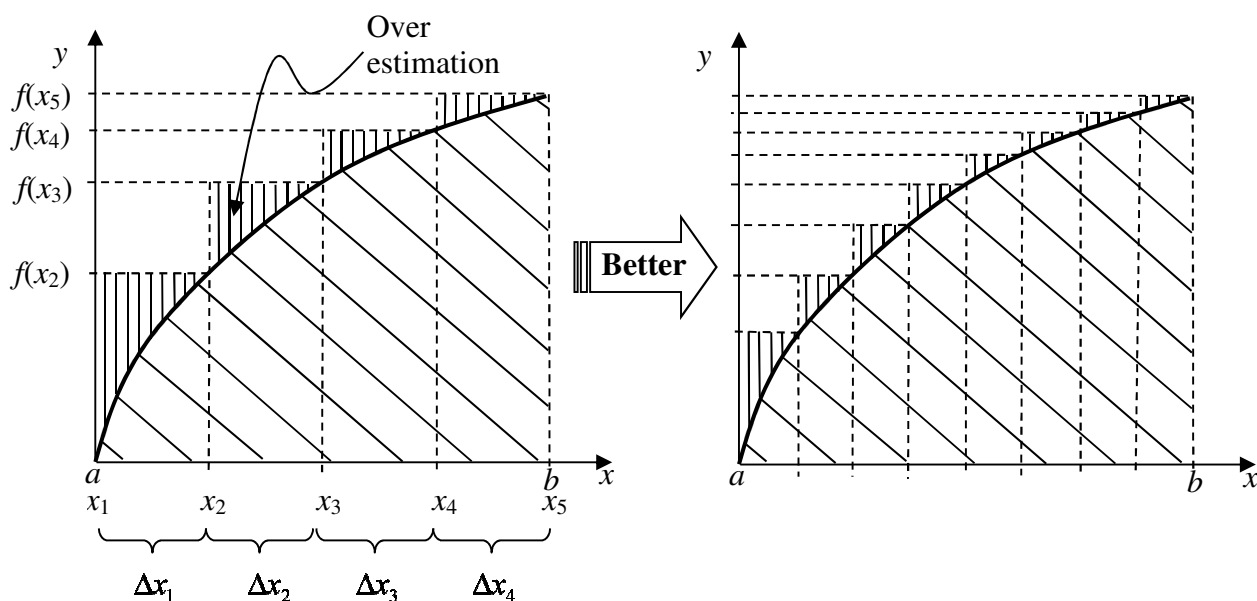
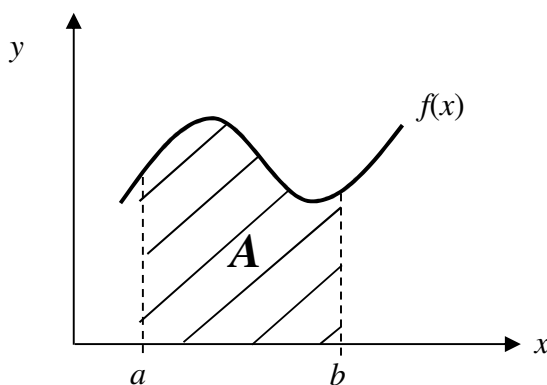
Ex.25: Find $\sum_{i=1}^5 i$ and $\sum_{i=1}^{100} i$.

[ANS: 15, 5050]

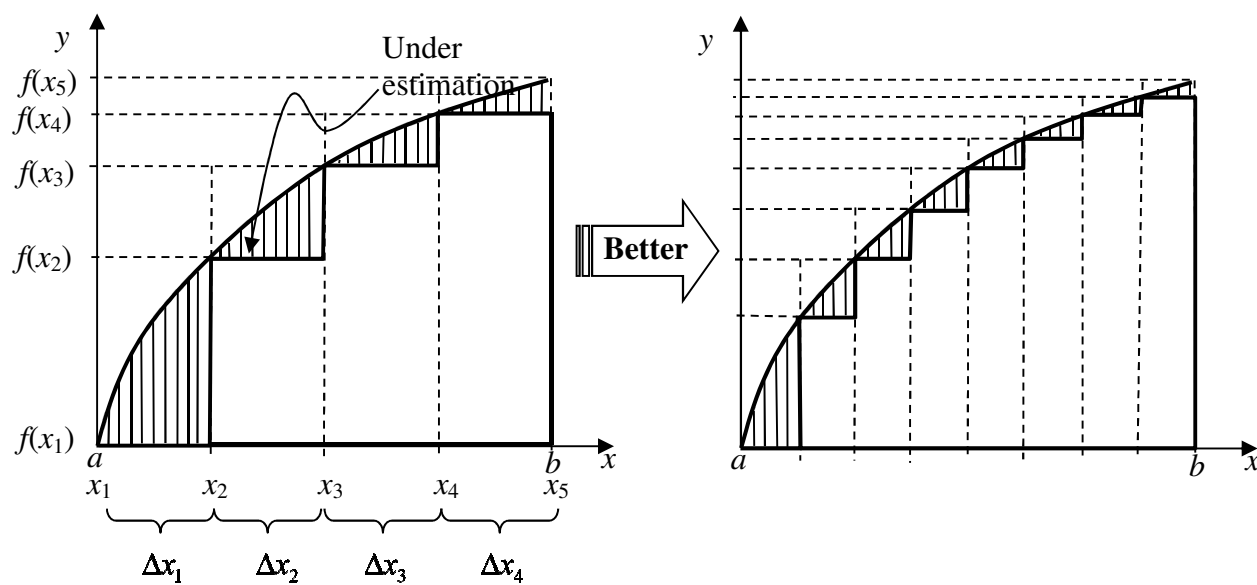
6.6 Definite Integral as an Area under a Curve

Area under a curve can be expressed as a limit of a sum of terms called a **definite integral**.

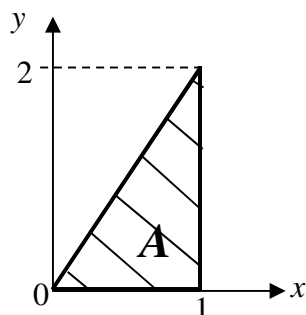
$$A = \int_a^b f(x) dx = \text{Area under a curve between } x = [a, b]$$



$$A^* = f(x_2)\Delta x_1 + f(x_3)\Delta x_2 + f(x_4)\Delta x_3 + f(x_5)\Delta x_4 = \sum_{i=1}^4 f(x_{i+1})\Delta x_i$$

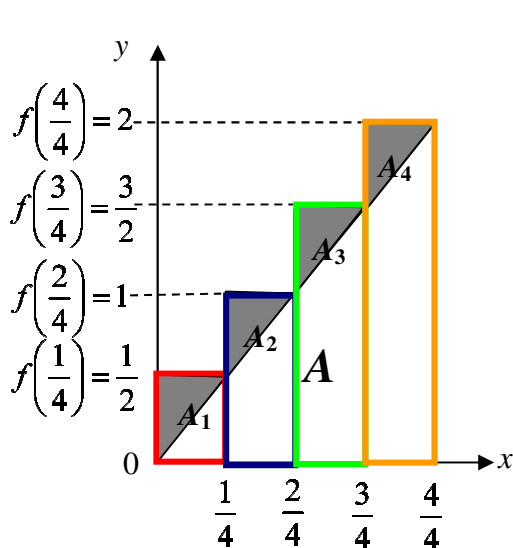


$$A^* = f(x_1)\Delta x_1 + f(x_2)\Delta x_2 + f(x_3)\Delta x_3 + f(x_4)\Delta x_4 = \sum_{i=1}^4 f(x_i)\Delta x_i$$



$$f(x) = 2x$$

$$A = \frac{1}{2} \cdot 1 \cdot 2 = 1$$



$$A_1 = \frac{1}{4} \times \frac{1}{2} = \frac{1}{8}$$

$$A_2 = \frac{1}{4} \times 1 = \frac{1}{4}$$

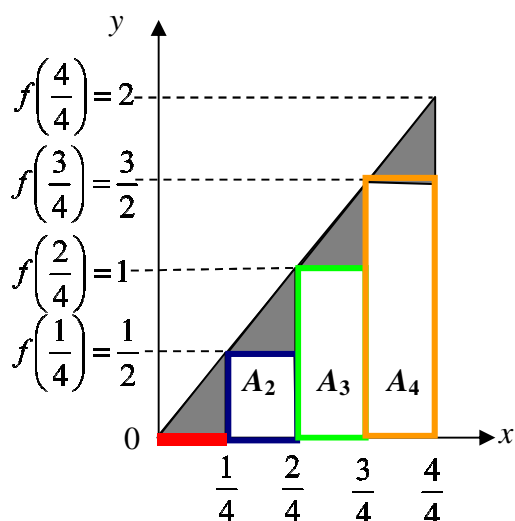
$$A_3 = \frac{1}{4} \times \frac{3}{2} = \frac{3}{8}$$

$$A_4 = \frac{1}{4} \times 2 = \frac{1}{2}$$

Estimated area (Over estimate)

$$\bar{S}_4 = A_1 + A_2 + A_3 + A_4 = \left(\frac{1+2+3+4}{8} \right) = \frac{5}{4}$$

$$\text{Over estimate by} = \frac{5}{4} - 1 = \frac{1}{4}$$



$$A_1 = 0$$

$$A_2 = \frac{1}{4} \times \frac{1}{2} = \frac{1}{8}$$

$$A_3 = \frac{1}{4} \times 1 = \frac{1}{4}$$

$$A_4 = \frac{1}{4} \times \frac{3}{2} = \frac{3}{8}$$

Estimated area (Under estimate)

$$\underline{S}_4 = A_1 + A_2 + A_3 + A_4 = \left(\frac{0+1+2+3}{8} \right) = \frac{3}{4}$$

$$\text{Under estimate by} = 1 - \frac{3}{4} = \frac{1}{4}$$

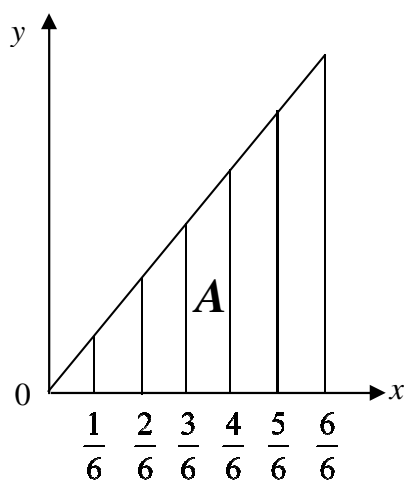
$$y = f(x) = 2x, \quad x = [0,1], \quad \Delta x = \frac{1}{4}$$

$$\bar{S}_4 = \frac{1}{4} f\left(\frac{1}{4}\right) + \frac{1}{4} f\left(\frac{2}{4}\right) + \frac{1}{4} f\left(\frac{3}{4}\right) + \frac{1}{4} f\left(\frac{4}{4}\right) \quad \underline{S}_4 = \frac{1}{4} f(0) + \frac{1}{4} f\left(\frac{1}{4}\right) + \frac{1}{4} f\left(\frac{2}{4}\right) + \frac{1}{4} f\left(\frac{3}{4}\right)$$

$$\bar{S}_4 = \frac{1}{4} \left[2\left(\frac{1}{4}\right) + 2\left(\frac{2}{4}\right) + 2\left(\frac{3}{4}\right) + 2\left(\frac{4}{4}\right) \right] \quad \underline{S}_4 = \frac{1}{4} \left[2(0) + 2\left(\frac{1}{4}\right) + 2\left(\frac{2}{4}\right) + 2\left(\frac{3}{4}\right) \right]$$

$$\bar{S}_4 = \frac{5}{4} > 1 \quad (\text{Exact area} = 1)$$

$$\underline{S}_4 = \frac{3}{4} < 1 \quad (\text{Exact area} = 1)$$



$$\bar{S}_4 = \sum_{i=1}^4 f(x_i) \Delta x$$

$$\underline{S}_4 = \sum_{i=1}^3 f(x_i) \Delta x$$

$$f(x) = 2x$$

$$x = [0,1]$$

$$\Delta x = \frac{1}{6}$$

$$\bar{S}_6 = \sum_{i=1}^6 f(x_i) \Delta x$$

$$\underline{S}_6 = \sum_{i=1}^5 f(x_i) \Delta x$$

$$\bar{S}_6 = \frac{1}{6} f\left(\frac{1}{6}\right) + \frac{1}{6} f\left(\frac{2}{6}\right) + \frac{1}{6} f\left(\frac{3}{6}\right) + \frac{1}{6} f\left(\frac{4}{6}\right) + \frac{1}{6} f\left(\frac{5}{6}\right) + \frac{1}{6} f\left(\frac{6}{6}\right)$$

$$\bar{S}_6 = \frac{1}{6} \left[2\left(\frac{1}{6}\right) + 2\left(\frac{2}{6}\right) + 2\left(\frac{3}{6}\right) + 2\left(\frac{4}{6}\right) + 2\left(\frac{5}{6}\right) + 2\left(\frac{6}{6}\right) \right]$$

$$\bar{S}_6 = \frac{1}{18} [1 + 2 + 3 + 4 + 5 + 6] = \frac{21}{18} = \frac{7}{6}$$

$$\text{Over estimate by} = \frac{7}{6} - 1 = \frac{1}{6}$$

$$\underline{S}_6 = \frac{1}{6} f(0) + \frac{1}{6} f\left(\frac{1}{6}\right) + \frac{1}{6} f\left(\frac{2}{6}\right) + \frac{1}{6} f\left(\frac{3}{6}\right) + \frac{1}{6} f\left(\frac{4}{6}\right) + \frac{1}{6} f\left(\frac{5}{6}\right)$$

$$\underline{S}_6 = \frac{1}{6} \left[2(0) + 2\left(\frac{1}{6}\right) + 2\left(\frac{2}{6}\right) + 2\left(\frac{3}{6}\right) + 2\left(\frac{4}{6}\right) + 2\left(\frac{5}{6}\right) \right]$$

$$\underline{S}_6 = \frac{1}{18} [0 + 1 + 2 + 3 + 4 + 5] = \frac{15}{18} = \frac{5}{6}$$

$$\text{Under estimate by} = 1 - \frac{5}{6} = \frac{1}{6}$$

The estimated result is getting closer as the interval (Δx) gets **SMALLER!**

General Term $f(x) = 2x$ Divide $x = [0,1]$ into n subintervals. Hence, $\Delta x = \frac{1}{n}$

$$\bar{S}_n = \sum_{i=1}^n f(x_i) \Delta x$$

$$\bar{S}_n = \frac{1}{n} f\left(\frac{1}{n}\right) + \frac{1}{n} f\left(\frac{2}{n}\right) + \frac{1}{n} f\left(\frac{3}{n}\right) + \dots + \frac{1}{n} f\left(\frac{n}{n}\right)$$

$$\bar{S}_n = \frac{1}{n} \left[2\left(\frac{1}{n}\right) + 2\left(\frac{2}{n}\right) + 2\left(\frac{3}{n}\right) + \dots + 2\left(\frac{n}{n}\right) \right]$$

$$\bar{S}_n = \frac{2}{n^2} \cdot [1 + 2 + 3 + \dots + n]$$

$$\bar{S}_n = \frac{2}{n^2} \cdot \left[\frac{n(n+1)}{2} \right] = \frac{n+1}{n}$$

Note: $\sum_{k=1}^n k = \frac{n(n+1)}{2}$

$$\underline{S}_n = \sum_{i=1}^{n-1} f(x_i) \Delta x$$

$$\underline{S}_n = \frac{1}{n} f\left(\frac{0}{n}\right) + \frac{1}{n} f\left(\frac{1}{n}\right) + \frac{1}{n} f\left(\frac{2}{n}\right) + \dots + \frac{1}{n} f\left(\frac{n-1}{n}\right)$$

$$\underline{S}_n = \frac{1}{n} \left[2\left(\frac{0}{n}\right) + 2\left(\frac{1}{n}\right) + 2\left(\frac{2}{n}\right) + \dots + 2\left(\frac{n-1}{n}\right) \right]$$

$$\underline{S}_n = \frac{2}{n^2} \cdot [0 + 1 + 2 + \dots + (n-1)]$$

$$\underline{S}_n = \frac{2}{n^2} \cdot \left[\frac{(n-1)n}{2} \right] = \frac{n-1}{n}$$

Note: $\sum_{k=1}^n k = \frac{n(n+1)}{2}$

As subinterval is getting smaller ($\Delta x \rightarrow 0$) $\rightarrow n$ is larger $\rightarrow \bar{S}_n, \underline{S}_n$ closer to the exact area A .

$$\bar{S}_n = \frac{n+1}{n} \qquad \underline{S}_n = \frac{n-1}{n}$$

Question: For $\bar{S}_n = \frac{n+1}{n}$ and $\underline{S}_n = \frac{n-1}{n}$, what happen if $n \rightarrow \infty$?

$$\lim_{n \rightarrow \infty} \bar{S}_n = \lim_{n \rightarrow \infty} \frac{n+1}{n} = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right) = 1$$

$$\lim_{n \rightarrow \infty} \underline{S}_n = \lim_{n \rightarrow \infty} \frac{n-1}{n} = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n}\right) = 1$$

$$\therefore \lim_{n \rightarrow \infty} \bar{S}_n = \lim_{n \rightarrow \infty} \underline{S}_n = 1$$

As $n \rightarrow \infty$, $\bar{S}_n \rightarrow A$ and $\underline{S}_n \rightarrow A$ ($A = 1$, exact area)

$$\underline{S}_n \leq A \leq \bar{S}_n$$

Common limit of $\underline{S}_n$ and $\bar{S}_n \rightarrow$ Definite integral of $f(x)$

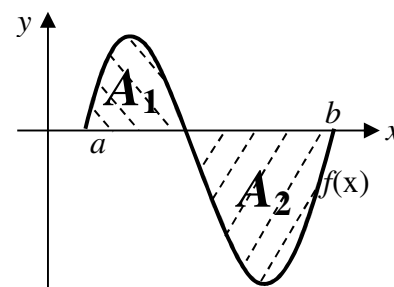
$$\therefore A = \int_0^1 2x dx = 1$$

The **common** limit of $\underline{S}_n$ and $\bar{S}_n$ as $n \rightarrow \infty$, if it exists, is called the **definite integral** of $f(x)$ over (a, b) .

$$A = \int_a^b f(x) dx$$

For example,

$$\int_0^1 2x dx = [x^2]_0^1 = 1^2 - 0^2 = 1 = \lim_{n \rightarrow \infty} \underline{S}_n = \lim_{n \rightarrow \infty} \bar{S}_n$$

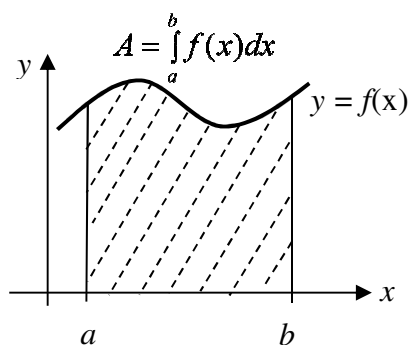


NOTE: Definite integral **DOES NOT** always represents an area.

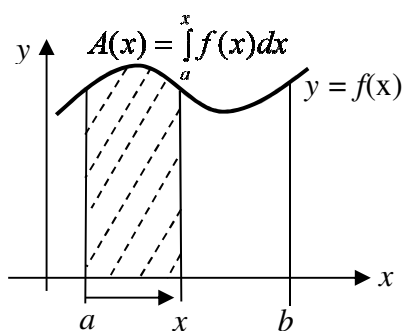
$$\int_a^b f(x) dx = A_1 - A_2 = \text{NEGATIVE. Area can't be negative!}$$

6.7 Fundamental Theorem of Integration Calculus

- To informally develop the fundamental theorem of integral calculus and to use it to complete definite integrals.
- To obtain a change in function values when the rate of change of the function is known.

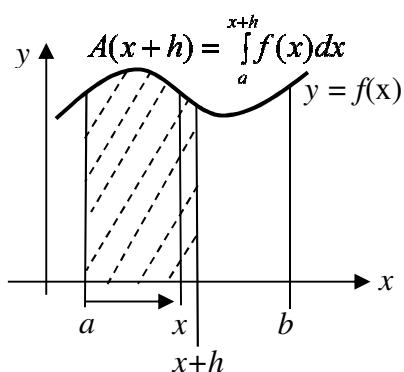


A = Area between the curve $y = f(x)$, $x = a$, $x = b$, and the x axis.



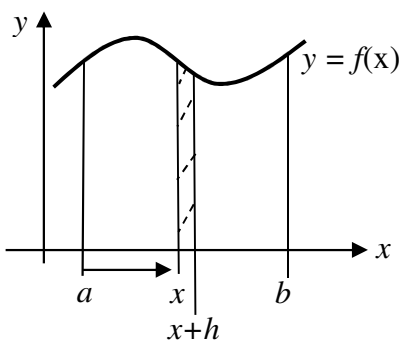
A = Area function

$A(x)$ = Area between the curve $y = f(x)$, $x = a$, $x = b$, and the x axis.



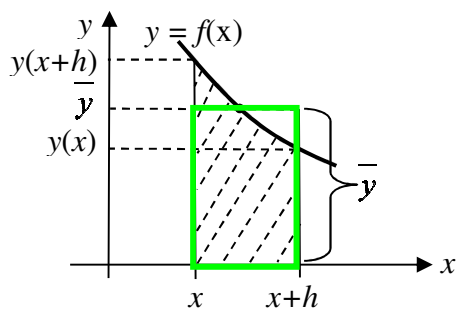
Increase x by h unit

$A(x+h)$ = Area between $y = f(x)$, $x = (x+h)$, $x = b$, and the x axis.



The different in area

$$\Delta A = A(x+h) - A(x)$$



As $h \rightarrow 0$, area  is approximately rectangle.

Estimate to rectangle with height $\bar{y}$

$$\Delta A = A(x+h) - A(x) = \bar{y} \cdot h$$

$$\bar{y} = \frac{A(x+h) - A(x)}{h}$$

$$\bar{y} = \frac{A(x+h) - A(x)}{h}$$

$$h \rightarrow 0 \text{ and } \bar{y} \rightarrow f(x)$$

$$\therefore \lim_{h \rightarrow 0} \left[\frac{A(x+h) - A(x)}{h} \right] = f(x)$$

This is the definition of a derivative of A .

$$\therefore A'(x) = f(x)$$

A **derivative** of an area A is $f(x)$ the function itself.

A is an **anti-derivative** of $f(x)$.

$$\therefore \int f(x) dx = A(x) + C_1 \quad (1)$$

The integral of a function $f(x)$ is the area function $A(x)$. Anti-derivative of $f(x)$ is $F(x)$.

$$\therefore \int f(x) dx = F(x) + C_2 \quad (2)$$

$$(1) = (2),$$

$$A(x) + C_1 = F(x) + C_2$$

$$A(x) = F(x) + C \quad (3)$$

$$A(0) = 0 = F(a) + C \Rightarrow C = -F(a)$$

(3) becomes

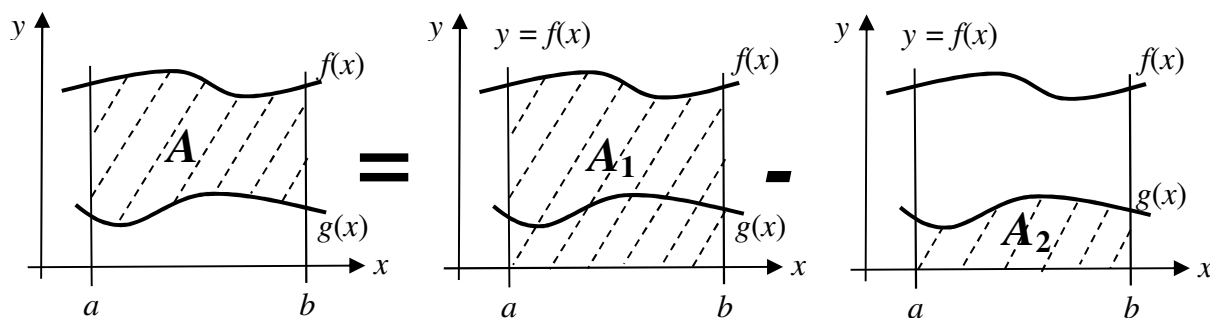
$$A(x) = F(x) - F(a) \quad (4)$$

If $x = b$, (4) becomes

$$A(x) = F(b) - F(a) = \int_a^b f(x) dx$$

Hence, if f is continuous on interval $[a, b]$, $F(x)$ is any anti-derivative of $f(x)$ on the interval then

$$\boxed{\int_a^b f(x) dx = F(b) - F(a)}$$



6.7.1 Area between Curves

Find area bounded by $f(x)$, $g(x)$, $x = a$, and $x = b$.

$$A = A_1 - A_2$$

$$A = \int_a^b f(x)dx - \int_a^b g(x)dx$$

$$A = \int_a^b [f(x) - g(x)] dx$$

$$\therefore \int_a^b [f(x) \pm g(x)] dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$$

Similarly others properties of definite integral can be derived from area between curves.

6.7.2 Properties of Definite Integral

$$1) \quad \int_a^b f(x)dx = -\int_b^a f(x)dx$$

$$2) \quad \int_a^a f(x)dx = 0$$

$$3) \quad \int_a^b kf(x)dx = k \int_a^b f(x)dx \quad (k \text{ is a constant})$$

$$4) \quad \int_a^b [f(x) \pm g(x)] dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$$

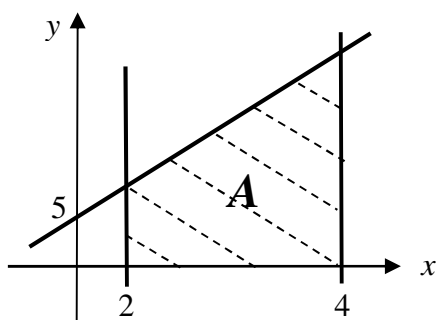
$$5) \quad \int_a^b f(x)dx = -\int_b^a f(t)dt \quad \text{Any variable will give the same result.}$$

$$6) \quad \int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$$

Ex.26: Use a definite integral to find the area of the region bounded by the given curve, the x axis, and the given lines.

(Note: We need to sketch the curve to make sure that there will be no negative area!)

(i) $y = x + 5$, $x = 2$, and $x = 4$



$$A = \int_2^4 (x + 5) dx = \left[\frac{x^2}{2} + 5x \right]_2^4$$

$$A = \left(\frac{4^2}{2} + 5(4) \right) - \left(\frac{2^2}{2} + 5(2) \right)$$

$$A = 28 - 12 = 16$$

(ii) $y = e^x$, $x = 1$, and $x = 3$

[Ans: $e^3 - e$]

(iii) $y = x^2 - 2x$, $x = 1$, and $x = 3$

[Ans: 2]

(iv) $y = x^3$, $x = -2$, and $x = 4$

[ANS: 68]

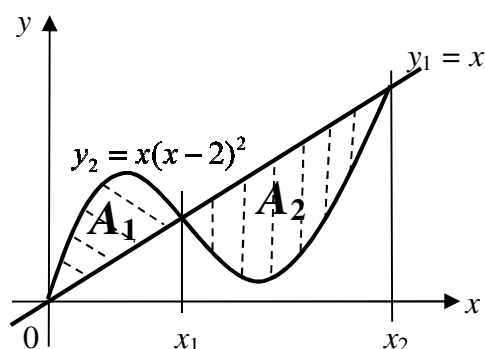
(v) $y = \frac{1}{x}$, $x = 1$, and $x = e$

[ANS: 1]

(vi) $y = \frac{1}{x}$, $x = 1$, and $x = e^2$

[ANS: 2]

Ex.27: Express the area in terms of integrals



Find intersection points

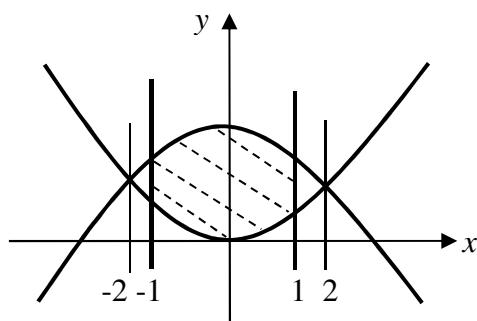
$$x(x-2)^2 = x$$

$A =$

Ex.28: Find the area of the region bounded by the graphs of the given equations. Be sure to find any needed points of intersection.

(i) $y = 8 - x^2$, $y = x^2$, $x = -1$, and $x = 1$

[Ans: 44/3]



Intersection: $8 - x^2 = x^2$

$$2x^2 = 8 \Rightarrow x = \pm 2$$

$$A = \int_{-1}^1 [(8 - x^2) - x^2] dx = \frac{44}{3}$$

(ii) $y = 2 - x^2$, $y = x$

[Ans: $\int_{-2}^1 [(2 - x^2) - x] dx = \frac{9}{2}$]

(iii) $y = x^2 + 2$, $y = 8$

[Ans: $\int_{-\sqrt{6}}^{\sqrt{6}} [8 - (x^2 + 2)] dx = 8\sqrt{6}$]

6.7.3 The Average Value of a Function

Let $f(x)$ be a function that is continuous on the interval $a \leq x \leq b$. Then the **average value** V of $f(x)$ over $a \leq x \leq b$ is given by the definite integral

$$V = \frac{1}{b-a} \int_a^b f(x) dx$$

Ex.29: A manufacture determines that t moths after introducing a new product, the company's sales will be $S(t)$ thousand dollars, where

$$S(t) = \frac{750t}{\sqrt{4t^2 + 25}}$$

What are the average monthly sales of the company over **the first 6 months** after the introduction of the new product? [Ans:

250]

6.8 Definite Integral (With numerical limit, the answer is real number.)

If $F(x)$ is any anti-derivative of the function $f(x)$, the 'definite' integral of $f(x)$ between a and b , ($a \leq x \leq b$), is

$$\int_a^b f(x) dx = F(b) - F(a) = [F(x)]_a^b \quad a < b$$

a is the lower limit of integration and b is the upper limit of integration. **Definite** integral will give a **definite** numerical value, **NO x or C** . C will cancel itself out during the calculation.

Ex.30: Show that $\int_1^5 3x^2 dx = 124$ and $\int_0^2 (4 - x^2) dx = \frac{16}{3}$.

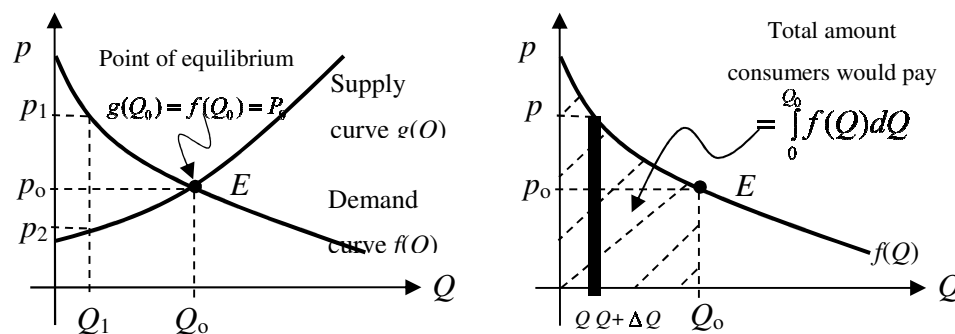
When using integration by substitution for a definite integral, we need to **substitute the limit** as well or **substitute before taking limits**.

Ex. 31: Show that $\int_1^2 (2x^3 - 1)^2 (6x^2) dx = \frac{3374}{3}$.

Ex. 32: Show that $\int_9^{64} \left(\frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}} \right) dx = \frac{76}{3}$.

6.9 Consumers' and Producers' Surplus

Economists are interested in studying how consumers' and producers' welfare are influenced by changes in the economic parameters. A common measure of such welfare changes is the consumers' and producers' surplus.



p = price **per unit** (example of unit is bahts/unit) Q = Quantity buy or sell

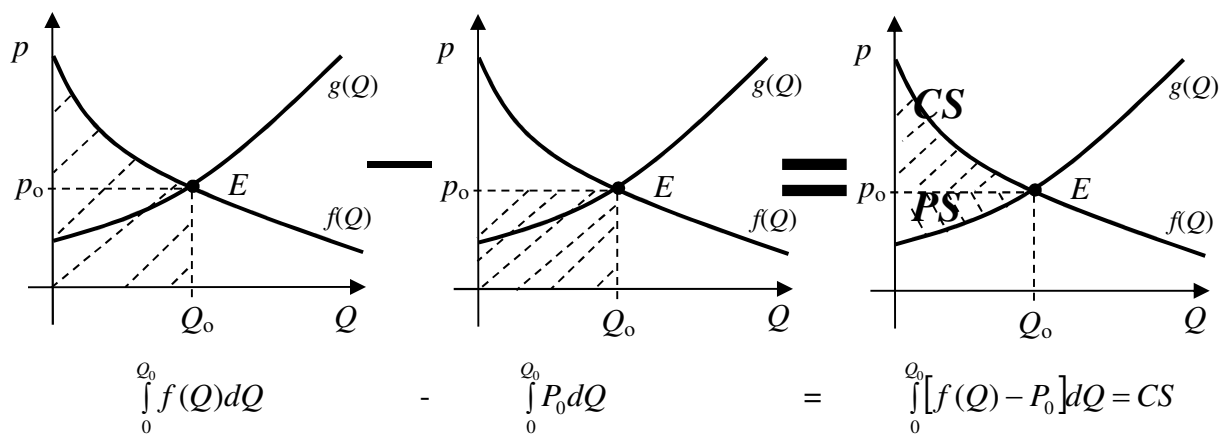
- Supply curve $[g(Q)]$ indicates the price p per unit at which the **producers** will **sell** (or **supply**) Q units.
- Demand curve $[f(Q)]$ indicates the price p per unit at which the **consumers** will **purchase** (or **demand**) Q units.

- **Point of equilibrium** (E) indicates the consumers to purchase precisely the same amount (Q_0) that the producers are willing to offer at the price per unit (p_0).

Demand and supply curves are intercepted.

$$p_0 = q(Q_0) = f(Q_0)$$

- If the market is at equilibrium ($p = p_0$), there are consumers who are willing to **pay more** than p_0 . For example, at the price of p_1 , consumers would buy Q_1 unit. These consumers **benefiting** from the lower price p_2 .
- $p\Delta Q$ is the total amount of money that customers would spend by buying ΔQ unit if the price per unit was p . In fact, the price is at the equilibrium p_0 so the customers only buy $p_0\Delta Q$. Thus, the customers **save** or **benefit** $(p - p_0)\Delta Q$.
- Area under the curve is the amount of money to sell or buy the product.



- For those who are willing to pay for commodity at price p_0 or higher, the total amount they are willing to pay is the total area $\int_0^{Q_0} f(Q) dQ$. In fact, consumers only pay $\int_0^{Q_0} P_0 dQ$. Thus, the customers **save**

$$CS = \int_0^{Q_0} [f(Q) - P_0] dQ.$$

- **Consumers' Surplus [CS]** is the total amount **save** by all the customers buying the commodity at a price lower than what they are maximally willing to pay.
- Some producers are also benefiting from the equilibrium price, since they are willing to supply at prices **lower** than p_0 . Similarly,

$$PS = \int_0^{Q_0} [P_0 - f(Q)] dQ$$

- **Producers' Surplus [PS]** is the total benefit of all the producers who obtain a price higher than the price they are willing to sell it for.

Ex.33: A demand function is $q = 10000e^{-0.02p}$ unit. At what price will revenue be maximised?

[Ans: 50 units]

Ex.34: A demand function is $p = f(q) = 100 - 0.05q$ unit. A supply function is $p = g(q) = 10 + 0.1q$ bahts/unit. Determine consumers' and producers' surplus under market equilibrium.

[Ans: 600 units, 70 bahts/unit, 9000 bahts, 18000 bahts]

Solution: At equilibrium point (q_0, p_0) , the demand and supply functions are intercepted.

$$100 - 0.05q = 10 + 0.1q$$

$$0.15q = 90$$

$$q_0 = 600 \text{ units}$$

$$p_0 = 10 + 0.1q_0 = 10 + 0.1(600)$$

$$p_0 = 70 \text{ bahts/unit}$$

$$\begin{aligned} CS &= \int_0^{q_0} [f(q) - p_0] dq \\ &= \int_0^{600} [\{100 - 0.05q\} - 70] dq \\ &= \left[30q - \frac{0.05q^2}{2} \right]_0^{600} \\ &= \left[30(600) - \frac{0.05(600)^2}{2} \right] - 0 \\ &= 9,000 \text{ bahts} \end{aligned}$$

$$\begin{aligned} PS &= \int_0^{q_0} [p_0 - f(q)] dq \\ &= \int_0^{600} [70 - \{10 + 0.1q\}] dq \\ &= \left[60q - \frac{0.1q^2}{2} \right]_0^{600} \\ &= \left[60(600) - \frac{0.1(600)^2}{2} \right] - 0 \\ &= 18,000 \text{ bahts} \end{aligned}$$

Ex.35: A demand function is $p = \frac{50}{q+5}$ unit. A supply function is $p = \frac{q}{10} + 4.5$ bahts/unit.

Determine consumers' and producers' surplus under market equilibrium.

[Ans: 5 units, 5 bahts/unit, $50\ln 2 - 25$ bahts, 1.25 bahts]

Ex.36: A demand function is $p = 2^{11-q}$. A supply function is $p = 2^{q+1}$. Determine consumers' surplus under market equilibrium.

[Ans: 5 unit, 64 bahts/unit, $\left(\frac{2^{11}}{\ln 2} - \frac{64}{\ln 2} - 320 \right)$ bahts]