1. Two individuals agree at date 0 to a forward contract that matures at date 2. The contract is written on an underlying asset that pays a dividend at date 1 equal to D_1. Let f_2 be the date 2 random payoff (profit) to the individual who is the long party in the forward contract. Also let m_{0i} be the stochastic discount factor over the period from dates 0 to i where $i = 1, 2$, and let $E_0[\cdot]$ be the expectations operator at date 0. What is the value of $E_0[m_{02} f_2]$? Explain your answer.

$$S_0 = E_0[m_{01} D_1] + E_0[m_{02} S_2] = D_0 + E_0[m_{02} S_2]$$

$$f_2 = S_2 - F_{02}$$

$$E_0[m_{01} f_2] = E_0[m_{02}(S_2 - F_{02})] = E[m_{02} S_2] - E[m_{02} F_{02}]$$

Note: $E[m_{02} F_{02}] = E[m_{02}] F_{02} = R_f^{-2} F_{02}$

Therefore,

$$E_0[m_{02} f_2] = E[m_{02} S_2] - E[m_{02} F_{02}]$$
$$= S_0 - D_0 - R_f^{-2} F_{02}$$

So, $F_{02} = R_f^2 (S_0 - D_0)$ which $E_0[m_{02} f_2] = 0$

2. Assume that there is an economy populated by infinitely lived representative individuals who maximize the lifetime utility function

$$E_0\left[\sum_{t=0}^{\infty} -\delta^t e^{-ac_t}\right]$$

where c_t is consumption at date t and $a > 0$, $0 < \delta < 1$. The economy is a Lucas endowment economy (Lucas 1978) having multiple risky assets paying date t dividends that total d_t per capita. Write down an expression for the equilibrium per capita price of the market portfolio in terms of the assets' future dividends.

$$P_0 = E_0\left[\sum_{t=1}^{\infty} \frac{u_c(C_t, t)}{u_c(C_0, 0)} d_t\right]$$

$$u(C_t, t) = -\delta^t e^{-aC_t}$$
$$= a\delta^t e^{-aC_t}$$

$\therefore C_t = d_t$

$$= E_0\left[\sum_{t=1}^{\infty} \delta^t e^{-a(d_t - d_0)} d_t\right]$$

3. **For the Lucas model with labor income, show that assumptions (6.25) and (6.26) lead to the pricing relationship (6.27) and (6.28).**

$$P_t = E_t\left[\sum_{t=1}^{\infty} \delta^j \left(\frac{C^*_{t+j}}{C^*_t}\right)^{\gamma-1} d_{t+j}\right]$$

$$P_t/d_t = E_t\left[\sum_{j=1}^{\infty} \delta^j \left(\frac{C^*_{t+j}}{C^*_t}\right)^{\gamma-1} \left(\frac{d_{t+j}}{d_t}\right)\right]$$

$$= E_t\left[\sum_{j=1}^{\infty} \delta^j e^{(\gamma-1)\ln(C_{t+j}/C_t)+\ln(d_{t+j}/d_t)}\right.$$

NOW, $\ln(C_{t+j}/C_t) = j\cdot\mu_c + \sigma_c \sum_{j=1}^{j} \eta_{t+j}$

$\ln(D_{t+j}/D_t) = j\cdot\mu_d + \sigma \sum_{j=1}^{j} \varepsilon_{t+j}$

So,

$$P_t/d_t = E_t\left[\sum_{j=1}^{\infty} \delta^j e^{(\gamma-1)(j\cdot\mu_c + \sigma_c \sum_{j=1}^{j}\eta_{t+j}) + j\cdot\mu_d + \sigma\sum_{j=1}^{j}\varepsilon_{t+j}}\right.$$

$$= E_t\left[\sum_{j=1}^{\infty} \delta^j e^{j((\gamma-1)\mu_c + \mu_d) + \sum_{j=1}^{j}[(\gamma-1)\sigma_c\eta_{t+j} + \sigma_c\varepsilon_{t+j}]}\right]$$

$$= \sum_{j=1}^{\infty} \delta^j e^{j\{(\gamma-1)\mu_c + g_d\}} e^{j((1-\gamma)^2\sigma_c^2 + \sigma_d^2 - 2(1-\gamma)\sigma_c\sigma_d\rho)}$$

$$= \sum_{j=1}^{\infty} e^{(\ln\delta - (1-\gamma)\mu_c + \mu_d + \frac{1}{2}((1-\gamma)^2\sigma_c^2 + \sigma_d^2 - (1-\gamma)\sigma_c\sigma_d\rho]}$$

$$= \frac{1}{1-\delta e^{-(1-\gamma)\mu_c + \frac{1}{2}[(1-\gamma)^2\sigma_c^2 + \sigma_d^2] - (1-\gamma)\rho\sigma_c\sigma_d}} - 1$$

So, $$P_t = d_t \frac{\delta e^a}{1-\delta e^a}$$

where $$a \equiv \mu_d - (1-\gamma)\mu_c + \frac{1}{2}[(1-\gamma)^2\sigma_c^2 + \sigma_d^2] - (1-\gamma)\rho\sigma_c\sigma_d$$

4. Consider a special case of the model of rational speculative bubbles discussed in this chapter. Assume that infinitely lived investors are risk-neutral and that there is an asset paying a constant, one-period risk-free return of $R_f = \delta^{-1} > 1$. There is also an infinitely lived risky asset with price p_t at date t. The risky asset is assumed to pay a dividend of d_t that is declared at date t and paid at the end of the period, date $t+1$. Consider the price $p_t = f_t + b_t$ where

$$f_t = \sum_{i=0}^{\infty} \frac{E_t[d_{t+i}]}{R_f^{\,i+1}} \tag{1}$$

and

$$b_{t+1} = \begin{cases} \frac{R_f}{q_t} b_t + e_{t+1} & \text{with probability } q_t \\ z_{t+1} & \text{with probability } 1 - q_t \end{cases} \tag{2}$$

where $E_t[e_{t+1}] = E_t[z_{t+1}] = 0$ and where q_t is a random variable as of date $t-1$ but realized at date t and is uniformly distributed between 0 and 1.

a. Show whether or not $p_t = f_t + b_t$, subject to the specifications in (1) and (2), is a valid solution for the price of the risky asset.

$$E_t[b_{t+1}] = \frac{R_f}{q_t} b_t q_t + E_t[e_{t+1}] q_t + (1 - q_t) E_t[z_{t+1}]$$

$$= R_f b_t$$

valid solution ✓

b. Suppose that p_t is the price of a barrel of oil. If $p_t \geq p_{solar}$, then solar energy, which is in perfectly elastic supply, becomes an economically efficient perfect substitute for oil. Can a rational speculative bubble exist for the price of oil? Explain why or why not.

Since $E_t[b_{t+1}] = R_f b_t$

$$\lim_{j \to \infty} E_t[b_{t+j}] = \begin{cases} +\infty & \text{if } b_t > 0 \\ -\infty & \text{if } b_t < 0 \end{cases}$$

For limited liability assets, such as oil, we cannot have a bubble path w/ a price becoming negative, so we need to consider only bubble w/ $b_t > 0$. In this case, we can see from the from the above equation that for a bubble solution to exist, the bubble component must be expected to increase infinitely. But this cannot be a rational expectation if there is an upper bound on the price of oil, as would be the case if there was a perfect substitute in perfectly elastic supply. P_t cannot rise above P_{solar}, b_t cannot rise above $P_{solar} - P^*_t$. Thus, a bubble path where b_t must be expected to increase to ∞ cannot possibly occur.

c. **Suppose p_t is the price of a bond that matures at date $T < \infty$. In this context, the d_t for $t \leq T$ denotes the bond's coupon and principal payments. Can a rational speculative bubble exist for the price of this bond? Explain why or why not.**

for similar reasons, a rational speculative bubble cannot exist for the price of a bond. since, at maturity, the bond's price must be $p_T = d_T$ and zero after date T, its price cannot rationally be expected to satisfy equation and increase ∞, A bubble path is invalid, and the only rational price is $p_t = p_t^*$,

5. **Consider an endowment economy with representative agents who maximize the following objective function:**

$$\max_{C_s(\omega_s), \forall s, i} E_t\left[\sum_{s=t}^{T} \delta^s u(C_s)\right]$$

where $T < \infty$. Explain why a rational speculative asset price bubble could not exist in such an economy.

w/ the economy, assets, having a finite horizon, asset prices could not have the form $p_t = f_t + b_t$ w/ $b_t \neq 0$ b/c at date T, $p_T = f_T = d_T$ which is an asset's final dividend payment. since $b_T = 0$ w/ certainty, then the bubble process $E_t(b_{t+1}) = \delta^{-1} b_t$ implies $E_{T-1}[b_T] = E_{T-1}[0] = \delta^{-1} b_{T-1}$, or $b_{T-1} = 0$. A similar argument implies $b_t = 0$ for all previous date, $t < T-1$.