Practice problem set 4
Single-variable optimization
EE320 Semester 1/2016
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Question 1 Total product function

Suppose that $TP(L) = 120L^2 - L^3$, where TP is the total output. Answer the following questions

a. Find the expression for AP and MP.
$AP(L) = 120L - L^2 \ and \ MP(L) = 240L - 3L^2$

b. Find the value of L that maximizes AP, MP and TP, respectively.

Maximize AP ⇒ $FOC: \ AP'(L) = 120 - 2L = 0 \ ==> L = 60$
$SOC: \ AP''(60) = -2 < 0)$

Maximize MP ⇒ $FOC: MP'(L) = 240 - 6L = 0 \ ==> L = 40$
$SOC: \ (MP''(40) = -6 < 0)$

Maximize TP ⇒
$FOC: \ TP'(L) = MP(L) = 240L - 3L^2 = 0 ==> L = 0, \ 80.$
$SOC: \ (TP''(L) = 240 - 6L \ ==> TP''(80) < 0.$

c. Define the domain set of L that justifies the production function.

One should be noted that (i) L must be greater than or equal to zero. (negative L doesn't make sense. Second, firm will not produce in the region that L is greater than 80 units. When L exceeds 80 units, TP decreases.

Question 2

Let the total cost function be:

$$TC(Q) = 2Q^2 - 8Q + 10.$$

a) Determine whether *TC(Q)* is a convex or concave function.

Ans. TC'(Q) = 4Q-8. ⇒ TC''(Q)=4>0. Thus, TC(Q) is a convex function.

b) Find the quantity Q^* that minimizes the total cost.

Ans. $TC'(Q) = 4Q-8=0. \Rightarrow Q^*=2.$

c) Verify that $TC(Q^*)$ is the lowest cost by using the second derivative test.

Ans. $TC''(2)=4>0$. Thus, $TC(2) =2$ is the minimum.

Question 3

Let the demand function be given by $Q = 60 - 2P$ where P is the unit price and Q is the amount of quantity. Assume that the production function of a firm is given by $Q = 5\sqrt{L}$ where L is the number of workers hired. Consider the following questions.

a. Suppose that wage is $50 per each worker hired. This is fixed, regardless to how many workers hired by a firm. Write down the equation that summarizes the total cost of hiring labor by "L" workers.

   $Cost = W * L = 50L$

b. Normally, our cost function is written in terms of Q. Based on "a", can you find a way to represent the total cost equation in the terms of Q, rather than L?

   Note that $Q = 5\sqrt{L} \Rightarrow Q^2 = 25L \Rightarrow L = \frac{1}{25}Q^2$.

   $Cost = 50L = 50(\frac{1}{25}Q^2) = 2Q^2$

c. Use the cost function in "b", and solve for the level of profit-maximizing output/price. Verify your answer

   First, rewrite demand into p-equal form. That is, $P = \frac{60-Q}{2}$

   $\pi(Q) = \frac{60-Q}{2}Q - 2Q^2$

   FOC: $\pi'(Q) = 30 - Q - 4Q = 0$

   $Q = 6$.

   SOC: $\pi''(Q) = -5 < 0 \Rightarrow$ This warrants that $Q = 6$ is actually the profit-maximizing level of output.

d. At the level of profit-maximizing output, how many workers should the firm hire?

$Q = 6 \implies L = \frac{1}{25}(6)^2 = 1.44\ units.$

Question 4 *Deriving the market supply*

Suppose the cost function of a representative firm can be given by: $C(Q) = 3Q^2 + 5Q + 75$ where Q is the level of output produced. Answer the following question

a. Derive the expression for MC, AVC, and ATC.

$MC(Q) = 6Q + 5$

$AVC(Q) = \frac{TVC}{Q} = 3Q + 5$

$ATC(Q) = 3Q + 5 + \frac{75}{Q}$

b. Find the level of Q that results in the lowest total cost.

That is, when MC = 0.

$MC(Q) = 6Q + 5 = 0 \implies Q = -5/6.$

However, it is not possible to have negative Q.

As you can see from the equation, MC is always positive. That means, C(Q) is a strictly increasing function for positive Q.

As a result, the lowest cost would be when Q = 0. That's firm only pays for the fixed cost.

c. Find the supply equation of the representative firm. Make sure you specify the range of price that justifies your equation as the one representing the supply equation.

$P = MC(Q)$

$P = 6Q + 5$

When Q = 0 , P = 5. That is, firm needs to earn at least $5 if it would stay active in the market.

d. If there are 60 identical firms in the market, derive the market supply curve.

$Q = \frac{P-5}{6}$ ==> $Q^s_M = 60 * (\frac{P-5}{6}) = 10(P-5)$ when P is greater than or equal to zero. $P = 5 + \frac{Q}{10}$

Question 5

Given the following function

$$f(x) = 2x^3 + 8x^2 - 32x - 50$$

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a. Find the critical value(s) of *x* and the corresponding stationary value(s) of *f(x)*.

Ans. $f'(x) = 6x^2 + 16x - 32 = 0 \Rightarrow x^* = -4, 4/3$

b. Evaluate whether the stationary value(s) found in part a) are relative maxima or minima or inflection points by using *the first-derivative test*.

Ans. f(-4) is a maximum because f'(-5)=38 > 0 and f'(-3)=-26 < 0.

f(4/3) is a minimum because f'(1)=-10 < 0 and f'(2)=24 > 0.

Question 6

A competitive firm receives a price *p* for each unit of its output, and pays a price *w* for each unit of its only variable input. It also incurs set up costs of *F*. Its output from using *x* units of variable input is $f(x) = \sqrt{x}$. Determine the firm's revenue, cost, and profit functions.

Ans. For x > 0, $R = p\sqrt{x}$; $C = wx + F$; $\pi(x) = p\sqrt{x} - wx - F$.

$\pi'(x) = 0$ when $w = p/2\sqrt{x} \Rightarrow x = p^2/4w^2$. Check: $\pi''(x) < 0$

Question 7

The price a firm obtains for a commodity varies with demand *Q* according to the formula *P(Q) = 18 – 0.006Q*. Total cost is $C(Q) = 0.004Q^2 + 4Q + 4500$.

a. Find the firm's profit and the value Q which maximizes profit.

Ans. $\pi(Q) = Q.P(Q) - C(Q) = -0.01Q^2 + 14Q - 4500 \;\Rightarrow Q^* = 700$

b. Find a formula for the elasticity of *P(Q)* w.r.t. *Q*, and find the particular value *Q** of *Q* at which the elasticity is equal to -1.

Ans. This is the price elasticity w.r.t. demand (Q=Qd):

$$E_d = \frac{Q}{P}\frac{dP}{dQ} = \frac{Q}{18-0.006Q}(-0.006) = -1 \Rightarrow Q^* 1500$$

c. Show that the marginal revenue is 0 at Q*.

Ans. MR = 18 -0.012Q ⇒ at Q*=1500, MR = 0.

<u>Question 8</u> Monopoly and Subsidy program

A monopolist firm faces the market demand equation given by $P = 150 - 0.5Q$ and operates under a technology with cost function given by $TC = 100 + 3Q + 7Q^2$. Consider the following questions

a. By using the derivative method, find the level of profit-maximizing output and price. Verify that your answer is the correction solution that results in maximized profit.

$MR = 150 - Q$

$MC = 3 + 14Q$

FOC: $MR = MC$

$150 - Q = 3 + 14Q$

$Q = \frac{147}{15}$

SOC:

$\pi'' = MR' - MC' < 0$

$MR' = -1$

$MC' = 14$

$\pi'' = -15 < 0$

So, $Q = \frac{147}{15}$ is the level of profit-maximizing output.

Continue with all the information given above, but now add another assumption to the questions. That is, we now assume that government subsidizes the monopolist for $3 for each unit of output.

b. Write the cost function of the monopolist when subsidization is taken into account.

$TC = 100 + 3Q + 7Q^2 - 3Q = 100 + 7Q^2$.

c. Find the level of profit-maximizing output and price under the subsidy program.

Redo the same as in (a), but now we use the new TC as obtained in (b).

FOC:

$$MR = MC$$
$$150 - Q = 14Q$$
$$Q = 10.$$

SOC. $\pi'' = -15 < 0$. Thus, $Q = 10$ is the level of profit-maximizing output. Given this Q, price would be \$145. (This can be obtained by plugging Q into the demand equation.)

Question 9

A monopolist is operating under a cost function given by,

$$TC = 2Q^3 - 12Q^2 + 94Q + 50,$$

where Q is the quantity of output.

Suppose that the market demand for the goods produced by this monopolist is given by: $P = 81 - Q^2$, where P is the price per unit of output.

Consider the following problems.

a. Determine the *absolute* value of the price elasticity of demand when P = \$54.

abs(elasticity) = 1

b. Without solving for the optimization, determine the revenue-maximizing level of output.

Where elasticity of demand is equal to 1. That is, Q = $\sqrt{27}$

c. Derive the profit function, and solve for the profit-maximizing level of output. Verify your answer by using the second-order derivative.

$profit = -3Q^3 + 12Q^2 - 12Q - 50$

$Q = 2; \quad \pi'' = -12 < 0$

d. Would the monopolist change the level of output if government taxes the monopolist based on the profit level that he earns?

Output will remain the same.