Optimization without Constraint : The Case of More than One Choice Variable I

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CW Ch.11

Outline

- The Differential Version of Optimization Conditions
- Extreme Values of a Function of Two Variables
- Quadratic Forms – An Excursion*
- Objective Functions with More than Two Variables*
- Economic Applications*
- Comparative-Static Aspects of Optimization*

The Differential Version of Optimization Conditions

- **First-Order Condition**

  Given a function $z = f(x)$, we can write the differential
  $$dz = f'(x)dx$$
  If $f'(x) \neq 0$, it must be the case that $dz \neq 0$ for any value of $dx \neq 0$. We can still alter the value of z by changing the value of x.

  Hence, a necessary condition for z to attain an extremum (a stationary point) is $dz = 0$ for an arbitrary nonzero dx.

  Thus the first-order *derivative* condition "$f'(x) = 0$" can be translated into the first-order differential condition "$dz = 0$ for an arbitrary nonzero dx."

  Recall that we still have inflection points, which make this first-order condition be just *necessary* condition, but not *sufficient*.

The Differential Version of Optimization Conditions

- **Second-Order Condition**

Given that $dz = f'(x)dx$, we can obtain the ***second-order differential***, denoted by d^2z (read: "d-two z"), by:

$$d^2z \equiv d(dz) = d[f'(x)dx]$$

$$= [df'(x)]dx$$

$$= [f''(x)dx]dx = f''(x)dx^2.$$

The term dx^2 (read: "d x squared") is always positive.

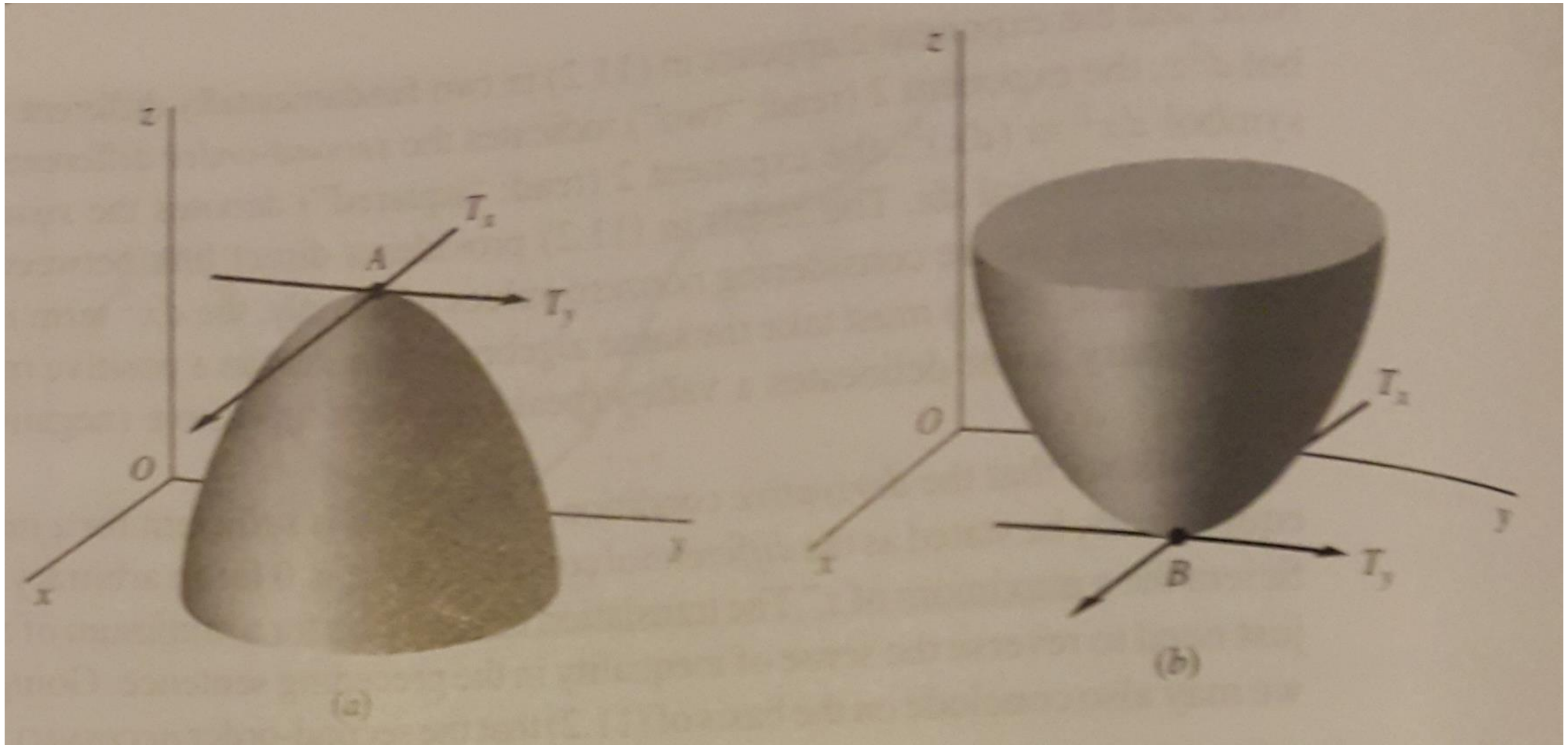
It follows that the *derivative* conditions can equivalently be stated as the differential conditions

"$d^2z < 0$ for an arbitrary nonzero dx" for a maximum of z

"$d^2z > 0$ for an arbitrary nonzero dx" for a minimum of z

Extreme Values of a Function of Two Variables

- With two choice variables, the graph of the function $z = f(x, y)$ becomes a surface in a 3-space as follows:



Extreme Values of a Function of Two Variables

- **First-Order Condition**

  For the function

$$z = f(x, y),$$

  the *first-order necessary* condition for an extremum now becomes

"$dz = 0$ for arbitrary values of dx and dy, not both zero."

  From the total differential

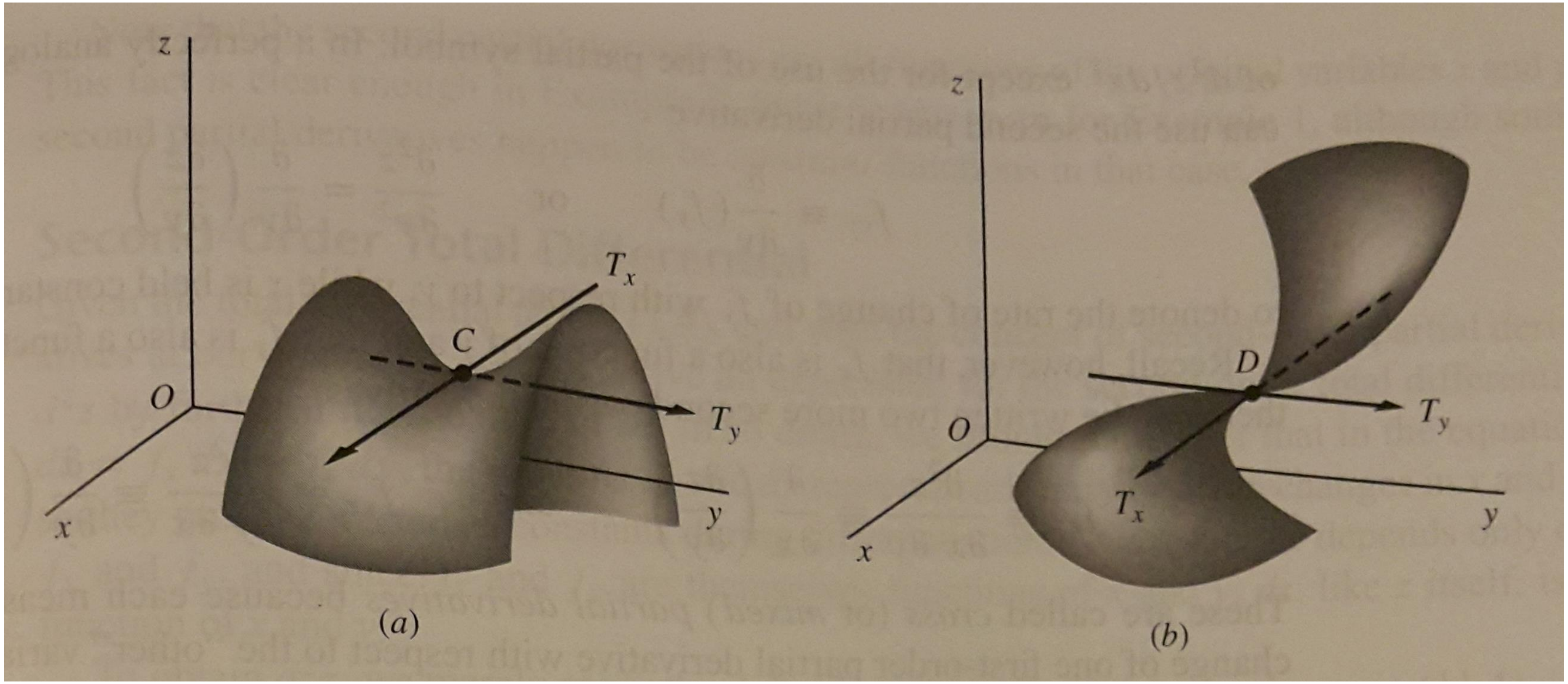
$$dz = f_x dx + f_y dy,$$

  the first-order necessary condition requires that f_x and f_y be simultaneously equal to zero. Thus the equivalent derivative version of the above first-order condition is:

"$f_x = f_y = 0$."

Extreme Values of a Function of Two Variables

- The first-order condition is necessary, but *not sufficient*. Look at the below diagrams. Point C in diagram (a) is referred to as a ***saddle point***. Point D in diagram (b) is an ***inflection point***.



Extreme Values of a Function of Two Variables

- **Second-Order Partial Derivatives**

  From the function $z = f(x, y)$, let define the second-order partial derivatives as below:

$$f_{xx}\left(\text{or } \frac{\partial^2 z}{\partial x^2}\right) \equiv \frac{\partial}{\partial x}(f_x) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)$$

$$f_{yy}\left(\text{or } \frac{\partial^2 z}{\partial y^2}\right) \equiv \frac{\partial}{\partial y}(f_y) = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$$

$$f_{xy} \equiv \frac{\partial^2 z}{\partial y \partial x} \equiv \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$$

$$f_{yx} \equiv \frac{\partial^2 z}{\partial x \partial y} \equiv \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$$

  Even though f_{xy} and f_{yx} are separately defined, they will – according to *Young's Theorem* – have the same values as long as the two cross partial derivatives are both continuous.

Example

- Find the four second-order partial derivatives of

$$z = x^3 + 5xy - y^2$$

Extreme Values of a Function of Two Variables

- **Second-Order Total Differential**

  Given the total differential $dz = f_x dx + f_y dy$, we can obtain the second-order total differential d^2z as:

$$d^2z \equiv d(dz) = \frac{\partial(dz)}{\partial x}dx + \frac{\partial(dz)}{\partial y}dy$$

$$= \frac{\partial}{\partial x}\left(f_x dx + f_y dy\right)dx + \frac{\partial}{\partial y}\left(f_x dx + f_y dy\right)dy$$

$$= \left(f_{xx}dx + f_{yx}dy\right)dx + \left(f_{xy}dx + f_{yy}dy\right)dy$$

$$= f_{xx}dx^2 + f_{yx}dydx + f_{xy}dxdy + f_{yy}dy^2$$

$$= f_{xx}dx^2 + 2f_{xy}dxdy + f_{yy}dy^2.$$

  Observe that the value of d^2z depends on all f_{xx}, f_{yy} as well as f_{xy}.

Example

- Given $z = x^3 + 5xy - y^2$, find dz and d^2z.

Extreme Values of a Function of Two Variables

- **Second-Order Condition**

  Once the first-order necessary condition is satisfied, the *second-order sufficient* condition for a ***maximum*** of $z = f(x, y)$ is

  "$d^2z < 0$ for arbitrary values of dx and dy, not both zero."

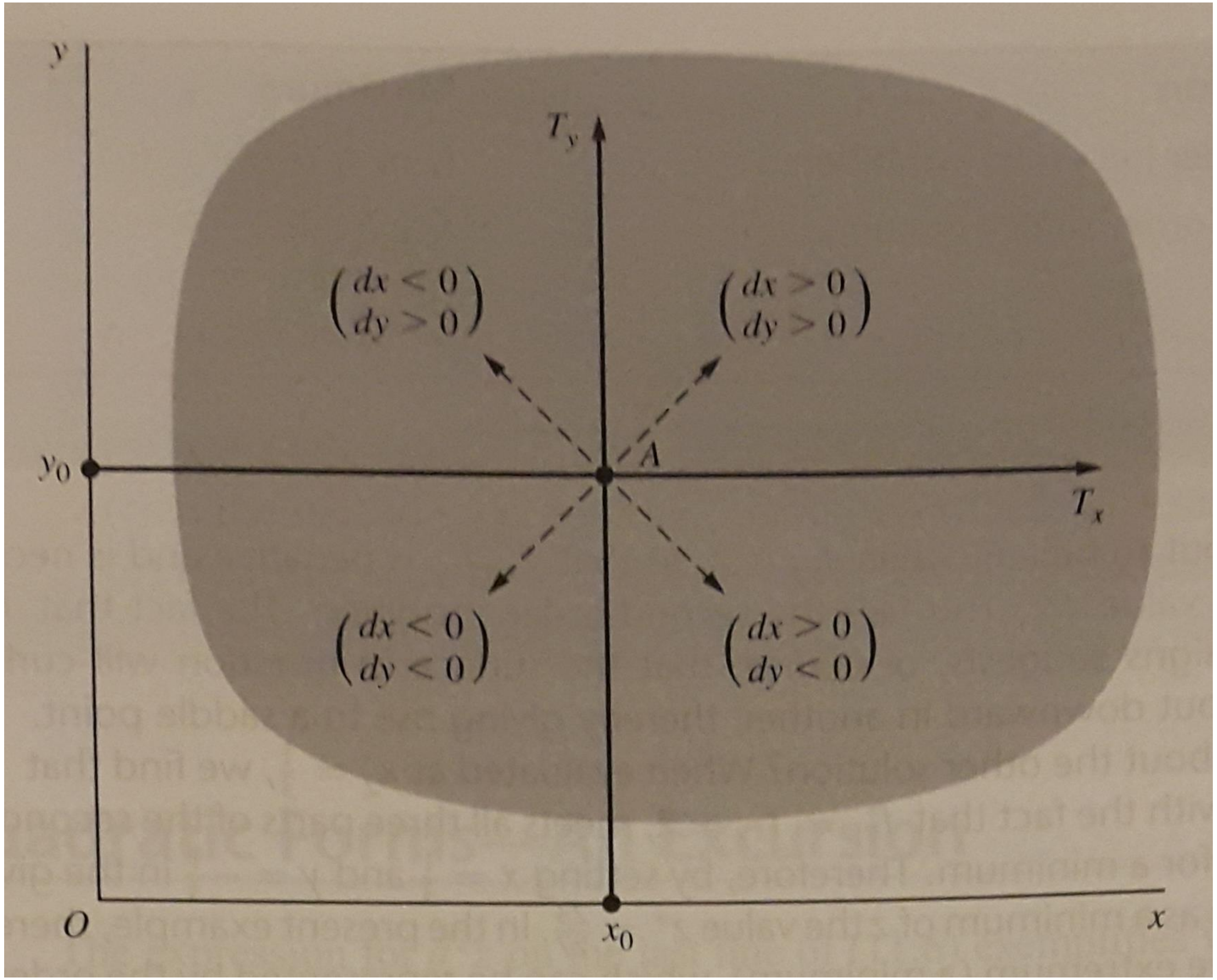
  And the *second-order sufficient* condition for a ***minimum*** of $z = f(x, y)$ is

  "$d^2z > 0$ for arbitrary values of dx and dy, not both zero."

  Note: we still have not dealt with the case of $d^2z = 0$.

Extreme Values of a Function of Two Variables

Note that the sign of d^2z depends not only on f_{xx} and f_{yy}, but also on the cross partial derivative f_{xy}. The role played by this latter partial derivative is to ensure that the surface in question will yield (two-dimensional) cross sections with the same shape (hill or valley) for all the possible directions.



Extreme Values of a Function of Two Variables

The second-order differential conditions can be translated into equivalent conditions on second-order derivatives. The actual translation requires a knowledge of quadratic forms (next section), but we may first state the main result here:

“For any values of dx and dy, not both zero,

$$d^2z\begin{cases} < 0 & \text{iff } f_{xx} < 0, f_{yy} < 0, \text{and } f_{xx}f_{yy} > f_{xy}^2. \quad (maximum) \\ > 0 & \text{iff } f_{xx} < 0, f_{yy} < 0, \text{and } f_{xx}f_{yy} > f_{xy}^2. \quad (minimum) \end{cases}$$

Example

- Find the extreme value(s) of $z = 8x^3 + 2xy - 3x^2 + y^2 + 1$.

FOC: $f_x = 24x^2 + 2y - 6x = 0$

$f_y = 2x + 2y = 0$

These equations yield two possible solutions:

$$x_1^* = 0, y_1^* = 0 \quad \text{and} \quad x_2^* = \frac{1}{3}, y_2^* = -\frac{1}{3}.$$

SOC: $f_{xx} = 48x - 6, f_{yy} = 2, f_{xy} = 2$

For $x_1^* = 0, y_1^* = 0$, $f_{xx} = -6$ ➔ cannot be an extremum

For $x_2^* = \frac{1}{3}, y_2^* = -\frac{1}{3}$, $f_{xx} = 10$, $f_{xx}f_{yy} > f_{xy}^2$ ➔ a relative minimum

In this example, there exists only one relative minimum, which is

$$(x^*, y^*, z^*) = \left(\frac{1}{3}, -\frac{1}{3}, \frac{23}{27}\right).$$

In Conclusion

Condition	Maximum	Minimum
First-order necessary	$f_x = f_y = 0$	$f_x = f_y = 0$
Second-order sufficient	$f_{xx}, f_{yy} < 0$ and $f_{xx} f_{yy} > 0$	$f_{xx}, f_{yy} > 0$ and $f_{xx} f_{yy} > 0$

Example

- Find the extreme value(s) of $z = x^2 + xy + 2y^2 + 3$

Example

- Find the extreme value(s) of $z = -x^2 - y^2 + 6x + 2y$