Matrices,
Algebra of Matrices
&
Elementary Operations

What is a Matrix ? (Not a movie trilogy staring Keanu Reaves)

A matrix is a rectangular array of numbers (or functions) enclosed in brackets. These numbers (or functions) are called the entries (or elements) of the matrix.

For example,

$$\mathbf{A} = [a_{ij}] = \begin{array}{c} \begin{array}{cccc} \text{Column1} & \text{Column2} & & \text{Column n} \end{array} \\ \left[\begin{array}{cccc} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & . & \cdots & a_{mn} \end{array}\right] \begin{array}{l} \text{Row1} \\ \text{Row 2} \\ \\ \text{Row m} \end{array} \end{array}$$

$\mathbf{A}$ is a matrix of dimension (size) *m* x *n* (*m* rows and *n* columns)

a_{ij} is an entry or an element of the matrix

a_{ii} is a main diagonal entry

$\begin{bmatrix} 0.3 & 1 & -5 \\ 0 & -0.2 & 16 \end{bmatrix}$ is a 2 x 3 matrix (Rectangular Matrix)

$\begin{bmatrix} e^{-x} & 2x^2 \\ e^{6x} & 4x \end{bmatrix}$ is a 2 x 2 matrix (Square Matrix)

$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is a 3 x 3 matrix (Square Matrix)

$\begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \vdots & 0 \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$ is a diagonal matrix

$\mathbf{I} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & 0 \\ 0 & 0 & \cdots & 1 \end{bmatrix}$ is an identity matrix or a unit matrix

$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & 1 \end{bmatrix}$ is an upper triagular matrix

$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 3 & 4 & 5 \end{bmatrix}$ is a lower triagular matrix

$\mathbf{0} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix}$ is a zero matrix

$\begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}$ is a 1 x 3 matrix (we usually call it a row vector)

$\begin{bmatrix} 4 \\ \frac{1}{2} \end{bmatrix}$ is a 2 x 1 matrix (we usually call it a column vector)

Vector

A **vector** is a matrix with only one row or column. Its entries are called the **components** of the vector. We shall denote vectors by ***lowercase*** boldface letters **a, b,** $\cdots$ or by its general component in brackets, $\mathbf{a} = [a_j]$, and so on. Our special vectors in (1) suggest that a (general) **row vector** is of the form

$$\mathbf{a} = [a_1 \quad a_2 \quad \cdots, \quad a_n]. \qquad \text{For instance,} \qquad \mathbf{a} = [-2 \quad 5 \quad 0.8 \quad 0 \quad 1].$$

A **column vector** is of the form

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}. \qquad \text{For instance,} \qquad \mathbf{b} = \begin{bmatrix} 4 \\ 0 \\ -7 \end{bmatrix}.$$

Basic operations of matrices

Equality of Matrices

Two matrices $\mathbf{A} = [a_{jk}]$ and $\mathbf{B} = [b_{jk}]$ are **equal,** written $\mathbf{A} = \mathbf{B}$, if and only if they have the same size and the corresponding entries are equal, that is, $a_{11} = b_{11}$, $a_{12} = b_{12}$, and so on. Matrices that are not equal are called **different.** Thus, matrices of different sizes are always different.

Let

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 4 & 0 \\ 3 & -1 \end{bmatrix}.$$

Then

$$\mathbf{A} = \mathbf{B} \quad \text{if and only if} \quad \begin{aligned} a_{11} &= 4, & a_{12} &= 0, \\ a_{21} &= 3, & a_{22} &= -1. \end{aligned}$$

Addition of Matrices

The **sum** of two matrices $\mathbf{A} = [a_{jk}]$ and $\mathbf{B} = [b_{jk}]$ ***of the same size*** is written $\mathbf{A} + \mathbf{B}$ and has the entries $a_{jk} + b_{jk}$ obtained by adding the corresponding entries of **A** and **B.** Matrices of different sizes cannot be added.

$$\mathbf{A} + \mathbf{B} = \left[a_{jk} + b_{jk} \right]_{m \times n}$$

For example;

$$\begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & 6 \\ -6 & 10 & -5 \end{bmatrix} + \begin{bmatrix} 4 & 7 & -8 \\ 9 & 3 & 5 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 6 & -5 \\ 9 & 7 & 11 \\ -5 & 9 & -3 \end{bmatrix}$$

Scalar Multiplication (Multiplication by a Number)

The **product** of any $m \times n$ matrix $\mathbf{A} = [a_{jk}]$ and any **scalar** c (number c) is written $c\mathbf{A}$ and is the $m \times n$ matrix $c\mathbf{A} = [ca_{jk}]$ obtained by multiplying each entry of **A** by c.

$$c\mathbf{A} = \begin{bmatrix} ca_{11} & ca_{12} & \cdots & ca_{1n} \\ ca_{21} & ca_{22} & \cdots & ca_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ ca_{m1} & ca_{m2} & \cdots & ca_{mn} \end{bmatrix}$$

If $\mathbf{A} = \begin{bmatrix} 2.7 & -1.8 \\ 0 & 0.9 \\ 9.0 & -4.5 \end{bmatrix}$, then $-\mathbf{A} = \begin{bmatrix} -2.7 & 1.8 \\ 0 & -0.9 \\ -9.0 & 4.5 \end{bmatrix}$, $\frac{10}{9}\mathbf{A} = \begin{bmatrix} 3 & -2 \\ 0 & 1 \\ 10 & -5 \end{bmatrix}$, $0\mathbf{A} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$.

$$\mathbf{A} = \begin{bmatrix} 4 & 0 & 5 \\ -1 & 3 & 2 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 7 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 2 & -3 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{A} + \mathbf{B} =$$

$$\mathbf{A} + \mathbf{C} = \qquad\qquad 2\mathbf{B} =$$

$$\mathbf{A} - 2\mathbf{B}$$

Properties of Matrix Addition

Cummulative law (a) $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$

Associative law of addition (b) $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$ (written $\mathbf{A} + \mathbf{B} + \mathbf{C}$)

(c) $\mathbf{A} + \mathbf{0} = \mathbf{A}$

(d) $\mathbf{A} + (-\mathbf{A}) = \mathbf{0}$.

Properties of Scalar Multiplication

Distributive laws (a) $c(\mathbf{A} + \mathbf{B}) = c\mathbf{A} + c\mathbf{B}$

(b) $(c + k)\mathbf{A} = c\mathbf{A} + k\mathbf{A}$

(c) $c(k\mathbf{A}) = (ck)\mathbf{A}$ (written $ck\mathbf{A}$)

(d) $1\mathbf{A} = \mathbf{A}$.

Matrix multiplication

Three different ways with the same answer:

Method 1: Each entry of $\boldsymbol{AB}$ is the product of a row and a column.

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}.$$

$$\begin{bmatrix} & & & \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ & & & \end{bmatrix} \begin{bmatrix} & b_{1j} & \\ & b_{2j} & \\ & \vdots & \\ & b_{nj} & \end{bmatrix} = \begin{bmatrix} & & \\ & (AB)_{ij} & \\ & & \end{bmatrix}$$

$(\boldsymbol{AB})$ij=row i of A times column j of $\boldsymbol{B}$

This single entry is the inner product of the two vectors.

Example

$$\begin{bmatrix} 1 & 1 & 6 \\ 3 & 0 & 3 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix} =$$

EXAMPLE $A = \begin{bmatrix} 2 & 3 & 6 \\ -1 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -3 \\ 0 & 1 \\ 4 & -7 \end{bmatrix}$. Compute AB, if it is defined.

Solution: Since A is 2×3 and B is 3×2, then AB is defined and AB is _____×_____.

$$AB = \begin{bmatrix} \mathbf{2} & \mathbf{3} & \mathbf{6} \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} \mathbf{2} & -3 \\ \mathbf{0} & 1 \\ \mathbf{4} & -7 \end{bmatrix} = \begin{bmatrix} \mathbf{28} & \blacksquare \\ \blacksquare & \blacksquare \end{bmatrix}$$

EXAMPLE: If A is 4×3 and B is 3×2, then what are the sizes of AB and BA?

Solution:

$$AB = \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}\begin{bmatrix} * & * \\ * & * \\ * & * \end{bmatrix} = \begin{bmatrix} \quad & \quad \end{bmatrix}$$

BA would be $\begin{bmatrix} * & * \\ * & * \\ * & * \end{bmatrix}\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$ Which is ___________

$\boxed{\text{If } A \text{ is } m \times n \text{ and } B \text{ is } n \times p, \text{ then } AB \text{ is } m \times p.}$

Method 2: Each column of $\boldsymbol{AB}$ is the product of a matrix and a column

Suppose A is $m \times n$ and B is $n \times p$ where $B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_p \end{bmatrix}$;

$$\boxed{AB = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_p \end{bmatrix}}$$

$$A_{m\times n}B_{n\times p} = \left[A_{m\times n}\begin{bmatrix} b_{11} \\ b_{21} \\ \cdot \\ \cdot \\ b_{n1} \end{bmatrix} \quad A_{m\times n}\begin{bmatrix} b_{12} \\ b_{21} \\ \cdot \\ \cdot \\ b_{n2} \end{bmatrix} \quad \cdot \; \cdot \quad A_{m\times n}\begin{bmatrix} b_{1p} \\ b_{2p} \\ \cdot \\ \cdot \\ b_{np} \end{bmatrix} \right]$$

Column j of $\boldsymbol{AB}$ = A times column j of $\boldsymbol{B}$

The number of columns in $\boldsymbol{A}$ has to equal the number of rows in $\boldsymbol{B}$.

Example $\begin{bmatrix} 1 & 1 & 6 \\ 3 & 0 & 3 \\ 1 & 1 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix} =$

Example

$$\begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix} =$$

EXAMPLE: Compute AB where $A = \begin{bmatrix} 4 & -2 \\ 3 & -5 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -3 \\ 6 & -7 \end{bmatrix}$.

Solution:

$$A\mathbf{b}_1 = \begin{bmatrix} 4 & -2 \\ 3 & -5 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 6 \end{bmatrix} = \begin{bmatrix} -4 \\ -24 \\ 6 \end{bmatrix}, \qquad A\mathbf{b}_2 = \begin{bmatrix} 4 & -2 \\ 3 & -5 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -3 \\ -7 \end{bmatrix} = \begin{bmatrix} 2 \\ 26 \\ -7 \end{bmatrix}$$

$$\Rightarrow AB = \begin{bmatrix} -4 & 2 \\ -24 & 26 \\ 6 & -7 \end{bmatrix}$$

Note that $A\mathbf{b}_1$ is a linear combination of the columns of A and $A\mathbf{b}_2$ is a linear combination of the columns of A.

Method 3: Each row of AB is the product of a row and a matrix

row i of AB = row i of A times B

$$\begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 5 & -1 & 0 \end{bmatrix} =$$

Properties of Matrix Multiplication

Cautions !

AB is <u>not always</u> equal to **BA**

Try

$$A = \begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix}$$

If **AB** = **AC** , **B** is not necessary equal to **C**

eg.

$$A = \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix}, B = \begin{bmatrix} 8 & 4 \\ 5 & 5 \end{bmatrix} \text{ and } C = \begin{bmatrix} 5 & -2 \\ 3 & 1 \end{bmatrix}$$

$B \neq C$ but $AB = AC$

If **AB** = **0** , **A** or **B** is not necessary equal to **0**

eg.

$$A = \begin{bmatrix} 3 & -6 \\ -1 & 2 \end{bmatrix}, B = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \qquad AB = 0$$

Properties of matrix multiplication

Let A be $m \times n$ and let B and C have sizes for which the indicated sums and products are defined.

a. $A(BC) = (AB)C$ (associative law of multiplication)

b. $A(B + C) = AB + AC$ (left - distributive law)

c. $(B + C)A = BA + CA$ (right-distributive law)

d. $r(AB) = (rA)B = A(rB)$ for any scalar r

e. $I_m A = A = AI_n$ (identity for matrix multiplication)

Transposition of Matrices and Vectors

The transpose of an $m \times n$ matrix $\mathbf{A} = [a_{jk}]$ is the $n \times m$ matrix $\mathbf{A}^\mathsf{T}$ (read *A transpose*) that has the first *row* of $\mathbf{A}$ as its first *column*, the second *row* of $\mathbf{A}$ as its second *column*, and so on.

$$\mathbf{A} = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & \cdot & \cdots & a_{mn} \end{bmatrix} \qquad \mathbf{A}^T = [a_{ji}] = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}$$

Transposition of Matrices and Vectors

If $\mathbf{A} = \begin{bmatrix} 5 & -8 & 1 \\ 4 & 0 & 0 \end{bmatrix}$, then $\mathbf{A}^\mathsf{T} = \begin{bmatrix} 5 & 4 \\ -8 & 0 \\ 1 & 0 \end{bmatrix}$.

EXAMPLE: Let $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ -2 & 4 \end{bmatrix}$. Compute AB, $(AB)^T$, A^TB^T and B^TA^T.

Solution:

$$AB = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}$$

$$(AB)^T = \begin{bmatrix} \quad \end{bmatrix}$$

$$A^TB^T = \begin{bmatrix} 1 & 3 \\ 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 3 & 10 \\ 2 & 0 & -4 \\ 2 & 1 & 4 \end{bmatrix}$$

$$B^TA^T = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}$$

$$\boxed{(AB)^T = \quad}$$

Properties of Matrix Transposition

Let A and B denote matrices whose sizes are appropriate for the following sums and products.

a. $(A^T)^T = A$ (I.e., the transpose of A^T is A)

b. $(A+B)^T = A^T + B^T$

c. For any scalar r, $(rA)^T = rA^T$

d. $(AB)^T = B^TA^T$ (I.e. the transpose of a product of matrices equals the product of their transposes in reverse order.)

EXAMPLE: Prove that $(ABC)^T =$ ________.

Solution: By Theorem 3d,

$$(ABC)^T = ((AB)C)^T = C^T(\qquad)^T$$
$$= C^T(\qquad) = \text{\_\_\_\_\_\_\_\_}.$$

Solution of System of Linear Equations

System of Linear Equations

A linear equation

Unknown Variables

$$a_1 x_1 + a_2 x_2 + \ldots + a_n x_n = b$$

Given Coefficients

Given Constant

A linear equation

$$a_1 x_1 + a_2 x_2 + \ldots + a_n x_n = b \qquad (*)$$

eg.

$$2x_1 - x_2 + 5x_3 = 2\sqrt{5}$$

Non-linear equation

Anything that is not in the form of a linear equation (*)

eg.

$$2x_1 - x^2{}_2 + 5x_3 \sin(x_1) = 25$$

System of Linear Equations

A collection of one or more linear equations involving the same set of variables.

A system of linear equations with 2 variables:

$$\begin{aligned} ax + by &= h \\ cx + dy &= k \end{aligned} \qquad \text{E.g.} \quad \begin{aligned} 2x + y &= 8 \\ x + 3y &= 9 \end{aligned}$$

A system of linear equations with 3 variables:

$$\begin{aligned} 6x_1 + x_2 + x_3 &= 6 \\ 5x_1 + x_2 + 2x_3 &= 4 \\ 4x_1 + x_2 - x_3 &= -2 \end{aligned}$$

The whole idea of linear algebra is to solve

$$\underline{\mathbf{A}}\underline{\mathbf{x}} = \underline{\mathbf{b}}$$

A system of linear equations can be written in matrix form

$$\begin{aligned} 6x_1 + x_2 + x_3 &= 6 \\ 5x_1 + x_2 + 2x_3 &= 4 \\ 4x_1 + x_2 - x_3 &= -2 \end{aligned}$$

$$\begin{bmatrix} 6 & 1 & 1 \\ 5 & 1 & 2 \\ 4 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ -2 \end{bmatrix} \quad \text{or} \quad \left[\begin{array}{ccc|c} 6 & 1 & 1 & 6 \\ 5 & 1 & 2 & 4 \\ 4 & 1 & -1 & -2 \end{array}\right]$$

A matrix equation: $\underline{\mathbf{A}}\underline{\mathbf{x}} = \underline{\mathbf{b}}$

An augmented matrix form $[\mathbf{A}|\mathbf{b}]$

A system of linear equations with n variables:

$$
\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots\ldots a_{1n}x_n = b_1 \\
&a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots\ldots a_{2n}x_n = b_2 \\
&a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \ldots\ldots a_{3n}x_n = b_3 \\
&\quad\vdots \\
&a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots\ldots a_{nn}x_n = b_n
\end{aligned}
\qquad (1)
$$

$x_1, x_2, x_3, \ldots\ldots, x_n$ Is a set of unknown variables

If <u>all b_i are zero</u> then the system is called <u>"Homogeneous system"</u>

If b_i <u>are not all zero</u> then the system is called <u>"Non homogeneous system"</u>

If the system (1) is homogeneous, it has at least the trivial solution x_1=0,…,x_n =0

<u>Solutions of System of Linear Equations</u>

A **linear system of** m **equations in** n **unknowns** $x_1, \cdots, x_n$ is a set of equations of the form

$$
\begin{aligned}
a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + \cdots + a_{2n}x_n &= b_2 \\
\cdots\cdots\cdots&\cdots\cdots\cdots \\
a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
$$

Can be written in a form of matrix equation as

$$\mathbf{Ax} = \mathbf{b} \longrightarrow [\mathbf{A}|\mathbf{b}]$$

$$
\underset{m \times n}{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}
\underset{n \times 1}{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}
=
\underset{m \times 1}{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}}
$$

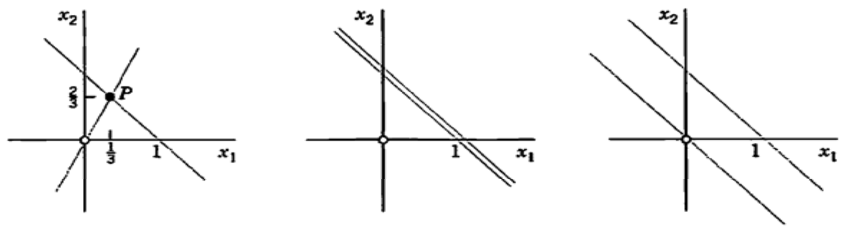
Geometric Interpretation. Existence and Uniqueness of Solutions

If $m = n = 2$, we have two equations in two unknowns x_1, x_2

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 &= b_1 \\
a_{21}x_1 + a_{22}x_2 &= b_2.
\end{aligned}
$$

There are 3 possible cases

Case (a)	Case (b)	Case (c)
$x_1 + x_2 = 1$	$x_1 + x_2 = 1$	$x_1 + x_2 = 1$
$2x_1 - x_2 = 0$	$2x_1 + 2x_2 = 2$	$x_1 + x_2 = 0$

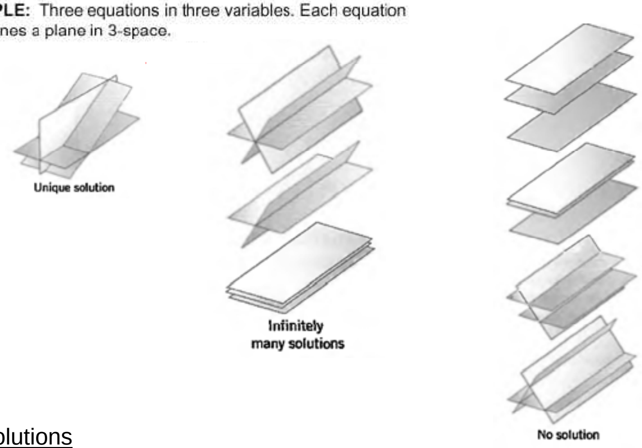


(a) Precisely one solution if the lines intersect.

(b) Infinitely many solutions if the lines coincide.

(c) No solution if the lines are parallel

EXAMPLE: Three equations in three variables. Each equation determines a plane in 3-space.



<u>Solutions</u>

Unique or Infinitely many solutions ⟶ Consistent system

No solution ⟶ Inconsistent system

How can we know ?

Strategy for solving a linear system

Replace one system with **an equivalent system** (one with the same solution set) that is easier to solve.

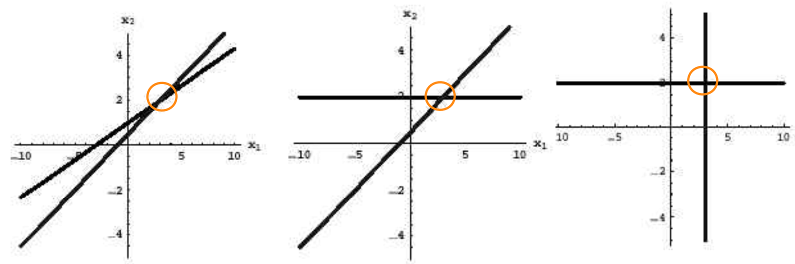
example

a)
$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 3x_2 &= 3 \end{aligned}$$

b)
$$\begin{aligned} x_1 - 2x_2 &= -1 \\ x_2 &= 2 \end{aligned}$$

c)
$$\begin{aligned} x_1 &= 3 \\ x_2 &= 2 \end{aligned}$$

Equivalent systems



$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 3x_2 &= 3 \end{aligned}$$
a)

$$\begin{aligned} x_1 - 2x_2 &= -1 \\ x_2 &= 2 \end{aligned}$$
b)

$$\begin{aligned} x_1 &= 3 \\ x_2 &= 2 \end{aligned}$$
c)

Solving a linear system

Elementary Row Operations:
1. *(Replacement)* Add one row to a multiple of another row.
2. *(Interchange)* Interchange two rows.
3. *(Scaling)* Multiply all entries in a row by a nonzero constant.

Note: **Row equivalent matrices:** Two matrices where one matrix can be transformed into the other matrix by a sequence of elementary row operations.

Fact about Row Equivalence: If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set.

Replacement: *k* x Row *i* adds to Row *j* and **replace Row *j* (the one that is not multiplied.)**

Example Solving a system of linear equations using augmented matrix methods.

$$3x_1 + 4x_2 = 1$$
$$x_1 - 2x_2 = 7$$

1. Augmented matrix corresponding to the system of linear equations.

$$\left[\begin{array}{cc|c} 3 & 4 & 1 \\ 1 & -2 & 7 \end{array}\right]$$

2. $R_1 \leftrightarrow R_2$ (To get a 1 in the upper left corner.)

$$\left[\begin{array}{cc|c} 1 & -2 & 7 \\ 3 & 4 & 1 \end{array}\right]$$

3. $(-3)R_1 + R_2 \to R_2$ (To get a 0 in the lower left corner.)

$$\left[\begin{array}{cc|c} 1 & -2 & 7 \\ 0 & 10 & -20 \end{array}\right]$$

4. $\left(\frac{1}{10}\right)R_2 \to R_2$ (To get a 1 in the lower right corner.)

$$\left[\begin{array}{cc|c} 1 & -2 & 7 \\ 0 & 1 & -2 \end{array}\right]$$

5. $(2)R_2 + R_1 \to R_1$ (To get a 0 in the upper right corner.)

$$\left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -2 \end{array}\right]$$

Hence, $x_1 = 3$ and $x_2 = -2$.

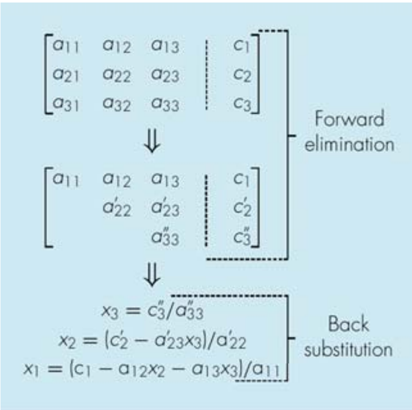
We can stop the row operation process at step 4 and perform back substitution to obtain the solution set. This method is known as "**Gauss Elimination**" method.

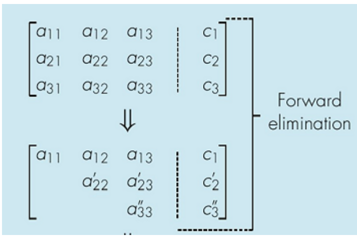
4. $\left(\frac{1}{10}\right)R_2 \to R_2$ (To get a 1 in the lower right corner.)

$$\left[\begin{array}{cc|c} 1 & -2 & 7 \\ 0 & 1 & -2 \end{array}\right]$$

$$x - 2y = 7 \quad (1)$$
$$y = -2 \quad (2)$$

Solve for y first in eq. (2)
and then substitute y into eq.(1)
to solve for x





backward elimination

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & C_1'' \\ 0 & 1 & 0 & C_2'' \\ 0 & 0 & 1 & C_3'' \end{array}\right]$$

Example of system of linear equations with 3 variables

An augmented matrix

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right]$$

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -3x_2 + 13x_3 &= -9 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array}\right]$$

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ -3x_2 + 13x_3 &= -9 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{array}\right]$$

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ -3x_2 + 13x_3 &= -9 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{array}\right]$$

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ x_3 &= 3 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\begin{aligned} x_1 - 2x_2 &= -3 \\ x_2 &= 16 \\ x_3 &= 3 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\begin{aligned} x_1 &= 29 \\ x_2 &= 16 \\ x_3 &= 3 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Check: Is $(29, 16, 3)$ a solution of the *original* system?

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned}$$

$$\begin{aligned} (29) - 2(16) + 3 &= 29 - 32 + 3 &= 0 \\ 2(16) - 8(3) &= 32 - 24 &= 8 \\ -4(29) + 5(16) + 9(3) &= -116 + 80 + 27 &= -9 \end{aligned}$$

Row operations are reversible.

If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution sets.

Example 1

$$6x_1 + x_2 + x_3 = 6$$
$$5x_1 + x_2 + 2x_3 = 4$$
$$4x_1 + x_2 - x_3 = -2$$

$$x_1 = 1, x_2 = -13, x_3 = 1$$

Example 2

$$\begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \\ -4 \end{bmatrix}$$

$$x_1 = -1 + \frac{4}{3}x_3$$
$$x_2 = 2$$

Example 3

$$3x_1 + 2x_2 + 2x_3 - 5x_4 = 8$$
$$0.6x_1 + 1.5x_2 + 1.5x_3 - 5.4x_4 = 2.7$$
$$1.2x_1 - 0.3x_2 - 0.3x_3 + 2.4x_4 = 2.1$$

$$x_2 = 1 - x_3 + 4x_4$$
$$x_1 = 2 - x_4$$

Example 4

$$3x_2 - 6x_3 = 8$$
$$x_1 - 2x_2 + 3x_3 = -1$$
$$5x_1 - 7x_2 + 9x_3 = 0$$

Inconsistent

Example 5

$$2x_1 - 4x_2 + x_3 = -4$$
$$4x_1 - 8x_2 + 7x_3 = 2$$
$$-2x_1 - 4x_2 - 3x_3 = 5$$

Inconsistent

Example 6 : For what values of h will the following system be consistent?

$$\begin{aligned} 3x_1 - 9x_2 &= 4 \\ -2x_1 + 6x_2 &= h \end{aligned}$$

Solution: Reduce to triangular form.

$$\left[\begin{array}{cc|c} 3 & -9 & 4 \\ -2 & 6 & h \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -3 & \frac{4}{3} \\ -2 & 6 & h \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -3 & \frac{4}{3} \\ 0 & 0 & h + \frac{8}{3} \end{array}\right]$$

The second equation is $0x_1 + 0x_2 = h + \frac{8}{3}$. System is consistent only if $h + \frac{8}{3} = 0$ or $h = \frac{-8}{3}$.

Example 7 Give an example of **b** that will make the linear system consistent.

$$\begin{aligned} x_1 + 2x_2 + 2x_3 &= b_1 \\ 2x_1 + 4x_2 + 6x_3 &= b_2 \\ 3x_1 + 6x_2 + 8x_3 &= b_3 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & 2 & 2 & b_1 \\ 2 & 4 & 6 & b_2 \\ 3 & 6 & 8 & b_3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & b_1 \\ 0 & 0 & 2 & b_2 - 2b_1 \\ 0 & 0 & 2 & b_3 - 3b_1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 2 & b_1 \\ 0 & 0 & 2 & b_2 - 2b_1 \\ 0 & 0 & 0 & -b_2 + 2b_1 + b_3 - 3b_1 \end{array}\right]$$

For consistent system 0 = b3-b2-b1

Eg. $\mathbf{b} = \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix}$

Inverse of Matrices

Matrix Inverses

The inverse of a real number a is denoted by a^{-1}. For example, $7^{-1} = 1/7$ and

$$7 \cdot 7^{-1} = 7^{-1} \cdot 7 = 1$$

An $n \times n$ matrix A is said to be **invertible** if there is an $n \times n$ matrix C satisfying

$$CA = AC = I_n$$

where I_n is the $n \times n$ identity matrix. We call C the **inverse** of A.

example $A = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}$; $C = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}$ $AC=$ $CA=$

The inverse of A is usually denoted by A^{-1}.

We have

$$\boxed{AA^{-1} = A^{-1}A = I_n}$$

Not all** $n \times n$ **matrices are invertible. A matrix which is *not* invertible is sometimes called a **singular** matrix. An invertible matrix is called **nonsingular** matrix.

Fact 1 If A is invertible, then the inverse is unique.

Proof: Assume B and C are both inverses of A. Then

$$B = BI = B(\_\_\_\_\_) = (\_\_\_\_\_)\_\_\_\_\_ = I\_\_\_\_\_ = C.$$

So the inverse is unique since any two inverses coincide.■

Fact 2 The inverse of A^{-1} is A itself.

Fact 3 Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

If $ad - bc = 0$, then A is not invertible.

Assume A is any invertible matrix and we wish to solve $A\mathbf{x} = \mathbf{b}$. Then

$$\_\_\_\_ A\mathbf{x} = \_\_\_\_\mathbf{b} \quad \text{and so}$$

$$I\mathbf{x} = \_\_\_\_\_\_ \text{ or } \mathbf{x} = \_\_\_\_\_\_.$$

Suppose $\mathbf{w}$ is also a solution to $A\mathbf{x} = \mathbf{b}$. Then $A\mathbf{w} = \mathbf{b}$ and

$$\_\_\_\_ A\mathbf{w} = \_\_\_\_\mathbf{b} \quad \text{which means} \quad \mathbf{w} = A^{-1}\mathbf{b}.$$

So, $\mathbf{w} = A^{-1}\mathbf{b}$, which is in fact the same solution.

We have proved the following result:

Fact 4 If A is an invertible $n \times n$ matrix, then for each $\mathbf{b}$ in $\mathbf{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

EXAMPLE: Use the inverse of $A = \begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix}$ to solve

$$\begin{aligned} -7x_1 + 3x_2 &= 2 \\ 5x_1 - 2x_2 &= 1 \end{aligned}.$$

Solution: Matrix form of the linear system:

$$\begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{14-15}\begin{bmatrix} -2 & -3 \\ -5 & -7 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}.$$

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}\begin{bmatrix} \\ \end{bmatrix} = \begin{bmatrix} \\ \end{bmatrix}$$

Properties of Inverses

Suppose A and B are invertible. Then the following results hold:

a. A^{-1} is invertible and $(A^{-1})^{-1} = A$ (i.e. A is the inverse of A^{-1}).

b. AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$

c. A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$

Partial proof of part b:

$$(AB)(B^{-1}A^{-1}) = A(\_\_\_\_\_\_)A^{-1}$$

$$= A(\_\_\_\_\_\_)A^{-1} = \_\_\_\_\_\_ = \_\_\_\_\_.$$

Similarly, one can show that $(B^{-1}A^{-1})(AB) = I$.

Proof part c

$$A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I$$
$$(A^{-1})^T A^T = (AA^{-1})^T = I^T = I$$

Matrix inversion algorithm

$$AA^{-1} = I$$

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

➔Take a column at a time, that equation determines the columns of A^{-1}

A times column j of A^{-1} = column j of I

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad ; \quad \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{array}\right]$$

Carry out elimination on *all systems simultaneously.*

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{array}\right]$$

$$[A \mid I]$$

$$\downarrow$$

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 7 & -3 \\ 0 & 1 & -2 & 1 \end{array}\right]$$

$$[I \mid A^{-1}]$$

The Gauss-Jordan method

Matrix inversion algorithm

Place A and I side-by-side to form an augmented matrix $[A \mid I]$. Then perform row operations on this matrix (which will produce identical operations on A and I).

$[A \mid I]$ will row reduce to $[I \mid A^{-1}]$

or A is not invertible.

EXAMPLE: Find the inverse of $A = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, if it exists.

Solution:

$$[A \mid I] = \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \sim \cdots \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & \frac{3}{2} & 1 & 0 \end{array}\right]$$

So $A^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 \\ \frac{3}{2} & 1 & 0 \end{bmatrix}$

Find the inverse of the matrix $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$, if it exists

Determinant of Matrices

Determinants

Determinant of a square matrix **A** is denoted by $|\mathbf{A}|$ or det **A** is a uniquely defined SCALAR associated with that matrix. Determinants are defined only for square matrices

A second-order determinant

For a 2x2 matrix $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$$\det \mathbf{A} = |\mathbf{A}| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$ is a scalar

Example : Given

$$\mathbf{A} = \begin{bmatrix} 6 & -3 \\ 5 & 9 \end{bmatrix}$$

$$\det \mathbf{A} = |\mathbf{A}| = (6)(9) - (-3)(5) = 69$$

A Third-order determinant

For a 3x3 matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\det \mathbf{A} = |\mathbf{A}| = (a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{32}a_{21}) - (a_{13}a_{31}a_{22} + a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33})$$

or

$$\det \mathbf{A} = |\mathbf{A}| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$$

OR

Subdeterminant or Minor

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{12}a_{23}a_{31} - a_{12}a_{21}a_{33} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} \quad [= \text{a scalar}]$$

A Third-order determinant

Example :

$$\begin{vmatrix} 2 & 1 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = (2)(5)(9) + (1)(6)(7) + (3)(8)(4) - (2)(8)(6) - (1)(4)(9) - (3)(5)(7) = -9$$

$$\begin{vmatrix} -7 & 0 & 3 \\ 9 & 1 & 4 \\ 0 & 6 & 5 \end{vmatrix} = (-7)(1)(5) + (0)(4)(0) + (3)(6)(9) - (-7)(6)(4) - (0)(9)(5) - (3)(1)(0) = 295$$

An n^{th} -order determinant

For an *n x n* matrix

$$\mathbf{A}=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}=\left[a_{ij}\right]_{n\times n}$$

By expanding

Any row *i* $\quad \det \mathbf{A} \text{ or } |\mathbf{A}| = a_{i1}C_{i1}+a_{i2}C_{i2}+\ldots+a_{in}C_{in} = \sum_{j=1}^{n} a_{ij}C_{ij}$

Any column *j* $\quad \det \mathbf{A} \text{ or } |\mathbf{A}| = a_{1j}C_{1j}+a_{2j}C_{2j}+\ldots+a_{nj}C_{nj} = \sum_{i=1}^{n} a_{ij}C_{ij}$

C_{ij} is the "cofactor" of the element

M_{ij} is the "minor" of the element a_{ij}

Obtained by deleting the *i* th row and *j* th column of a given determinant

The **(i, j)-cofactor** of A is the number C_{ij} where

$$C_{ij} = (-1)^{i+j} \det A_{ij}.$$

$$\det A = a_{11}.c_{11} + a_{12}c_{12} + \ldots\ldots a_{1n}c_{1n}$$

A cofactor expansion across the first row of A

Use a matrix of signs to determine $(-1)^{i+j}$

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} = 1C_{11} + 2C_{12} + 0C_{13}$$

(cofactor expansion across row 1)

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Example

Evaluate determinant of **A**, given

$$\mathbf{A}=\begin{bmatrix} 0 & 1 & 2 \\ 1 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix}$$

Choose row 1 for expansion, since there is 0 in row 1

$$\det \mathbf{A} = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$C_{12} = (-1)^{1+2}(M_{12}) \rightarrow M_{12} = \begin{vmatrix} 1 & 4 \\ 5 & 7 \end{vmatrix} = (1)(7)-(4)(5) = -13$$

$$C_{13} = (-1)^{1+3}(M_{13}) \rightarrow M_{13} = \begin{vmatrix} 1 & 3 \\ 5 & 6 \end{vmatrix} = (1)(6)-(3)(5) = -9$$

$$\det \mathbf{A} = (1)(3)+(2)(-9) = -5$$

EXAMPLE: Compute the determinant of $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{bmatrix}$

using cofactor expansion down column 3.

Solution

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} = 0\begin{vmatrix} 3 & -1 \\ 2 & 0 \end{vmatrix} - 2\begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix} + 1\begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} = 1.$$

Example

$$A=\begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix} \qquad \det A = ?$$

(-2)

EXAMPLE: Compute the determinant of $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 5 \end{bmatrix}$

(14)

Basic properties of determinants

(1) $\det \mathbf{A}^T = \det \mathbf{A}$

Example 1 $\begin{vmatrix} 4 & 3 \\ 5 & 6 \end{vmatrix} = \begin{vmatrix} 4 & 5 \\ 3 & 6 \end{vmatrix} = 9$

Example 2 $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix} = ad - bc$

Hence, column operations = row operations in determinant. (2)-(4)

(2) The interchange of any two rows (or any two columns) will alter the sign, but not the numerical value of the determinant.

Example

$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$ $\quad \begin{vmatrix} c & d \\ a & b \end{vmatrix} = cb - ad = -(ad - bc)$

$\begin{vmatrix} 0 & 1 & 3 \\ 2 & 5 & 7 \\ 3 & 0 & 1 \end{vmatrix} = -26.$ $\quad \begin{vmatrix} 3 & 1 & 0 \\ 7 & 5 & 2 \\ 1 & 0 & 3 \end{vmatrix} = 26.$

(3) The multiplication of any one row (or one column) by a scalar k will change the value of the determinant k-fold.

Example

$$\begin{vmatrix} ka & kb \\ c & d \end{vmatrix} = kad - kbc = k(ad - bc) = k\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$\begin{vmatrix} 15a & 7b \\ 12c & 2d \end{vmatrix} = 3\begin{vmatrix} 5a & 7b \\ 4c & 2d \end{vmatrix} = 3(2)\begin{vmatrix} 5a & 7b \\ 2c & d \end{vmatrix} = 6(5ad - 14bc)$$

(4) The addition (subtraction) of a multiple of any row (or column) to (from) another row will leave the value of the determinant unaltered.

Example

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc,$$

$$\begin{vmatrix} a & b \\ c + ka & d + kb \end{vmatrix} = a(d + kb) - b(c + ka) = ad - bc = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

EXAMPLE: Compute $\begin{vmatrix} 2 & 4 & 6 \\ 5 & 6 & 7 \\ 7 & 6 & 10 \end{vmatrix}$

Solution

$$\begin{vmatrix} 2 & 4 & 6 \\ 5 & 6 & 7 \\ 7 & 6 & 10 \end{vmatrix} = 2\begin{vmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 7 & 6 & 10 \end{vmatrix} = 2\begin{vmatrix} 1 & 2 & 3 \\ 0 & -4 & -8 \\ 0 & -8 & -11 \end{vmatrix}$$

$$= 2(-4)\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -8 & -11 \end{vmatrix} = 2(-4)\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 5 \end{vmatrix}$$

$$= 2(-4)(1)(1)(5) = -40$$

EXAMPLE: Compute $\begin{vmatrix} 2 & 3 & 0 & 1 \\ 4 & 7 & 0 & 3 \\ 7 & 9 & -2 & 4 \\ 1 & 2 & 0 & 4 \end{vmatrix}$ using a combination of row reduction and cofactor expansion.

(-12)

EXAMPLE: Compute $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 0 & 0 \\ 2 & 7 & 6 & 10 \\ 2 & 9 & 7 & 11 \end{vmatrix}$.

(-10)

Compute Det(**A**)

$$\mathbf{A} = \begin{bmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ -6 & 7 & -7 & 4 \\ -5 & -8 & 0 & 9 \end{bmatrix}$$

Det(**A**) = 0 when **A** is not invertible (singular).

(0)

(5) If one row (or column) is a multiple of another row (or column), the value of the determinant will be zero.

Example

$$\begin{vmatrix} 2a & 2b \\ a & b \end{vmatrix} = 2ab - 2ab = 0 \qquad \begin{vmatrix} c & c \\ d & d \end{vmatrix} = cd - cd = 0$$

(6) A zero row or column renders the value of a determinant zero.

Example

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 4 & 5 & 6 \end{vmatrix} = 0 = \begin{vmatrix} 0 & 1 & 5 \\ 0 & 2 & 8 \\ 0 & 3 & 9 \end{vmatrix}$$

Further Properties

(7) $$\det(\mathbf{AB}) = \det \mathbf{A} \cdot \det \mathbf{B}$$

EXAMPLE: Compute $\det A^3$ if $\det A = 5$.

Solution: $\det A^3 = \det(AAA) = (\det A)(\det A)(\det A)$

$= \underline{\hspace{3cm}} = \underline{\hspace{2cm}}.$

(8) If **A** is an *n* x *n* upper or lower triangular matrix

Triangular Matrices:

$$\begin{bmatrix} * & * & \cdots & * & * \\ 0 & * & \cdots & * & * \\ 0 & 0 & \ddots & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * \end{bmatrix} \qquad \begin{bmatrix} * & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ * & * & \ddots & 0 & 0 \\ * & * & \cdots & * & 0 \\ * & * & \cdots & * & * \end{bmatrix}$$

(upper triangular) (lower triangular)

$$\det \mathbf{A} = a_{11}a_{22}a_{33}\ldots a_{nn}$$

Example

$$[A] = \begin{bmatrix} 2 & 0 & 0 \\ 3 & 10 & 0 \\ 4 & 8 & 9 \end{bmatrix} \rightarrow |A| = (2)(10)(9) = \mathbf{180}$$

Applications of determinants

Cramer's Rule

Cramer's rule can be used to study how the solution of **Ax=b** affected by the changes in the entries of b.

Let A be an invertible $n \times n$ matrix. For any $\mathbf{b}$ in R^n, the unique solution $\mathbf{x}$ of $A\mathbf{x} = \mathbf{b}$ has entries given by

$$x_i = \frac{\det A_i(\mathbf{b})}{\det A}, \quad i = 1,2,3\ldots n$$

where $$A_i(\mathbf{b}) = [\mathbf{a}_1 \quad . \quad . \quad \underset{\uparrow \atop Col\ i}{\mathbf{b}} \quad . \quad . \quad \mathbf{a}_n]$$

Example

$$3x_1 - 2x_2 = 6$$
$$-5x_1 + 4x_2 = 8$$

Find the solution of the equation system

$$\begin{aligned} 7x_1 - x_2 - x_3 &= 0 \\ 10x_1 - 2x_2 + x_3 &= 8 \\ 6x_1 + 3x_2 - 2x_3 &= 7 \end{aligned}$$

The computation of A^{-1}

$$A^{-1} = \frac{1}{\det A}\begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix} \quad (4)$$

The matrix of cofactors on the right side of (4) is called the **adjugate** (or **classical adjoint**) of A, denoted by adj A. (The term *adjoint* also has another meaning in advanced texts on linear transformations.) The next theorem simply restates (4).

The adjoint of A_{nxn} is defined to be the transpose of the matrix of cofactors:

$$\boxed{adjA = [C_{ij}(A)]^T}$$

$$adjA ==> n \times n$$

Example

$$A = \begin{bmatrix} 3 & 0 & -1 \\ 4 & 7 & 3 \\ -2 & 8 & 5 \end{bmatrix}$$

$$adjA = \begin{bmatrix} 11 & -26 & 46 \\ -8 & 13 & -24 \\ 7 & -13 & 21 \end{bmatrix}^T = \begin{bmatrix} 11 & -8 & 7 \\ -26 & 13 & -13 \\ 46 & -24 & 21 \end{bmatrix}$$

Theorem

An Inverse Formula

Let A be an invertible $n \times n$ matrix. Then

$$A^{-1} = \frac{1}{\det A}\,\text{adj}\,A$$

EXAMPLE 3 Find the inverse of the matrix $A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 4 & -2 \end{bmatrix}$.

Solution The nine cofactors are

$$C_{11} = + \begin{vmatrix} -1 & 1 \\ 4 & -2 \end{vmatrix} = -2, \quad C_{12} = - \begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} = 3, \quad C_{13} = + \begin{vmatrix} 1 & -1 \\ 1 & 4 \end{vmatrix} = 5$$

$$C_{21} = - \begin{vmatrix} 1 & 3 \\ 4 & -2 \end{vmatrix} = 14, \quad C_{22} = + \begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix} = -7, \quad C_{23} = - \begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix} = -7$$

$$C_{31} = + \begin{vmatrix} 1 & 3 \\ -1 & 1 \end{vmatrix} = 4, \quad C_{32} = - \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = 1, \quad C_{33} = + \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} = -3$$

The adjugate matrix is the *transpose* of the matrix of cofactors. [For instance, C_{12} ge in the (2, 1) position.] Thus

$$\text{adj } A = \begin{bmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{bmatrix}$$

We could compute det A directly, but the following computation provides a check the calculations above *and* produces det A:

$$(\text{adj } A) \cdot A = \begin{bmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 4 & -2 \end{bmatrix} = \begin{bmatrix} 14 & 0 & 0 \\ 0 & 14 & 0 \\ 0 & 0 & 14 \end{bmatrix} = 14I$$

Since $(\text{adj } A)A = 14I$, Theorem 8 shows that det $A = 14$ and

$$A^{-1} = \frac{1}{14} \begin{bmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{bmatrix} = \begin{bmatrix} -1/7 & 1 & 2/7 \\ 3/14 & -1/2 & 1/14 \\ 5/14 & -1/2 & -3/14 \end{bmatrix}$$