Question 0:

Group 5

Consider the function f defined for all (x,y) such that

$$f\ (x, y; a) = \frac{1}{2}x^2 - x + ay(x-1) - \frac{1}{3}y^3 + a^2y^2,$$

where a is a constant.

a. Prove that $(x^*, y^*) = (1 - a^3, a^2)$ is a stationary point of $f\ (x, y; a)$. ✓
b. State the condition under which the above stationary point is a global maximum.
c. Given that $G(a) = f\ (x^*, y^*; a)$, show the derivative of G with respect to a.
d. Calculate $\frac{\partial f\ (x,y;a)}{\partial a}$ and evaluate its value where $(x^*, y^*) = (1 - a^3, a^2)$. Compare your answer with the answer obtained in b. Are they the same?
e. Determine the domain of (x,y) in the xy-plan where f is convex.

a. $f(x, y; a) = \frac{1}{2}x^2 - x + ay(x-1) - \frac{1}{3}y^3 + a^2y^2$

$$\frac{\partial f}{\partial x} = \frac{1}{2}(2)x^{2-1} - 1 + ay = 0$$

$$x - 1 + ay = 0$$

$$x^* = 1 - ay$$
$$= 1 - a(a^2)$$
$$= 1 - a^3$$

$$\frac{\partial f}{\partial y} = ax - a - \frac{1}{3}(3)y^{3-1} + a^2(2)y^{2-1} = 0$$

$$ax - a - y^2 + 2a^2y = 0$$

$$a(1 - ay) - a - y^2 + 2a^2y = 0$$

$$\cancel{a} - a^2y - \cancel{a} - y^2 + 2a^2y = 0$$

$$-y^2 + a^2y = 0$$
$$y^2 - a^2y = 0$$
$$y(y - a^2) = 0$$
$$y = 0$$
$$y^* = a^2$$

$\therefore (x^*, y^*) = (1 - a^3, a^2)$ is a stationary point of $f(x, y; a)$.

b. $\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$

$f_{xx} = \frac{\partial^2 f}{\partial x^2} = 1$

$f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = a = f_{yx}$

$f_{yy} = \frac{\partial^2 f}{\partial y^2} = -2y + 2a^2$

$|H_1| = 1 > 0$

$|H_2| = (f_{xx})(f_{yy}) - (f_{xy})(f_{yx})$

$= (1)(-2y + 2a^2) - (a)(a)$

$= -2y + 2a^2 - a^2$

$= -2y + a^2$

the stationary point $(x^*, y^*) = (1-a^3, a^2)$ will be a global maximum when $|H_1| > 0$ & $|H_2| < 0$ for $\forall x$ & $\forall y$. $|H_1|$ will always be positive, while $|H_2|$ will be negative only if $-2y + a^2 < 0$ or when $y > \frac{a^2}{2}$

c. $G(a) = f(x^*, y^*; a)$

$$\frac{\partial G}{\partial a} = \frac{1}{2}(1-a^3)^2 - (1-a^3) + a(a^2)[(1-a^3) - 1] - \frac{1}{3}(a^2)^3 + a^2(a^2)^2$$

$$= \frac{1}{2}(1 - 2(1)(a^3) + (a^3)^2) - (1-a^3) + a^3[(1-a^3) - 1] - \frac{1}{3}(a^6) + a^6$$

$$= \frac{1}{2}(1 - 2a^3 + a^6) - 1 + a^3 + a^3 - a^6 - a^3 - \frac{1}{3}a^6 + a^6$$

$$= \frac{1}{2} \cancel{- a^3} + \frac{a^6}{2} - 1 \cancel{+ a^3} - \frac{a^6}{3}$$

$$= -\frac{1}{2} - \frac{a^6}{3} + \frac{a^6}{2}$$

d. $\frac{\partial f(x,y;a)}{\partial a} = xy - y + 2ay^2$

given $x^* = 1 - a^3,\ y = a^2$

$$\frac{\partial f(x,y;a)}{\partial a} = (1-a^3)a^2 - a^2 + 2a(a^2)^2$$
$$= a^2 - a^5 - a^2 + 2a^5$$
$$= a^5$$

$$\therefore \frac{\partial f(x,y;a)}{\partial a} \neq \frac{\partial f(x^*,y^*;a)}{\partial a}$$

e. f is convex when $\partial^2 f > 0$
or when $|H_1| > 0$ or $|H_2| > 0$
$|H_1|$ will always be positive
but $|H_2|$ will be positive only
when $y < \frac{a^2}{2}$

$\therefore$ Domain of x is $(-\infty, \infty)$
Domain of y is $\left(-\infty, \frac{a^2}{2}\right)$

Question 1

Consider a linear demand equation faced by a monopolist.

$$Q = 2000 + 4\sqrt{A} - 20P,$$

where A is the dollar amount of the expenditure on advertisement, P is the unit price, and Q is the amount of quantity demanded by consumers. The demand equation is an extended version of the traditional one. The proposed function captures an idea that advertising can boost the total demand in the market as potential customers get more information about the product.

Suppose that the monopolist cost function is given by $C(Q, A) = c(Q) + A$ where $c(Q) = 2Q + 1000$

Consider the following problem.

a) Construct the profit function.
b) Determine the optimal pricing (P*) and optimal advertisement (A*) that maximize the profit of the monopolist.
c) Confirm the result with the second-order differential test, i.e. hessian-matrix method.
d) What is the maximum level of profit?

a) $\pi = TR - TC$

$= PQ - C(Q,A)$

$= P(2000 + 4\sqrt{A} - 20P) - [(2Q + 1000) + A]$

$= P(2000 + 4\sqrt{A} - 20P) - [(2(2000 + 4\sqrt{A} - 20P) + 1000) + A]$

$= 2000P + 4\sqrt{A}P - 20P^2 - (4000 + 8\sqrt{A} - 40P + 1000 + A)$

$= 2000P + 4\sqrt{A}P - 20P^2 - 4000 - 8\sqrt{A} + 40P - 1000 - A$

$= -20P^2 + 2040P + 4\sqrt{A}P - 8\sqrt{A} - A - 5000$

b) $\frac{\partial \pi}{\partial P} = -40P + 2040 + 4\sqrt{A} = 0$

$-40P = -2040 - 4\sqrt{A}$

$P = 51 + \frac{1}{10}\sqrt{A}$

$= 51 + \frac{1}{10}\sqrt{(2P-8)^2}$

$= 51 + \frac{(2P-8)}{10}$

$10P = 510 + 2P - 8$

$8P = 502$

$P^* = \underline{62.75}$

$\frac{\partial \pi}{\partial A} = 4P\left(\frac{1}{2}\right)A^{-\frac{1}{2}} - 8A^{-\frac{1}{2}} - 1 = 0$

$\boxed{\frac{2P}{\sqrt{A}} - \frac{8}{\sqrt{A}}} - 1 = 0$

$2PA^{-\frac{1}{2}} - 8A^{-\frac{1}{2}}$

$\frac{2P-8}{\sqrt{A}} = 1$

$A = (2P-8)^2$

$= 4P^2 - 2(2P)(8) + 64$

$= 4P^2 - 32P + 64$

$= P^2 - 8P + 16$

$= 4(\underline{62.75})^2 - 32(\underline{62.75}) + 64$

$A^* = 13806.25$

c) $\begin{vmatrix} \pi_{PP} & \pi_{PA} \\ \pi_{AP} & \pi_{AA} \end{vmatrix} = \begin{vmatrix} -40 & \frac{2}{\sqrt{A}} \\ \frac{2}{\sqrt{A}} & \frac{-P+4}{A\sqrt{A}} \end{vmatrix}$

$\frac{\partial \pi}{\partial A}$ $2PA^{-\frac{1}{2}} - 8A^{\frac{1}{2}-1} = 0$

$\frac{\partial^2 \pi}{\partial A \partial A}$ $2P\left(\frac{-1}{2}\right)A^{-\frac{1}{2}-1} - 8\left(\frac{-1}{2}\right)A^{-\frac{1}{2}-1}$

$-PA^{-\frac{3}{2}} + 4A^{-\frac{3}{2}} = \frac{-P+4}{A\sqrt{A}}$

$\pi_{PP} = -40$

$\pi_{PA} = \pi_{AP} = 4\left(\frac{1}{2}\right)A^{-\frac{1}{2}} = \frac{2}{\sqrt{A}}$

$\pi_{AA} = -PA^{-\frac{3}{2}} + 4A^{-\frac{3}{2}} = \frac{-P+4}{A\sqrt{A}}$

$|H_1| = -40 < 0$ for $\forall P, A$

$|H_2| = -40\left(\frac{-P+4}{A\sqrt{A}}\right) - \left(\frac{2}{\sqrt{A}}\right)\left(\frac{2}{\sqrt{A}}\right) = \frac{40P-160}{A\sqrt{A}} - \frac{4}{A}$

$= \frac{40P - 160 - 4\sqrt{A}}{A\sqrt{A}}$

$= \frac{40(13806.25) - 160 - 4\sqrt{13806.25}}{13806.25\sqrt{13806.25}} = 0.34 > 0$

$\because |H_1| < 0$, $|H_2| > 0$ $\therefore$ it's negative definite,
the function is concave at $P^* = 62.75$ and $A^* = 13,806.25$ $(\partial^2\pi < 0)$,
so we prove that P^* and A^* can maximize the profit of the monopolist

d) $\pi = -20P^2 + 2040P + 4\sqrt{A}P - 8\sqrt{A} - A - 5000$

$= -20(62.75)^2 + 2040(62.75) + 4\underbrace{\boxed{\sqrt{13,806.25}}}_{= 117.5}(62.75) - 8\sqrt{13,806.25} - (13,806.25) - 5,000$

$= -78,751.25 + 128,010 + 29,492.5 - 940 - 13,806.25 - 5,000$

$\therefore \pi = 59,005$

Hence, the maximum level of profit is 59,005 *

Question 3:

Consider a simple two-market model where demand and supply for each market is given below. (Notationally, let's name the two markets as A and B, respectively.)

Market A:

Demand: $p_A = 10 - 2Q_A$

Supply: $p_A = 1 + Q_A$

Market B:

Demand: $p_B = 20 - Q_B$

Supply: $p_B = 2 + 2Q_B$

✓ a. Derive the market equilibrium
✓ b. Suppose the government imposes unit tax on consumers in both markets at the rate of t_A and t_B. Solve for the after-tax equilibrium as the function of t_A and t_B.
✓ c. How much revenue can the government collect from the taxation?
✓ d. Determine the level of t_A and t_B that maximizes government's revenue.
✓ e. Use the second-order conditions test and show that the answer obtained in "d" is a global solution.

3. a) Market Equilibrium

Market A. Demand: $p_A^d = 10 - 2Q_A \rightarrow Q_A^d = \frac{10 - p_A^d}{2}$

Supply: $p_A^s = 1 + Q_A \rightarrow Q_A^s = p_A^s - 1$

Set $Q_D = Q_S \rightarrow \frac{10 - p_A^d}{2} = p_A^s - 1$

at equilibrium $p^d = p^s$

$10 - p_A = 2p_A - 2$

$\therefore p_A^* = 12 \Rightarrow Q_A^d = 12 - 1 = 11$

Market B: Demand: $p_B = 20 - Q_B \rightarrow Q_B = 20 - p_B$

Supply: $p_B = 2 + 2Q_B \rightarrow Q_B = \frac{p_B - 2}{2}$

Set $Q_D = Q_S \rightarrow 20 - p_B = \frac{p_B - 2}{2} \Rightarrow 40 - 2p_B = p_B - 2$

$3p_B = 42 \rightarrow p_B^* = 14 \Rightarrow Q_B^* = 20 - 14 = 6$ #

b. Suppose the government imposes unit tax on consumers in both markets at the rate of t_A and t_B. Solve for the after-tax equilibrium as the function of t_A and t_B.

3. b) Imposed tax on consumers, Solve for the after-tax equilibrium

$p^s + t = p^d$ (tax on Buyer)

Market A: Demand: $p^d_A = 10 - 2Q_A$

Supply: $p^s_A = 1 + Q_A$

Imposed unit tax on consumer,

$p^s_A + t_A = p^d_A = 10 - 2Q_A$

$p^s_A = 1 + Q_A$

$p^s_A = (10 - t_A) - 2Q_A$

$p^s_A = 1 + Q_A$

$\Rightarrow$ Solve $(10 - t_A) - 2Q_A = 1 + Q_A$

$3Q^*_A = 10 - t_A - 1$

$\therefore Q^*_A = \frac{9 - t_A}{3} = 3 - \frac{t_A}{3}$, $p^*_A = 1 + 3 - \frac{t_A}{3} = 4 - \frac{t_A}{3}$

Market B: Demand: $P^d_B = 20 - Q^d_B$

Supply: $P^s_B = 2 + 2Q^s_B$

Imposed unit tax on consumer,

$$P^s_B + t_B = P^d_B = 20 - Q^d_B$$

$$P^s_B = 2 + 2Q^s_B$$

$$P^s_B = (20 - t_B) - Q^d_B$$

$$P^s_B = 2 + 2Q^s_B$$

$\Rightarrow$ Solve $(20 - t_B) - Q_B = 2 + 2Q_B$

$$3Q_B = 20 - 2 - t_B$$

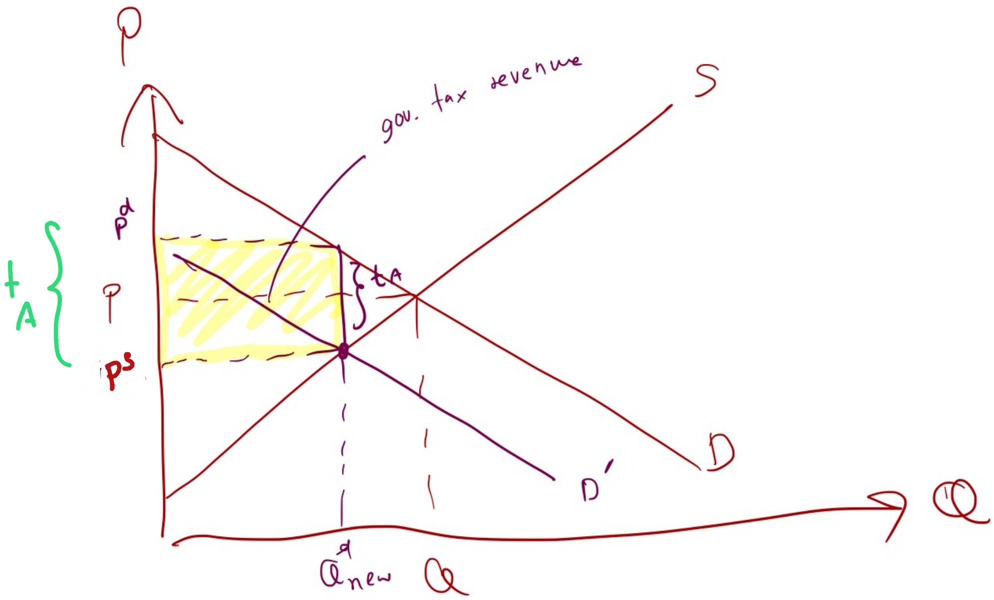
$$\therefore Q^*_B = \frac{18 - t_B}{3} = 6 - \frac{t_B}{3}, \quad P^*_B = 2 + 2\left(6 - \frac{t_B}{3}\right)$$

$$= 2 + 12 - \frac{2t_B}{3}$$

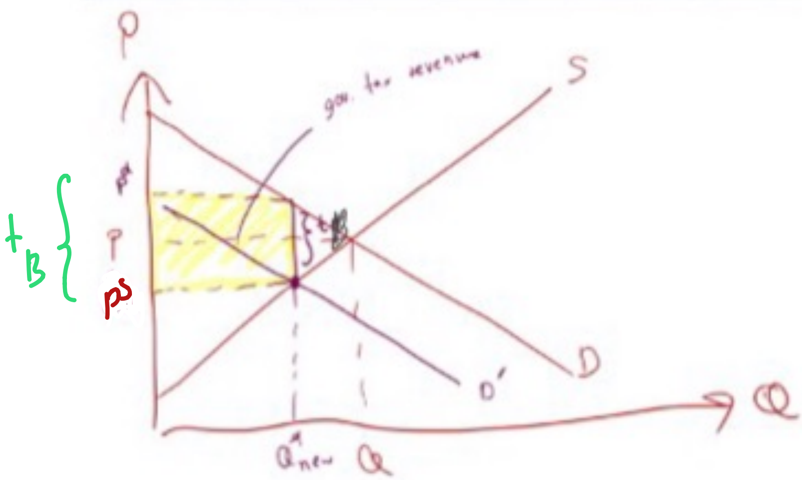
$$= 14 - \frac{2t_B}{3}$$

#

c. How much revenue can the government collect from the taxation?



$$Q^*_A \times t_A = (3 - \tfrac{1}{3} t_A) t_A = 3t_A - \tfrac{1}{3} t_A^2$$



$$Q^*_B \times t_B = (6 - \tfrac{t_B}{3}) t_B = 6t_B - \tfrac{t_B^2}{3}$$

d. Determine the level of t_A and t_B that maximizes government's revenue.

Market A, Government's Revenue $= 3t_A - \frac{1}{3}t_A^2$

$$\frac{\partial f_G}{\partial t_A} = 3 - \frac{2}{3}t_A = 0$$

$$\frac{2}{3}t_A = 3$$

$$t_A = \frac{3 \times 3}{2} = 4.5$$

Market B, Government's Revenue $= 6t_B - \frac{t_B^2}{3}$

$$\frac{\partial f_G}{\partial t_B} = 6 - \frac{2}{3}t_B = 0$$

$$\frac{2}{3}t_B = 6$$

$$t_B = \cancel{6}^{3} \times \frac{3}{\cancel{2}} = 9$$

e. Use the second-order conditions test and show that the answer obtained in "d" is a global solution.

$$|H| = \begin{vmatrix} f_{AA} & f_{AB} \\ f_{BA} & f_{BB} \end{vmatrix}$$

$$\left.\begin{aligned} f_{AA} &= -\frac{2}{3} < 0 \\ f_{AB} &= 0 \\ f_{BA} &= 0 \\ f_{BB} &= -\frac{2}{3} \end{aligned}\right\} \therefore |H| = \begin{vmatrix} -\frac{2}{3} & 0 \\ 0 & -\frac{2}{3} \end{vmatrix}$$

$$|H_1| = -\frac{2}{3} < 0 \text{ for } \forall t_A, t_B$$

$$|H_2| = \left(-\frac{2}{3}\right)\left(-\frac{2}{3}\right) - 0 \text{ for } \forall t_A, t_B$$

$$= \frac{4}{9} > 0$$

So, our Hessian matrix is negative definite and globally concave so, the answer in 'd' is globally maximized.