Answer to Problem Set 1

The starting point is the value function for t=1. Since $x_2=0$, the transition equation for period 1 is given as

$$x_2 = 0 = x_2 - u_2 \tag{1}$$

The transition equation for t=1 provides us with the optimal decision rule for the control variable u in period 1.

$$u_1 = x_1 \tag{2}$$

Equation 2 simply says to extract and sell all the oil that remains at the beginning of t=1 in that same period. It is independent of any price variables or any other variables dates t<1. It is an application of the principle of optimality that underlying dynamic programming: we can do no better than just follow the derived optimality policy rule regardless of what happens to control and state variables in earlier periods.

To obtain the optimal decision rules for the control variable in the period prior to t=1, we will make use of the recursive nature of Bellman equation. One period at a time, we will move backward in time until we have found the decision rule for t=0.

The first step in finding the decision rule for setting the control variable in t=0 consists of substituting the optimal policy function for t=1 into the value function for t=1,

$$V_1(x_1) = \left(p_1 u_1 - 0.05u_1^2\right) + 0.9V_0(x_2)$$

By equation 2, this is simplified to

$$V_1(x_1) = \left(p_1 x_1 - 0.05x_1^2\right) \tag{3}$$

Next, we substituting equation 3 into the Bell equation for period t=0

$$V_2(x_0) = \max_{u_0}\left\{\left(p_0 u_0 - 0.05u_0^2\right) + 0.9V_1(x_1)\right\}$$

To obtain

$$V_2(x_0) = \max_{u_0}\left\{\left(p_0 u_0 - 0.05u_0^2\right) + 0.9\left(p_1 x_1 - 0.05x_1^2\right)\right\} \tag{4}$$

Before we can maximize the right-hand side of equation 4 with respect to u_0 we need to consider that x_1 is connected to u_0 via the transition function

$$x_1 = x_0 - u_0 \tag{5}$$

Substituting equation 5 into 4 we obtain

$$V_2(x_0) = \max_{u_0}\left\{\left(p_0 u_0 - 0.05u_0^2\right) + 0.9\left(p_1(x_0 - u_0) - 0.05(x_0 - u_0)^2\right)\right\} \tag{6}$$

Now, only the decision variable u_0 and variables that are known at time t=0 are in equation 6. The right-hand side of equation 6 can be maximized with respect to the control variable u_0,

$$\frac{d\left[\left(p_0 u_0 - 0.05u_0^2\right) + 0.9\left(p_1\left(x_0 - u_0\right) - 0.05\left(x_0 - u_0\right)^2\right)\right]}{du_0} = 0$$

$$p_0 - 0.19u_0 - 0.9p_1 + 0.09u_0 = 0.$$

Solving the above first-order condition for u_2 gives the optimal policy rule for period 0,

$$u_0 = 0.4737x_0 + 5.263p_0 - 4.737p_1 \qquad (7)$$

This complete the calculation for t=0.