FN211 Financial Mathematics and Statistics

Final Exercise

Semester 1/2020

1. Risk and Return

Problem 1: A stock's return has the following distribution:

Demand for the Company's Products	Probability of This Demand Occurring	Rate of Return If This Demand Occurs (%)
Weak	0.1	−50%
Below average	0.2	-5
Average	0.4	16
Above average	0.2	25
Strong	0.1	60

Calculate the stock's expected return, standard deviation, and coefficient of variation.

Sol:

Expected return	Variance	Standard Deviation	Coefficient of Variation
11.4	712.44	26.69	2.3414

Problem 2: The market and Stock J have the following probability distributions:

Probability	R_M	R_J
0.3	15%	20%
0.4	9	5
0.3	18	12

a. Calculate the expected rates of return for the market and Stock J.
b. Calculate the standard deviations for the market and Stock J.
c. Calculate the coefficients of variation for the market and Stock J.

Sol:

a. Expected rates of return for the market = 13.5%
   Expected rates of return for Stock J = 11.6%
b. Standard deviations for the market = 14.85
   Standard deviations for Stock J = 38.64
c. Coefficients of variation for the market = 0.28545
   Coefficients of variation for the market = 0.535871

Problem 3: Consider the following two scenarios for the economy, and the returns in each scenario for the market portfolio, an aggressive stock A, and a defensive stock D.

Scenario	Market Aggressive	Rate of Return Stock A	Defensive Stock D
Bust	–8%	–10%	–6%
Boom	32	38	24

a. If each scenario is equally likely, find the expected rate of return on the market portfolio and on each stock.
b. Which stock seems to be a better buy based on your answers to (a)?

Sol:

a. Expected return on the market portfolio = 12%
   Expected return on Stock A = 14%
   Expected return on Stock D = 9%

b. Standard Deviation of Stock A = 24
   Standard Deviation of Stock D = 15
   Coefficient of variation of Stock A = 1.7143
   Coefficient of variation of Stock D = 1.6667
   Therefore, Stock D seems to a better buy stock.

Problem 4: Consider the possible rates of return that you might earn next year on a $50,000 investment in stock A or on a $50,000 investment in stock B, depending upon the states of the economy: recession, normal, and prosperity.

For stock A:

State of Economy	Return (R_i)	Probability (P_i)
Recession	-5%	0.2
Normal	20%	0.6
Prosperity	40%	0.2

For stock B:

State of Economy	Return (R_i)	Probability (P_i)
Recession	10%	0.2
Normal	15%	0.6
Prosperity	20%	0.2

Calculate the stocks' expected return, variance, and standard deviation. Then, identify which stock is more attractive to invest?

Sol:

Then the expected rate of return ($\bar{r}$) for stock A is computed as follows:

$$\bar{r} = \sum_{i=1}^{n} r_i p_i = (-5\%)(0.2) + (20\%)(0.6) + (40\%)(0.2) = 19\%$$

Stock B's expected rate of return is:

$$\bar{r} = (10\%)(0.2) + (15\%)(0.6) + 20\%(0.2) = 15\%$$

		Step 1	Step 2		Step 3
Return (r_i) (%)	**Probability (p_i)**	**$r_i p$ (%)**	**$(r_i - \bar{r})$(%)**	**$(r_i - \bar{r})^2$**	**$(r_i - \bar{r})^2 p_i$(%)**
−5	0.2	−1	−24	576	115.2
20	0.6	12	1	1	0.6
40	0.2	8	21	441	88.2
		$\bar{r} = 19$			$\sigma^2 = 204$

For stock B:

Return (r_i) (%)	Probability (p_i)	Step 1 $r_i p$ (%)	Step 2 $(r_i - \bar{r})$(%)	$(r_i - \bar{r})^2$	Step 3 $(r_i - \bar{r})^2 p_i$(%)
10	0.2	2	−5	25	5
15	0.6	9	0	0	0
20	0.2	4	5	25	5
		$\bar{r} = 15$			$\sigma^2 = 10$

Problem 5: Assuming the following probability distribution of the possible returns, calculate the expected return and the standard deviation of the returns.

Probability (P_i)	*Return (R_i)*
0.1	-20%
0.2	5%
0.3	10%
0.4	25%

Sol:

$$r = \sum r_i p_i$$

$$\sigma = \sqrt{\sum (r_i - \bar{r})^2 p_i}$$

It is convenient to set up the following table:

r_i (%)	p_i	$r_i p_i$ (%)	$(r_i - \bar{r})$(%)	$(r_i - \bar{r})^2$	$(r_i - \bar{r})^2 p_i$ (%)
−20	0.1	−2	−32	1,024	102.4
5	0.2	1	−7	49	9.8
10	0.3	3	−2	4	1.2
25	0.4	10	13	169	67.6
		$\bar{r} = 12$			$\sigma^2 = 181$

Since $\sigma^2 = 181$, $\sigma = \sqrt{181} = 13.45\%$.

Problem 6: Stocks A and B have the following probability distributions of possible future returns:

Probability (pi)	*A (%)*	*B (%)*
0.1	-15	-20
0.2	0	10
0.4	5	20
0.2	10	30
0.1	25	50

(*a*) Calculate the expected rate of return for each stock and the standard deviation of returns for each stock, (*b*) Calculate the coefficient of variation, (*c*) Which stock is less risky? Explain.

Sol:

(a) For stock A:

r_i (%)	p_i	r_ip_i (%)	$(r_i-\bar{r})$(%)	$(r_i-\bar{r})^2$	$(r_i-\bar{r})^2p_i$ (%)
−15	0.1	−1.5	−20	400	40
0	0.2	0	−5	25	5
5	0.4	2	0	0	0
10	0.2	2	5	25	5
25	0.1	2.5	20	400	40
		$\bar{r} = 5.0$			$\sigma^2 = 90$

Since $\sigma^2 = 90$, $\sigma = \sqrt{90} = 9.5\%$.

For stock B:

r_i (%)	p_i	r_ip_i (%)	$(r_i-\bar{r})$(%)	$(r_i-\bar{r})^2$	$(r_i-\bar{r})^2p_i$ (%)
−20	0.1	−2	−39	1,521	152.1
10	0.2	2	−9	81	16.2
20	0.4	8	1	1	0.4
30	0.2	6	11	121	24.2
50	0.1	5	31	961	96.1
		$\bar{r} = 19$			$\sigma^2 = 289$

Since $\sigma^2 = 289$, $\sigma = \sqrt{289} = 17\%$.

(b) The coefficient of variation is $\sigma/\bar{r}$. Thus, for stock A:

$$\frac{9.5\%}{5\%} = 1.9$$

For stock B:

$$\frac{17.0\%}{19\%} = 0.89$$

(c) Stock B is less risky than stock A since the coefficient of variation (a measure of relative risk) is smaller for stock B.

Problem 7: Ken Parker must decide which of two securities is best for him. By using probability estimates, he computed the following statistics:

Statistic	Security X	Security Y
Expected return	12%	8%
Standard deviation	20%	10%

(a) Compute the coefficient of variation for each security, and (b) explain why the standard deviation and coefficient of variation give different rankings of risk. Which method is superior and why?

Sol:

(a) For the X coefficient of variation $(\sigma / \bar{r})$ is 20/12=1.67. For Y it is 10/8=1.25.

(b) Unlike the standard deviation, the coefficient of variation considers the standard deviation of securities relative to their average return. The coefficient of variation is therefore the more useful measure of relative risk. The lower the coefficient of variation, the less risky the security relative to the expected return. Thus, in this problem, security Y is relatively less risky than security X.

Problem 8: A portfolio consists of assets A and B. Asset A makes up one-third of the portfolio and has an expected return of 18 percent. Asset B makes up the other two-thirds of the portfolio and is expected to earn 9 percent. What are the expected return and risk on this portfolio?

Sol:

Portfolio Expected Return = (1/3)x18% + (2/3)x9% = 12%

Problem 9: The securities of firms A and B have the expected return and standard deviations given below; the expected correlation between the two stocks (ρ_{AB}) is 0.1.

Stock	*Expected Return*	*Standard Deviation*
A	14%	20%
B	9%	30%

Compute the return and risk for each of the following portfolios: (*a*) 100 percent A; (*b*) 100 percent B; (*c*) 60 percent A– 40 percent B; and (*d*) 50 percent A–50 percent B.

Additional Question: assume the expected correlation between the two stocks (ρ_{AB}) = -1.0. How does this have an effect on previous answers?

Sol:

(*a*) 100 percent A: $\bar{r} = 14\%;\ \sigma = 20\%;\ \sigma/\bar{r} = \frac{20}{14} = 1.43$

(*b*) 100 percent B: $\bar{r} = 9\%;\ \sigma = 30\%;\ \sigma/\bar{r} = \frac{30}{9} = 3.33$

(*c*) 60 percent A – 40 percent B:

$$r_p = w_A r_A + w_B r_B = (0.6)(14\%) + (0.4)(9\%) = 12\%$$

$$\sigma_p = \sqrt{w_A^2\sigma_B^2 + w_B^2\sigma_B^2 + 2w_A w_B \rho_{AB}\sigma_A\sigma_B}$$

$$= \sqrt{(0.6)^2(0.2)^2 + (0.4)^2(0.3)^2 + 2(0.6)(0.4)\rho_{AB}(0.2)(0.3)}$$

$$= \sqrt{0.0144 + 0.0144 + 0.0288\rho_{AB}} = \sqrt{0.0288 + 0.0288(0.1)} = \sqrt{0.03168} = 0.1780 = 17.8\%$$

(*d*) 50 percent A – 50 percent B:

$$r_p = (0.5)(14\%) + (0.5)(9\%) = 11.5\%$$

$$\sigma_p = \sqrt{(0.5)^2(0.2)^2 + (0.5)^2(0.3)^2 + 2(0.5)(0.5)\rho_{AB}(0.2)(0.3)}$$

$$= \sqrt{0.01 + 0.0225 + 0.03\rho_{AB}} = \sqrt{0.0325 + 0.03\rho_{AB}}$$

$$= \sqrt{0.0325 + 0.03(0.1)} = \sqrt{0.0355} = 0.1884 = 18.84\%$$

Problem 10: What is the standard deviation of the following two-stock portfolio?

	Weighting	Standard Deviation	Correlation
Stock A	60%	0.11	0.7
Stock B	40%	0.14	

Sol

portfolio risk is:

$$\sigma_p = \sqrt{w_A^2\sigma_A^2 + w_B^2\sigma_B^2 + 2\rho_{AB}w_A w_B \sigma_A \sigma_B}$$

$$= \left[(0.6)^2(0.11)^2 + (0.4)^2(0.14)^2 + 2(0.7)(0.6)(0.4)(0.11)(0.14)\right]^{1/2} = (126.664)^{1/2} = 0.1125 = 11.25\%$$

Problem 11: Consider the following information:

State of Economy	Probability of State of Economy	Rate of Return if State Occurs		
		Stock A	**Stock B**	**Stock C**
Boom	.20	.30	.45	.33
Good	.40	.12	.10	.15
Poor	.30	.01	-.15	-.05
Bust	.10	-.06	-.30	-.09

a. Your portfolio is originally invested 40 percent in A and 60 percent in C. Calculate the expected return, variance, and standard deviation of the portfolio?

b. If you are thinking about adding stock B into your portfolio with 50 percent fraction, how will this affect the portfolio's expected return, variance, and standard deviation?

(Assuming coefficient of correlation of all cases is equal to 1)

Sol:

Expected Return			Variance		
Stock A	**Stock B**	**Stock C**	**Stock A**	**Stock B**	**Stock C**
0.105	0.055	0.102	0.013125	0.039909	0.009098

a. Portfolio expected return = 10.32%, Variance = 0.010621,
Standard Deviation = 0.103057

b. Portfolio expected return = 7.91%, Variance = 0.022927,
Standard Deviation = 0.151415

2. Basic Matrices

Problem 1: Nutrition A four-ounce serving of Campbell's® Pork & Beans contains 5 grams of protein and 21 grams of carbohydrates. A typical slice of "lite" rye bread contains 4 grams of protein and 12 grams of carbohydrates.

a. I am planning a meal of "beans-on-toast" and I want it to supply 20 grams of protein and 80 grams of carbohydrates. How should I prepare my meal?
b. If I require A grams of protein and B grams of carbohydrates, give a formula that tells me how many slices of bread and how many servings of Pork & Beans to use.

Problem 2: Nutrition According to the nutritional information on a package of Honey Nut Cheerios® brand cereal, each 1-ounce serving of Cheerios contains 3 grams protein and 24 grams carbohydrates. Each half-cup serving of enriched skim milk contains 4 grams protein and 6 grams carbohydrates.

a. I am planning a meal of cereal and milk and I want it to supply 26 grams of protein, and 78 grams of carbohy- drates. How should I prepare my meal?
b. If I require A grams of protein and B grams of carbohy- drates, give a formula that tells me how many servings of milk and Cheerios to use.

Problem 3: Resource Allocation You manage an ice cream factory that makes three flavors: Creamy Vanilla, Continental Mocha, and Succulent Strawberry. Into each batch of Creamy Vanilla go two eggs, one cup of milk, and two cups of cream. Into each batch of Continental Mocha go one egg, one cup of milk, and two cups of cream. Into each batch of Succulent Strawberry go one egg, two cups of milk, and one cup of cream. Your stocks of eggs, milk, and cream vary from day to day. How many batches of each flavor should you make in order to use up all of your ingredients if you have the following amounts in stock?

a. 350 eggs, 350 cups of milk, and 400 cups of cream
b. 400 eggs, 500 cups of milk, and 400 cups of cream
c. A eggs, B cups of milk, and C cups of cream

Problem 4: Resource Allocation The Arctic Juice Company makes three juice blends: PineOrange, using 2 quarts of pineapple juice and 2 quarts of orange juice per gallon; PineKiwi, using 3 quarts of pineapple juice and 1 quart of kiwi juice per gallon; and OrangeKiwi, using 3 quarts of orange juice and 1 quart of kiwi juice per gallon. The amount of each kind of juice the company has on hand varies from day to day. How many gallons of each blend can it make on a day with the following stocks?

a. 800 quarts of pineapple juice, 650 quarts of orange juice, 350 quarts of kiwi juice.
b. 650 quarts of pineapple juice, 800 quarts of orange juice, 350 quarts of kiwi juice.
c. A quarts of pineapple juice, B quarts of orange juice, C quarts of kiwi juice.

Problem 5: *Investing in Municipal Bond Funds Exercises a and b are based on the following data on three tax-exempt municipal bond funds.*

	2003 Yield
PNF (Pimco NY)	6%
FDMMX (Fidelity Spartan Mass)	5%
FFLIX (Fidelity Spartan Florida)	7%

a. You invested a total of $9000 in the three funds at the be- ginning of 2003, including an equal amount in FDMMX and FFLIX. Your 2003 yield for the year from the first two funds amounted to $400. How much did you invest in each of the three funds?
b. You invested a total of $6000 in the three funds at the be- ginning of 2003, including an equal amount in PNF and FDMMX. Your total yields for 2003 amounted to $360. How much did you invest in each of the three funds?

Problem 6: *Investing in Stocks Exercises a and b are based on the following data on three computer stocks.*

	Price	Dividend Yield
APPL (Apple Computer)	$25	0.6%
HPQ (Hewlett Packard)	25	1.2
DELL	35	0

a. You invested a total of $5800 in Apple, Hewlett Packard, and Dell shares at the above prices, and expected to earn $21 in annual dividends. If you purchased a total of 200 shares, how many shares of each stock did you purchase?
b. You invested a total of $6750 in Apple, Hewlett Packard, and Dell shares at the above prices, and expected to earn $45 in annual dividends. If you purchased a total of 250 shares, how many shares of each stock did you purchase?

Problem 7: Setting up the technology matrix.

a. Each unit of television news requires 0.2 units of television news and 0.5 units of radio news. Each unit of radio news re- quires 0.1 units of television news and no radio news. With Sector 1 as television news and Sector 2 as radio news, set up the technology matrix A.
b. Production of one unit of cologne requires no cologne and 0.5 units of perfume. Into one unit of perfume go 0.1 units of cologne and 0.3 units of perfume. With Sector 1 as cologne and Sector 2 as perfume, set up the technology matrix A.

Problem 8: Finding changes in production.

a. Given $A = \begin{bmatrix} 0.1 & 0.4 \\ 0.2 & 0.5 \end{bmatrix}$, find the changes in production required to meet an increase in demand of 50 units of Sector 1 products and 30 units of Sector 2 products.
b. Given $A = \begin{bmatrix} 0.5 & 0.4 \\ 0 & 0.5 \end{bmatrix}$, find the changes in production required to meet an increase in demand of 20 units of Sector 1 products and 10 units of Sector 2 products.

c. Let $(I - A)^{-1} = \begin{bmatrix} 1.5 & 0.1 & 0 \\ 0.2 & 1.2 & 0.1 \\ 0.1 & 0.7 & 1.6 \end{bmatrix}$, external demand for the products in Sector 1 increases by 1 unit. By how many units should each sector increase production? What do the columns of the matrix *(I − A)−1* tell you?

d. Let $(I - A)^{-1} = \begin{bmatrix} 1.5 & 0.1 & 0 \\ 0.1 & 1.1 & 0.1 \\ 0 & 0 & 1.3 \end{bmatrix}$, and assume that the external demand for the products in each of the sectors increases by 1 unit. By how many units should each sector increase production?

Problem 9: Campus Food The two campus cafeterias, the Main Dining Room and Bits & Bytes, typically use each other's food in doing business on campus. One weekend, the input-output table was as follows.

To	**Main DR**	**Bits & Bytes**
From **Main DR**	\$10,000	\$20,000
Bits & Bytes	5000	0
Total Output	50,000	40,000

Given that the demand for food on campus last weekend was \$45,000 from the Main Dining Room and \$30,000 from Bits & Bytes, how much did the two cafeterias have to produce to meet the demand last weekend?

Problem 10: Plagiarism Two student groups at Enormous State University, the Choral Society and the Football Club, maintain files of term papers that they write and offer to students for re- search purposes. Some of these papers they use themselves in generating more papers. In order to avoid suspicion of plagiarism by faculty members (who seem to have astute memories), each paper is given to students or used by the clubs only once (no copies are kept). The number of papers that were used in the production of new papers last year is shown in the following input-output table:

To	**Choral Soc.**	**Football Club**
From **Choral Soc.**	20	10
Football Club	10	30
Total Output	100	200

Given that 270 Choral Society papers and 810 Football Club papers will be used by students outside of these two clubs next year, how many new papers do the two clubs need to write?

3. Regression Analyses

Problem 1: One data set contains data on births to women in the United States. Two variables of interest are the dependent variable, infant birth weight in ounces (bwght), and an explanatory variable, average number of cigarettes the mother smoked per day during pregnancy (cigs). The following simple regression was estimated using data on n = 1,388 births:

$$\widehat{bwght} = 119.77 - 0.514\ cigs$$

a. What is the predicted birth weight when cigs □ 0? What about when cigs = 20 (one pack per day)? Comment on the difference.
b. Does this simple regression necessarily capture a causal relationship between the child's birth weight and the mother's smoking habits? Explain.
c. To predict a birth weight of 125 ounces, what would cigs have to be? Comment.
d. The proportion of women in the sample who do not smoke while pregnant is about 0.85. Does this help reconcile your finding from part (iii)?

Problem 2: In the linear consumption function

$$\widehat{cons} = \hat{\beta}_0 + \hat{\beta}_1 inc,$$

the (estimated) marginal propensity to consume (MPC) out of income is simply the slope, while the average propensity to consume (APC) is $\widehat{cons/inc} = \hat{\beta}_0/inc + \hat{\beta}_1$. Using 101 observations for 100 families on annual income and consumption (both measured in dollars), the following equation is obtained:

$$\widehat{cons} = -124.84 + 0.853\ inc$$

$$n = 100,\ R^2 = 0.692.$$

a. Interpret the intercept in this equation, and comment on its sign and magnitude.
b. What is the predicted consumption when family income is $30,000?
c. With inc on the x-axis, draw a graph of the estimated MPC and APC.

Problem 3: Using data from 1988 for houses sold in Andover, Massachusetts, from Kiel and McClain (1995), the following equation relates housing price (price) to the distance from a recently built garbage incinerator (dist):

$$\widehat{\log(price)} = 9.40 + 0.312\log(dist)$$

$$n = 135,\ R^2 = 0.162.$$

a. Interpret the coefficient on log(dist). Is the sign of this estimate what you expect it to be?
b. Do you think simple regression provides an unbiased estimator of the ceteris paribus elasticity of price with respect to dist? (Think about the city's decision on where to put the incinerator.)
c. What other factors about a house affect its price? Might these be correlated with distance from the incinerator?

Problem 4: In an effort to produce a formula for estimating the age of large free-standing oak trees non-invasively, the girth x (in inches) five feet off the ground of 15 such trees of known age y (in years) was measured. The sample data are summarized by the following information.

$$n = 15 \quad \sum x = 3368 \quad \sum y = 6496$$

$$\sum xy = 1,933,219 \quad 74 \leq x \leq 395$$

$$\sum x^2 = 917,780 \quad \sum y^2 = 4,260,666$$

Compute the linear regression equation for these sample data and interpret its meaning in the context of the problem.

Problem 5: Construction standards specify the strength of concrete 28 days after it is poured. For 30 samples of various types of concrete the strength x after 3 days and the strength y after 28 days (both in hundreds of pounds per square inch) were measured. The sample data are summarized by the following information.

$$n = 30 \quad \sum x = 501.6 \quad \sum y = 1338.8$$

$$\sum xy = 23,246.55 \quad 11 \leq x \leq 22$$

$$\sum x^2 = 8724.74 \quad \sum y^2 = 61,980.14$$

Compute the linear regression equation for these sample data and interpret its meaning in the context of the problem.

Problem 6: Power-generating facilities used forecasts of temperature to forecast energy demand. The average temperature x (degrees Fahrenheit) and the day's energy demand y (million watt-hours) were recorded on 40 randomly selected winter days in the region served by a power company. The sample data are summarized by the following information.

$$n = 40 \quad \sum x = 2000 \quad \sum y = 2969$$

$$\sum xy = 143,042 \quad 40 \leq x \leq 60$$

$$\sum x^2 = 101,340 \quad \sum y^2 = 243,027$$

Compute the linear regression equation for these sample data and interpret its meaning in the context of the problem.

4. Optimization

Problem 1: A farmer has 600 m of fencing with which he plans to enclose a rectangular pasture adjacent to a long existing wall. She plans to build one fence parallel to the wall, two to form the ends of the enclosure, and a fourth (parallel to the ends of the enclosure) to divide it equally. What is the maximum area that can be enclosed?

Problem 2: A small island is 2 km off shore in a large lake. A woman on the island can row her boat 10 km/h and can run at a speed of 20 km/h. If she rows to the closest point of the straight shore, she will land 6 km from a village on the shore. Where should she land to reach the village most quickly by a combination of running and rowing?

Problem 3: A company has started selling a new type of smartphone at the price of $\$110 - 0.05x$, where x is the number of smartphones manufactured per day. The parts for each smartphone cost $50 and the labor and overhead for running the plant cost $6000 per day. How many smartphones should the company manufacture and sell per day to maximize profit?

Problem 4: The U.S. Postal Service states that the girth plus the length of Standard Post Package must not exceed 130". Given a rectangular box, the "length" is the longest side, and the "girth" is twice the sum of the width and the height. Given a rectangular box where the width and height are equal, what are the dimensions of the box that give the maximum volume subject to the constraint of the size of a Standard Post Package?