Student ID .... Name...section ....CR no. ......

PART A	
Question	Mark
1	/8
2	/~~12~~ 10
TOTAL PART A	/~~20~~ 18

PART A: (questions 1-2, ~~20~~ 18 marks in total)

1. John has inherited \$25,000 and put the money in three types of investment: a money market account, municipal bonds and a mutual fund. After one year, he received a total of \$1,620 in simple interest from the three investments. The money market paid 6% annually, the bonds paid 7% annually, and the mutually fund paid 8% annually. His investment in the bonds is \$6,000 more than that in the mutual funds. Find the amount John invested in each type of investment.
   (a) Write down the matrix equation ($\underline{\mathbf{A}}\,\underline{\mathbf{x}} = \underline{\mathbf{b}}$) representing the problem.
   (b) Use **Gauss-Jordan** method to obtain the inverse of the matrix $\underline{\mathbf{A}}$ in part (a)
   (c) Use the answer from (b) to determine the amount of each type of investment.

(8 marks)

$x_1 + x_2 + x_3 = 25{,}000$

$0.06x_1 + 0.07x_2 + 0.08x_3 = 1620$

$x_2 - x_3 = 6000$

a) $\begin{bmatrix} 1 & 1 & 1 \\ 0.06 & 0.07 & 0.08 \\ & 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 25{,}000 \\ 1620 \\ 6000 \end{bmatrix}$

b) $[A \mid I] = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0.06 & 0.07 & 0.08 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right]$

$R2 \times 100 = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 6 & 7 & 8 & 0 & 100 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right]$

$R2 - 6R1 = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -6 & 100 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right]$

$R3 - R2 = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -6 & 100 & 0 \\ 0 & 0 & -3 & 6 & -100 & 1 \end{array}\right]$

$R3 \div -3 = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -6 & 100 & 0 \\ 0 & 0 & 1 & -2 & \frac{100}{3} & -\frac{1}{3} \end{array}\right]$

$$R_2 - 2R_3 = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & \frac{100}{3} & \frac{2}{3} \\ 0 & 0 & 1 & -2 & \frac{100}{3} & -\frac{1}{3} \end{array}\right]$$

$$R_1 - R_3 = \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 3 & -\frac{100}{3} & \frac{1}{3} \\ 0 & 1 & 0 & -2 & \frac{100}{3} & \frac{2}{3} \\ 0 & 0 & 1 & -2 & \frac{100}{3} & -\frac{1}{3} \end{array}\right]$$

$$R_1 - R_2 = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 5 & -\frac{200}{3} & -\frac{1}{3} \\ 0 & 1 & 0 & -2 & \frac{100}{3} & \frac{2}{3} \\ 0 & 0 & 1 & -2 & \frac{100}{3} & -\frac{1}{3} \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 5 & -\frac{200}{3} & -\frac{1}{3} \\ -2 & \frac{100}{3} & \frac{2}{3} \\ -2 & \frac{100}{3} & -\frac{1}{3} \end{bmatrix}$$

C) $\underline{X} = \underline{A}^{-1}\underline{b}$

$$= \begin{bmatrix} 5 & -\frac{200}{3} & -\frac{1}{3} \\ -2 & \frac{100}{3} & \frac{2}{3} \\ -2 & \frac{100}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 25000 \\ 1620 \\ 6000 \end{bmatrix}$$

$$\underline{X} = \begin{bmatrix} 15000 \\ 8000 \\ 2000 \end{bmatrix}$$

2. Let $\underline{\mathbf{A}}^{-1} = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 2 & 0 \\ 0 & 2 & 5 \end{bmatrix}$ and $\underline{\mathbf{C}} = \begin{bmatrix} 1 & -1 & 0 \\ -4 & 2 & 3 \\ 0 & 1 & -2 \end{bmatrix}$

a) Find det $\underline{\mathbf{A}}^{-1}$ and adjoint of $\underline{\mathbf{A}}^{-1}$ (adj ($\underline{\mathbf{A}}^{-1}$))

b) Find $\underline{\mathbf{B}}^{-1}$ if $\left(\underline{\mathbf{A}}^{-1}(\underline{B}^{-1})^{T} - \underline{\mathbf{I}}\right)^{T} = \underline{\mathbf{C}}$.

c) Solve $\underline{\mathbf{A}}^{T}\underline{\mathbf{x}} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

~~12~~ 10 (marks)

a) $|A^{-1}| = \begin{vmatrix} 2 & 3 & 1 \\ 0 & 2 & 0 \\ 0 & 2 & 5 \end{vmatrix} = 2\begin{vmatrix} 2 & 0 \\ 2 & 5 \end{vmatrix} = 20$

Cofactor $C_{11} = +\begin{vmatrix} 2 & 0 \\ 2 & 5 \end{vmatrix} = 10$

$C_{12} = -\begin{vmatrix} 0 & 0 \\ 0 & 5 \end{vmatrix} = 0$

$C_{13} = +\begin{vmatrix} 0 & 2 \\ 0 & 2 \end{vmatrix} = 0$

$C_{21} = -\begin{vmatrix} 3 & 1 \\ 2 & 5 \end{vmatrix} = -13$

$C_{22} = +\begin{vmatrix} 2 & 1 \\ 0 & 5 \end{vmatrix} = 10$

$C_{23} = -\begin{vmatrix} 2 & 3 \\ 0 & 2 \end{vmatrix} = -4$

$C_{31} = +\begin{vmatrix} 3 & 1 \\ 2 & 0 \end{vmatrix} = -2$

$C_{32} = -\begin{vmatrix} 2 & 1 \\ 0 & 0 \end{vmatrix} = 0$

$C_{33} = +\begin{vmatrix} 2 & 3 \\ 0 & 2 \end{vmatrix} = 4$

$\text{Adj}(A^{-1}) = \begin{bmatrix} 10 & -13 & -2 \\ 0 & 10 & 0 \\ 0 & -4 & 4 \end{bmatrix}$

b)

$$\left(A^{-1}B^{-1^T} - I\right)^T = C$$

$$A^{-1}B^{-1^T} - I = C^T$$

$$A^{-1}B^{-1^T} = C^T + I$$

$$A\left(A^{-1}B^{-1^T}\right) = A\left(C^T + I\right)$$

$$B^{-1^T} = A\left(C^T + I\right)$$

$$B^{-1} = \left(A\left(C^T + I\right)\right)^T$$

$$= \left(C^T + I\right)^T A^T$$

$$= \left(C + I^T\right) A^T$$

$$= \left(C + I\right) A^T$$

$$B^{-1} = \begin{bmatrix} 2 & -1 & 0 \\ -4 & 3 & 3 \\ 0 & 1 & -1 \end{bmatrix} A^T$$

$$A = \frac{1}{|A|} \text{Adj } A^{-1} = \frac{1}{20}\begin{bmatrix} 10 & -13 & -2 \\ 0 & 10 & 0 \\ 0 & -4 & 4 \end{bmatrix}$$

$$A^T = \frac{1}{20}\begin{bmatrix} 10 & 0 & 0 \\ -13 & 10 & -4 \\ -2 & 0 & 4 \end{bmatrix}$$

$$B^{-1} = \frac{1}{20}\begin{bmatrix} 2 & -1 & 0 \\ -4 & 3 & 3 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} 10 & 0 & 0 \\ -13 & 10 & -4 \\ -2 & 0 & 4 \end{bmatrix}$$

$$= \frac{1}{20}\begin{bmatrix} 33 & -10 & 4 \\ -85 & 30 & 0 \\ -11 & 10 & -8 \end{bmatrix}$$

c)

$$A^T \underline{x} = \underline{b}$$

$$\underline{x} = A^{T^{-1}} \underline{b} = A^{-1^T} \underline{b}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 3 & 2 & 2 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 6 \end{bmatrix}$$

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PART B: (questions 3-4, ~~26~~ 23 marks in total)

PART B	
Question	Mark
3	~~/15~~ 13
4	~~/11~~ 10
TOTAL PART B	~~/26~~ 23

3. For a system of linear equation below where C and D are real constants

$$x_1 + 2x_2 + x_3 + 5x_5 + 2x_6 = 1 + x_4$$
$$2x_1 + 5x_2 + 2x_3 + x_6 = 2 - 3x_4 - 6x_5$$
$$x_1 + 3x_2 + Cx_3 + 4x_4 + x_5 = 5 - Dx_6$$

a) Write down the matrix equation ($\underline{\mathbf{A}}\underline{\mathbf{x}} = \underline{\mathbf{b}}$) and the augmented matrix that represents the above system of linear equations.

b) What are values of C and D that make this system <u>inconsistent</u>?

c) If $C = 1$, what are all possible values of D that make this system consistent?

d) If $D = 0$, choose a value of C that makes this system consistent. Also solve the given linear system with your specified value of C. **Write down the solution in vector form.**

e) If $C = 2$ and $D = 0$, is it possible for the linear system to have a unique solution? Give the reason to your answer.

(~~15~~ 13 marks)

a)
$$\begin{bmatrix} 1 & 2 & 1 & -1 & 5 & 2 \\ 2 & 5 & 2 & 3 & 6 & 1 \\ 1 & 3 & C & 4 & 1 & D \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

$$\left[\begin{array}{cccccc|c} 1 & 2 & 1 & -1 & 5 & 2 & 1 \\ 2 & 5 & 2 & 3 & 6 & 1 & 2 \\ 1 & 3 & C & 4 & 1 & D & 5 \end{array}\right]$$

b)
$R_2 - 2R_1$, $R_3 - R_1$
$$\left[\begin{array}{cccccc|c} 1 & 2 & 1 & -1 & 5 & 2 & 1 \\ 0 & 1 & 0 & 5 & -4 & -3 & 0 \\ 0 & 1 & C-1 & 5 & -4 & D-2 & 4 \end{array}\right]$$

$R_3 - R_2$
$$\left[\begin{array}{cccccc|c} 1 & 2 & 1 & -1 & 5 & 2 & 1 \\ 0 & 1 & 0 & 5 & -4 & -3 & 0 \\ 0 & 0 & C-1 & 0 & 0 & D+1 & 4 \end{array}\right]$$

Inconsistent $C - 1 = 0$ and $D + 1 = 0$
$C = 1$ and $D = -1$

c)

c) $c_{\le 1}$ $\left[\begin{array}{cccccc|c} & & & & & & \\ 0 & 0 & 0 & 0 & 0 & D+1 & 4 \end{array}\right]$

Consistent $D \neq -1$

d) $D=0$
Let $C=1$

$$\left[\begin{array}{cccccc|c} 1 & 2 & 1 & -1 & 5 & 2 & 1 \\ 0 & 1 & 0 & 5 & -4 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right]$$

↑ ↑ ↑ Free columns

Let $x_3 = c_3$
$x_4 = c_4$
$x_5 = c_5$

R3 : $x_6 = 4$

R2 : $x_2 = -3x_6 + 4x_5 - 5x_4$

$= -12 + 4c_5 - 5c_4$

R1 : $x_1 = 1 - 2x_6 - 5x_5 + x_4 - x_3 - 2x_2$

$= 1 - 8 - 5c_5 + c_4 - c_3 - 2(-12 + 4c_5 - 5c_4)$

$= 17 - 13c_5 + 11c_4 - c_3$

$$\underline{x} = \begin{bmatrix} 17 \\ -12 \\ 0 \\ 0 \\ 0 \\ 4 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_4 \begin{bmatrix} 11 \\ -5 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_5 \begin{bmatrix} -13 \\ 4 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad c_3, c_4, c_5 \in \mathbb{R}$$

e) $C=2, D=0$

$$\left[\begin{array}{cccccc|c} & & & & & & \\ 0 & 0 & 1 & 0 & 0 & 1 & 4 \end{array}\right]$$

No since there are free columns

OR Rank = 3, n = 6

4. Given $\mathbf{A} = \begin{bmatrix} 2 & 0 & 1 & 0 \\ 1 & 3 & 2 & 1 \\ 1 & 2 & 0 & 4 \\ 1 & 2 & 0 & 5 \end{bmatrix}$,

a) Use **cofactor expansion method together with row operations** to determine $|\mathbf{A}|$ and $|\mathbf{A}^{-1}|$

b) Let $\mathbf{B} = \begin{bmatrix} 0 & a & 2 \\ 0 & 0 & 9 \\ a & 1 & 2 \end{bmatrix}$; $a > 0$, if det $\mathbf{B} = 81$, find the value of a

c) Determine det $\mathbf{C}$ if $(\det \mathbf{B})^{\frac{1}{2}} \det(\mathbf{A}^2 \mathbf{A}^T) = \dfrac{\det(\mathbf{C}^3)}{\det(\mathbf{A}^{-1})}$

d) Using **Cramer's rule** to solve for $\underline{\mathbf{Bx}} = \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix}$

10

(~~11~~ marks)

a) $|A| = \begin{vmatrix} 2 & 0 & 1 & 0 \\ 1 & 3 & 2 & 1 \\ 1 & 2 & 0 & 4 \\ 1 & 2 & 0 & 5 \end{vmatrix}$

$R2 - R3$, $R4 - R3$ $= \begin{vmatrix} 2 & 0 & 1 & 0 \\ 0 & 1 & 2 & -3 \\ 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 1 \end{vmatrix}$

Expansion across R4 $\quad |A| = +1 \begin{vmatrix} 2 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{vmatrix}$

$= R1 - 2R3 \begin{vmatrix} 0 & -4 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{vmatrix}$

$= 1 \begin{vmatrix} -4 & 1 \\ 1 & 2 \end{vmatrix} = -9$

b) $|B| = a \begin{vmatrix} a & 2 \\ 0 & 9 \end{vmatrix} = a(9a) = 9a^2$

$81 = 9a^2$

$a = \pm 3$, $a > 0$ so $a = 3$

c) $$|B|^{\frac{1}{2}}\,|A^2|\,|A^T| = \frac{|C^3|}{|A^{-1}|}$$

$$9(-9)^2(-9) = |C|^3|A|$$

$$-9^4 = |C|^3(-9)$$

$$|C|^3 = 9^3$$

$$|C| = 9$$

d) $$BX = \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix} \quad ; \quad B = \begin{bmatrix} 0 & 3 & 2 \\ 0 & 0 & 9 \\ 3 & 1 & 2 \end{bmatrix}$$

$$X_1 = \frac{\begin{vmatrix} -3 & 3 & 2 \\ 0 & 0 & 9 \\ 2 & 1 & 2 \end{vmatrix}}{|B|} = \frac{-9\begin{vmatrix} -3 & 3 \\ 2 & 1 \end{vmatrix}}{81} = \frac{-9(-9)}{81} = 1$$

$$X_2 = \frac{\begin{vmatrix} 0 & -3 & 2 \\ 0 & 0 & 9 \\ 3 & 2 & 2 \end{vmatrix}}{81} = \frac{+3\begin{vmatrix} -3 & 2 \\ 0 & 9 \end{vmatrix}}{81} = \frac{3(-27)}{81} = -1$$

$$X_3 = \frac{\begin{vmatrix} 0 & 3 & -3 \\ 0 & 0 & 0 \\ 3 & 1 & 2 \end{vmatrix}}{81} = \frac{+3\begin{vmatrix} 3 & -3 \\ 0 & 0 \end{vmatrix}}{81} = 0$$

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PART D: (question 8-9, 29 marks in total)

PART D	
Question	Mark
~~8~~ 9	3/4
~~9~~ 10	/5
~~10~~ 11	/25
TOTAL PART D	/29

9. ~~8~~. If A is 4×4 matrix such that $\underline{A}\begin{bmatrix}0\\3\\1\\2\end{bmatrix}=\begin{bmatrix}1\\0\\0\\-1\end{bmatrix}$ and $\underline{A}\begin{bmatrix}1/2\\1\\1\\5/2\end{bmatrix}=\begin{bmatrix}-2\\0\\1\\1/2\end{bmatrix}$,

what is the product $\underline{A}\begin{bmatrix}1\\11\\5\\11\end{bmatrix}$? (Hints: you can obtain $\underline{A}\begin{bmatrix}1\\11\\5\\11\end{bmatrix}$ without finding the matrix $\underline{A}$)

(~~4~~ 3 marks)

$$\begin{bmatrix}1\\11\\5\\11\end{bmatrix} = m\begin{bmatrix}0\\3\\1\\2\end{bmatrix} + n\begin{bmatrix}1/2\\1\\1\\5/2\end{bmatrix}$$

R1: $1 = 0m + \frac{1}{2}n \rightarrow n = 2$

R2: $11 = 3m + n \rightarrow m = 3$

R3: $5 = m + n$ ✓

R4: $11 = 2m + \frac{5}{2}n$ ✓

$$A\begin{bmatrix}1\\11\\5\\11\end{bmatrix} = m\begin{bmatrix}1\\0\\0\\-1\end{bmatrix} + n\begin{bmatrix}-2\\0\\1\\1/2\end{bmatrix}$$

$$= \begin{bmatrix}3\\0\\0\\-3\end{bmatrix} + \begin{bmatrix}-4\\0\\2\\1\end{bmatrix}$$

$$= \begin{bmatrix}-1\\0\\2\\-2\end{bmatrix}$$

Q10

State **True or False** and give your **reason** to your answer briefly.

1.1 Every matrix is row equivalent to a unique matrix in reduced echelon form. TRUE

1.2 If a system of linear equations has no free variables, then it has a unique solution. False. It is not necessary. If a system has number of unknowns more than number so f equations, there is a possibility for no solution case as well.

1.3 If $\underline{\mathbf{A}}$ is an 4x6 matrix and the equation $\underline{\mathbf{A}}\underline{\mathbf{x}} = \underline{\mathbf{b}}$ is consistent for every $\underline{\mathbf{b}}$, then $\underline{\mathbf{A}}$ has 6 pivot columns.

False since the maximum number of pivots that the matrix A can have are 4.

1.4 If $\underline{\mathbf{A}}$ is an *mxn* matrix, if the equation $\underline{\mathbf{A}}\underline{\mathbf{x}} = \underline{\mathbf{b}}$ has at least two different solutions, and if the equation $\underline{\mathbf{A}}\underline{\mathbf{x}} = \underline{\mathbf{c}}$ is consistent, then the equation $\underline{\mathbf{A}}\underline{\mathbf{x}} = \underline{\mathbf{c}}$ has many solutions.

True.

1.5 An economist solves a homogeneous system of 50 equations in 54 variables and finds that exactly 4 of the unknowns are free variables. The economist can be certain that for any associated nonhomogeneous system (with the same coefficients) always have solutions?

TRUE. Since there are 50 pivots which fill up every row (i.e. there are pivots in all rows) hence no zero row in echelon forma of A. This implies there is no possibility for no solution case.