

## Solution: Assignment 4

1. Let  $a_1, a_2, a_3$ , and  $a_4$  be some constant real numbers such that

$$a_1 < a_2 < a_3 < a_4.$$

Determine the solution set for each of the following inequalities in terms of  $a_1, a_2, a_3$ , and  $a_4$ .

- (a)  $(x - a_1)(x - a_2)(x - a_3)(x - a_4) \geq 0$   
 (b)  $(x - a_1)^2(x - a_2)(x - a_3)^2(x - a_4)^3 \geq 0$

**Solution** To solve  $(x - a_1)(x - a_2)(x - a_3)(x - a_4) \geq 0$ , since  $a_1 < a_2 < a_3 < a_4$ , we will consider sub-intervals divided using the points  $x = a_1, a_2, a_3, a_4$  as shown in the table.

|                                        | $x \in (-\infty, a_1)$ | $x \in (a_1, a_2)$ | $x \in (a_2, a_3)$ | $x \in (a_3, a_4)$ | $x \in (a_4, \infty)$ |
|----------------------------------------|------------------------|--------------------|--------------------|--------------------|-----------------------|
| $x - a_1$                              | —                      | +                  | +                  | +                  | +                     |
| $x - a_2$                              | —                      | —                  | +                  | +                  | +                     |
| $x - a_3$                              | —                      | —                  | —                  | +                  | +                     |
| $x - a_4$                              | —                      | —                  | —                  | —                  | +                     |
| $(x - a_1)(x - a_2)(x - a_3)(x - a_4)$ |                        | +                  | —                  | +                  | —                     |

That is, the solution set is given by  $(-\infty, a_1] \cup [a_2, a_3] \cup [a_4, \infty)$ . ■

(b)

To solve  $(x - a_1)^2(x - a_2)(x - a_3)^2(x - a_4)^3 \geq 0$ , since  $a_1 < a_2 < a_3 < a_4$  and  $(x - a_1)^2, (x - a_3)^2 \geq 0$ , we will consider  $(x - a_2)(x - a_4)^3 \geq 0$  and use sub-intervals divided by the points  $x = a_2, a_4$  together with the set  $\{a_1, a_2, a_3, a_4\}$  as shown in the table.

|                        | $x \in (-\infty, a_2)$ | $x \in (a_2, a_4)$ | $x \in (a_4, \infty)$ |
|------------------------|------------------------|--------------------|-----------------------|
| $x - a_2$              | —                      | +                  | +                     |
| $x - a_4$              | —                      | —                  | +                     |
| $(x - a_2)(x - a_4)^3$ |                        | +                  | —                     |

That is, the solution set is given by  $(-\infty, a_2] \cup [a_4, \infty) \cup \{a_3\}$ .

Note that  $a_1 \in (-\infty, a_2]$ . ■

2. Let  $a$  and  $b$  be real numbers with  $|b| < |a| < 1$  and  $a > 0$ .

- (a) Show that  $a - 1 < 0 < b + a < 2a < 1 + a$ .  
 (b) (Modified) Show that

$$\frac{2a^2 - 2a}{a + 1} > \frac{a^2 - 1}{a + b}.$$

Remark: The original version  $\frac{2a^2 - 2a}{b + a} > \frac{a^2 - 1}{2a}$  is incorrect and this problem will not be graded.

**Solution:**(a)  $|a| < 1$  implies

$$-1 < a < 1 \Rightarrow -a > -1 \quad (1)$$

and  $|b| < |a|$  and  $a > 0$  imply

$$-|a| < b < |a| \quad \text{and} \quad -a < b < a \quad \text{since} \quad |a| = a. \quad (2)$$

From the inequalities in (1) and (2), we have

$$-1 < -a < b < a < 1$$

and adding  $a$  throughout the above inequalities gives

$$a - 1 < 0 < b + a < 2a < 1 + a.$$

(b) First notice the given inequality in (b) can be rewritten as

$$\frac{2a(a-1)}{b+a} > \frac{(a-1)(a+1)}{2a}.$$

From (a),

$$\begin{aligned} 0 &< 2a < a+1 \\ 0 < b+a < a+1 &\Rightarrow 0 < \frac{1}{a+1} < \frac{1}{a+b} \end{aligned}$$

Multiply the above two inequalities gives,

$$0 < \frac{2a}{a+1} < \frac{a+1}{a+b}$$

and since  $a-1 < 0$ ,

$$0 > \frac{2a(a-1)}{a+1} > \frac{(a+1)(a-1)}{a+b} \quad \text{or} \quad \frac{2a^2-2a}{a+1} > \frac{a^2-1}{a+b}.$$

■

3. Find the solution set for each of following inequalities.

(a)

$$\frac{1}{x} < 8$$

(b)

$$\frac{9-x}{x-2} \geq \frac{2-x}{x-9}$$

(c)

$$\frac{|2x+5|-2}{3|x|-1} \leq 1$$

(d)

$$\frac{|x^2 - x + 1|}{|3x + 1|} > 1$$

(e)

$$\frac{|x^2 - 2| + 2}{x^2 - 3|x| + 2} < 0$$

**Solution:**

(a)

$$\frac{1}{x} < 8$$

Answer:  $(-\infty, 0) \cup (1/8, \infty)$ 

$$\begin{aligned} \frac{1}{x} &< 8 \\ \frac{1}{x} - 8 &< 0 \\ \frac{1 - 8x}{x} &< 0 \\ \frac{x - 1/8}{x} &> 0 \end{aligned}$$

We consider the intervals divided by  $x = 0, 1/8$  as follows.

|                   | $x \in (-\infty, 0)$ | $x \in (0, 1/8)$  | $x \in (1/8, \infty)$ |
|-------------------|----------------------|-------------------|-----------------------|
| $\frac{x-1/8}{x}$ | $\frac{(-)}{(-)}$    | $\frac{(+)}{(-)}$ | $\frac{(+)}{(+)}$     |
|                   | +                    | -                 | +                     |

Hence, from the table, the solution set is  $(-\infty, 0) \cup (1/8, \infty)$ . ■

(b)

$$\frac{9 - x}{x - 2} \geq \frac{2 - x}{x - 9}$$

Answer:  $(2, 11/2] \cup (9, \infty)$

$$\begin{aligned}
\frac{9-x}{x-2} &\geq \frac{2-x}{x-9} \\
\frac{9-x}{x-2} - \frac{2-x}{x-9} &\geq 0 \\
\frac{-(x-9)^2 + (x-2)^2}{(x-2)(x-9)} = \frac{(x-2)^2 - (x-9)^2}{(x-2)(x-9)} &\geq 0 \\
\frac{(x-2-x+9)(x-2+x-9)}{(x-2)(x-9)} &\geq 0 \\
\frac{7(2x-11)}{(x-2)(x-9)} &\geq 0
\end{aligned}$$

We consider the intervals divided by  $x = 2, 11/2, 9$  as follows.

|                               | $x \in (-\infty, 2)$       | $x \in (2, 11/2)$          | $x \in (11/2, 9)$          | $x \in (9, \infty)$        |
|-------------------------------|----------------------------|----------------------------|----------------------------|----------------------------|
| $\frac{7(2x-11)}{(x-2)(x-9)}$ | $\frac{(-)}{(-)(-)} = (-)$ | $\frac{(-)}{(+)(-)} = (+)$ | $\frac{(+)}{(+)(-)} = (-)$ | $\frac{(+)}{(+)(+)} = (+)$ |

Note that we consider  $\geq$ , so  $\{11/2\}$  has to be included in the solution set, but not the points that give zero denominator. Hence, from the table, the solution set is  $(2, 11/2] \cup (9, \infty)$ . ■

(c)

$$\frac{|2x+5|-2}{3|x|-1} \leq 1$$

By using definition of absolute value, we have  $|2x+5| = \begin{cases} -(2x+5), & 2x+5 < 0 \Leftrightarrow x < -5/2 \\ 2x+5, & 2x+5 \geq 0 \Leftrightarrow x \geq -5/2 \end{cases}$ ,

and  $|x| = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$ , we will consider 3 cases:

Case I:  $x \in (-\infty, -5/2)$ , Case II:  $x \in [-5/2, 0)$ , Case III:  $x \in [0, \infty)$ .

**Case I:**  $x \in (-\infty, -5/2)$ .

$$\begin{aligned}
\frac{|2x+5|-2}{3|x|-1} &\leq 1 \\
\frac{-(2x+5)-2}{-3x-1} &\leq 1 \Leftrightarrow \frac{2x+7}{3x+1} \leq 1 \Leftrightarrow \frac{2x+7}{3x+1} - 1 \leq 0 \Leftrightarrow \frac{(2x+7)-(3x+1)}{3x+1} \leq 0 \\
\frac{-x+6}{3x+1} &\leq 0 \quad \text{or} \quad \frac{x-6}{3x+1} \geq 0
\end{aligned}$$

We consider the intervals divided by  $x = -1/3, 6$  as follows.

|                    | $x \in (-\infty, -1/3)$ | $x \in (-1/3, 6)$       | $x \in (6, \infty)$     |
|--------------------|-------------------------|-------------------------|-------------------------|
| $\frac{x-6}{3x+1}$ | $\frac{(-)}{(-)} = (+)$ | $\frac{(-)}{(+)} = (-)$ | $\frac{(+)}{(+)} = (+)$ |

Note that we consider  $\geq$ , so  $\{6\}$  has to be included in the solution set, but not the points that give zero denominator. Hence, from the table,  $(-\infty, -1/3) \cup [6, \infty)$ . That is, in this case, the solution set is  $(-\infty, -5/2) \cap \{(-\infty, -1/3) \cup [6, \infty)\} = (-\infty, -5/2)$ .

**Case II:**  $x \in [-5/2, 0)$ ,

$$\begin{aligned} \frac{|2x+5|-2}{3|x|-1} &\leq 1 \\ \frac{(2x+5)-2}{-3x-1} &\leq 1 \Leftrightarrow \frac{2x+3}{3x+1} \geq -1 \Leftrightarrow \frac{2x+3}{3x+1} + 1 \geq 0 \Leftrightarrow \frac{(2x+3)+(3x+1)}{3x+1} \geq 0 \\ \frac{5x+4}{3x+1} &\geq 0 \end{aligned}$$

We consider the intervals divided by  $x = -4/5, -1/3$  as follows.

|                     | $x \in (-\infty, -4/5)$ | $x \in (-4/5, -1/3)$    | $x \in (-1/3, \infty)$  |
|---------------------|-------------------------|-------------------------|-------------------------|
| $\frac{5x+4}{3x+1}$ | $\frac{(-)}{(-)} = (+)$ | $\frac{(-)}{(+)} = (-)$ | $\frac{(+)}{(+)} = (+)$ |

Note that we consider  $\leq$ , so  $\{-4/5\}$  has to be included in the solution set, but not the points that give zero denominator. Hence, from the table,  $(-1/3, 2]$ . That is, in this case, the solution set is  $[-5/2, 0) \cap \{(-\infty, -4/5] \cup (-1/3, \infty)\} = [-5/2, -4/5] \cup (-1/3, 0)$ .

**Case III:**  $x \in [0, \infty)$ .

$$\begin{aligned} \frac{|2x+5|-2}{3|x|-1} &\leq 1 \\ \frac{(2x+5)-2}{3x-1} &\leq 1 \Leftrightarrow \frac{2x+3}{3x-1} \leq 1 \Leftrightarrow \frac{2x+3}{3x-1} - 1 \leq 0 \Leftrightarrow \frac{(2x+3)-(3x-1)}{3x-1} \leq 0 \\ \frac{-x+4}{3x-1} &\leq 0 \quad \text{or} \quad \frac{x-4}{3x-1} \geq 0 \end{aligned}$$

We consider the intervals divided by  $x = 1/3, 4$  as follows.

|                    | $x \in (-\infty, 1/3)$  | $x \in (1/3, 4)$        | $x \in (4, \infty)$     |
|--------------------|-------------------------|-------------------------|-------------------------|
| $\frac{x-4}{3x-1}$ | $\frac{(-)}{(-)} = (+)$ | $\frac{(-)}{(+)} = (-)$ | $\frac{(+)}{(+)} = (+)$ |

Note that we consider  $\geq$ , so  $\{4\}$  has to be included in the solution set, but not the points that give zero denominator. Hence, from the table,  $(-\infty, 1/3) \cup [4, \infty)$ . That is, in this case, the solution set is  $[0, \infty) \cap \{(-\infty, 1/3) \cup [4, \infty)\} = [0, 1/3) \cup [4, \infty)$ .

From cases (I)-(III), we have that the solution set is given by  $(-\infty, -5/2) \cup \{[-5/2, -4/5] \cup (-1/3, 0)\} \cup \{[0, 1/3) \cup [4, \infty)\} = (-\infty, -4/5) \cup (-1/3, 1/3) \cup [4, \infty)$ . ■

(d)

$$\frac{|x^2 - x + 1|}{|3x + 1|} > 1$$

**Answer:**  $(-\infty, -1/3) \cup (-1/3, 0) \cup (4, \infty)$

There are many approaches for solving this problem. Two approaches are considered here.

**Approach 1:**

Since both sides are positive, the above inequality is equivalent to

$$\begin{aligned} \left( \frac{|x^2 - x + 1|}{|3x + 1|} \right)^2 > 1^2 &\Leftrightarrow \left( \frac{x^2 - x + 1}{3x + 1} \right)^2 - 1^2 > 0 \Leftrightarrow \\ &\left( \frac{x^2 - x + 1}{3x + 1} - 1 \right) \left( \frac{x^2 - x + 1}{3x + 1} + 1 \right) > 0 \\ \left( \frac{x^2 - x + 1 - 3x - 1}{3x + 1} \right) \left( \frac{x^2 - x + 1 + 3x + 1}{3x + 1} \right) &> 0 \\ \frac{(x^2 - 4x)(x^2 + 2x + 2)}{(3x + 1)^2} &> 0. \end{aligned}$$

Consider  $x^2 + 2x + 2$ . Since  $b^2 - 4ac = 4 - 4(1)(2) < 0$  and  $a = 1 > 0$ , then  $x^2 + 2x + 2 > 0 \quad \forall x \in \mathbb{R}$  and the inequality becomes:

$$\begin{aligned} x^2 - 2x + 2 > 0 \quad \forall x \in \mathbb{R} &\Rightarrow \frac{x(x - 4)}{(3x + 1)^2} > 0 \\ (3x + 1)^2 \geq 0 \quad \forall x \in \mathbb{R}, \text{ for } x \neq -1/3, &\Rightarrow x(x - 4) > 0 \end{aligned}$$

By setting  $x(x - 4) = 0$ , we consider the intervals divided by  $x = 0, 4$  as follows.

|            | $x \in (-\infty, 0)$ | $x \in (0, 4)$ | $x \in (4, \infty)$ |
|------------|----------------------|----------------|---------------------|
| $x(x - 4)$ | $(-)(-) = (+)$       | $(+)(-) = (-)$ | $(+)(+) = (+)$      |

Hence, from the table and from the fact that  $x \neq -1/3$ ,  $\{(-\infty, 0) \cup (4, \infty)\} - \{-1/3\} = (-\infty, -1/3) \cup (-1/3, 0) \cup (4, \infty)$ . ■

**Approach 2:**

Consider  $x^2 - x + 1$ . Since  $b^2 - 4ac = 1 - 4(1)(1) < 0$  and  $a = 1 > 0$ , then  $x^2 - x + 1 > 0 \quad \forall x \in \mathbb{R}$  and  $|x^2 - x + 1| = x^2 - x + 1$ . So the inequality becomes:

$$\frac{x^2 - x + 1}{|3x + 1|} > 1$$

and we only need to consider two cases:  $|3x + 1| = \begin{cases} -(3x + 1), & 3x + 1 < 0 \Leftrightarrow x < -1/3 \\ 3x + 1, & 3x + 1 \geq 0 \Leftrightarrow x \geq -1/3 \end{cases}$ ,

(\*) Note that, since  $|3x + 1|$  is the denominator, we must have  $x \neq -1/3$  and we consider (I)  $x < -1/3$  and (II)  $x > -1/3$ .

(I) For  $x < -1/3$ ,  $|3x + 1| = -(3x + 1)$

$$\frac{x^2 - x + 1}{|3x + 1|} > 1 \Leftrightarrow x^2 - x + 1 > |3x + 1| \Leftrightarrow x^2 - x + 1 > -3x - 1 \Leftrightarrow x^2 + 2x + 2 > 0$$

which is true for all real numbers  $x$  (consider  $x^2 + 2x + 2$ . Since  $b^2 - 4ac = 4 - 4(1)(2) < 0$  and  $a = 1 > 0$ , then  $x^2 + 2x + 2 > 0 \quad \forall x \in \mathbb{R}$ ). Hence, the solution set in this case is  $(-\infty, -1/3) \cap \mathbb{R} = (-\infty, -1/3)$ .

(II) For  $x > -1/3$ ,  $|3x + 1| = 3x + 1$

$$\frac{x^2 - x + 1}{|3x + 1|} > 1 \Leftrightarrow x^2 - x + 1 > |3x + 1| \Leftrightarrow x^2 - x + 1 > 3x + 1 \Leftrightarrow x^2 - 4x > 0.$$

By setting  $x(x - 4) = 0$ , we consider the intervals divided by  $x = 0, 4$  as follows.

|            | $x \in (-\infty, 0)$ | $x \in (0, 4)$ | $x \in (4, \infty)$ |
|------------|----------------------|----------------|---------------------|
| $x(x - 4)$ | $(-)(-) = (+)$       | $(+)(-) = (-)$ | $(+)(+) = (+)$      |

Hence, from the table and from the fact that  $x > -1/3$ ,  $(-1/3, \infty) \cap \{(-\infty, 0) \cup (4, \infty)\} = (-1/3, 0) \cup (4, \infty)$ .

From (I) and (II), the solution set is  $(-\infty, -1/3) \cup (-1/3, 0) \cup (4, \infty)$ . ■

(e)

$$\frac{|x^2 - 2| + 2}{x^2 - 3|x| + 2} < 0$$

Answer:  $(-2, -1) \cup (1, 2)$

Notice that  $|x^2 - 2| \geq 0 \Rightarrow |x^2 - 2| + 2 \geq 2 > 0$  for any real number  $x$ . That is, to find  $x$  that gives  $\frac{|x^2 - 2| + 2}{x^2 - 3|x| + 2} < 0$ , we can solve

$$\begin{aligned} \frac{1}{x^2 - 3|x| + 2} &< 0 \\ \frac{1}{|x|^2 - 3|x| + 2} &< 0 \\ \frac{1}{(|x| - 1)(|x| - 2)} &< 0. \end{aligned}$$

By setting  $(|x| - 1)(|x| - 2) = 0$ , we consider the intervals divided by  $|x| = 1, 2$  together with the fact that  $|x| \geq 0$  as shown below.

|                      | $ x  \in [0, 1)$ | $ x  \in (1, 2)$ | $ x  \in (2, \infty)$ |
|----------------------|------------------|------------------|-----------------------|
| $( x  - 1)( x  - 2)$ | $(-)(-) = (+)$   | $(+)(-) = (-)$   | $(+)(+) = (+)$        |

Hence, from the table  $|x| \in (1, 2)$  or

$$1 < |x| < 2 \Leftrightarrow 1 < |x| \text{ and } |x| < 2 \Leftrightarrow x \in (-\infty, -1) \cup (1, \infty) \text{ and } x \in (-2, 2).$$

That is, the solution set is  $\{(-\infty, -1) \cup (1, \infty)\} \cap (-2, 2) = (-2, -1) \cup (1, 2)$ . ■

4. A math class counts the midterm exam scores as  $1/3$  of the grade and the final exam scores as  $2/3$  of the grade. Paul scored 48% on the midterm.

To get an A, Paul must have the total score between 90% and 100% inclusive;  
to get a B, Paul must have the total score between 80% and 89% inclusive;  
to get a C, Paul must have the total score between 70% and 79% inclusive.

Suppose that all scores with decimal digits get rounded up to the closest higher integers.

(a) What range of scores (in %) on the final exam would make Paul get a C?

(b) What is the highest grade that Paul could get for this class?

**Solution:** (a)  $[81, 94.5]$ . (b) B

(a) Let  $x$  and  $y$  be the midterm and final scores out of  $100/3$  and  $200/3$ , respectively. Then

$$\frac{x}{100/3} = \frac{48}{100} \Rightarrow x = 16.$$

To get a C,

$$16 + y \in [70, 79] \Rightarrow 70 \leq 16 + y \leq 79 \Rightarrow 54 \leq y \leq 63.$$

Let  $Y$  be the final exam score in %. Then  $\frac{Y}{100} = \frac{y}{200/3}$  or  $y = \frac{2}{3}Y$ . That is, Paul will get a C if

$$54 \leq \frac{2}{3}Y \leq 63 \Rightarrow 81 \leq Y \leq 94.5.$$

(b) Since the maximum of the final score is  $y = 200/3$ , the maximum score for the grade is

$$\frac{200}{3}\% + 16\% = 82.6666...\% \in [80, 89].$$

That is, the highest grade that Paul could get for this class is B. ■

5. (Optional) The weekly demand (the number bought by consumers) for the a product is given by the formula

$$d = 9000 - 60p$$

where  $p$  is the price each in dollars.

(a) What is the demand when the price is 30 each?

(b) In what price range will the demand be above 6000?

6. (Optional) A store's revenue  $R(x)$  (in Baht) on the sale of  $x$  cupcakes is determined by the formula  $R(x) = 50x - x^2$ . The cost  $C(x)$  (in Baht) for producing  $x$  cupcakes is given by the formula  $C(x) = 2x + 400$ . For what values of  $x$  is the store's profit positive? Note: profit = revenue - cost.

7. (Optional) Find the solution set for each of the following inequities.

(a)  $|x - 1| < 7$

Ans:  $(-6, 8)$

(b)  $3 < |x + 2| < 4$

Ans:  $(-6, -5) \cup (1, 2)$

(c)  $|2x + 3| \leq 2$

Ans:  $[-5/2, -1/2]$

(d)  $|3x - 1| \geq 1$

Ans:  $(-\infty, 0] \cup [2/3, \infty)$

(e)  $\frac{|x+2|}{2} > |x|$

Ans:  $(-2/3, 2)$

- (f)  $|x + 1| - |2x - 1| < 0$   
Ans:  $(-\infty, 0) \cup (2, \infty)$
- (g)  $|2 - x| - x \leq 0$   
Ans:  $[1, \infty)$
- (h)  $|x + 1| > 3 - |x|$   
Ans:  $(-\infty, -2) \cup (1, \infty)$
- (i)  $x^2 - 2 \geq \frac{1}{2}|x - 1|$   
Ans:  $(-\infty, \frac{-1-\sqrt{41}}{4}) \cup (3/2, \infty)$
- (j)  $|x + 3| \geq 2 - x$   
Ans:  $[-1/2, \infty)$
- (k)  $|2x^2 + x - 1| \geq 2$   
Ans:  $(-\infty, -3/2] \cup [1, \infty)$
- (l)  $\frac{|x|}{2} + 3 > x^2$   
Ans:  $(-2, 2)$
- (m)  $|x| > \frac{2}{|x+1|}$   
Ans:  $(-\infty, -2) \cup (1, \infty)$
- (n)  $\frac{x^2}{x+1} < |x|$   
Ans:  $(-\infty, -1) \cup (-1/2, 0) \cup (0, \infty)$
- (o)  $\frac{4}{|x|} \leq \frac{1}{|x|}$   
Ans:  $\emptyset$
- (p)  $|x^2 + 1| < |x + 1|$   
Ans:  $(0, 1)$
- (q)  $(x^2 + 1)(|x + 2| - |x|) \geq 0$   
Ans:  $[-1, \infty)$
- (r)  $\frac{|1-x|}{x^3+2x^2+5x+4} \leq 0$   
Ans:  $(-\infty, -1) \cup \{1\}$