# Solution: Exercise 1 (Part 1)

1. Determine whether the statement forms are logically equivalent. In each case, construct a truth table to justify your answer.

   (a) $(p \wedge q) \to r$, $(p \to r) \wedge (q \to r)$

   (b) $p \to (q \to r)$, $(p \to q) \to r$

   (c) $p \to q \vee r$, $p \wedge \sim q \to r$, $p \wedge \sim r \to q$.

   Answer:
   (a) $(p \wedge q) \to r$, and $(p \to r) \wedge (q \to r)$

| $p$ | $q$ | $r$ | $p \to r$ | $q \to r$ | $p \wedge q$ | $(p \wedge q) \to r$ | $(p \to r) \wedge (q \to r)$ |
|---|---|---|---|---|---|---|---|
| T | T | T | T | T | T | T | T |
| T | T | F | F | F | T | F | F |
| T | F | T | T | T | F | T | T |
| T | F | F | F | T | F | T | F |
| F | T | T | T | T | F | T | T |
| F | T | F | T | F | F | T | F |
| F | F | T | T | T | F | T | T |
| F | F | F | T | T | F | T | T |

   Since rows 4 and 6 of the truth table for $(p \wedge q) \to r$, and $(p \to r) \wedge (q \to r)$ have different truth values, then these statement forms are not logically equivalent. ■

   (b)

   $p \to (q \to r)$, $(p \to q) \to r$

| $p$ | $q$ | $r$ | $q \to r$ | $p \to q$ | $p \to (q \to r)$ | $(p \to q) \to r$ |
|---|---|---|---|---|---|---|
| T | T | T | T | T | T | T |
| T | T | F | F | T | F | F |
| T | F | T | F | F | T | T |
| T | F | F | F | F | T | T |
| F | T | T | T | T | T | T |
| F | T | F | F | T | T | F |
| F | F | T | T | T | T | T |
| F | F | F | T | T | T | F |

   From the truth table, since there are cases when truth values of $p \to (q \to r)$ and $(p \to q) \to r$ (i.e. rows 6 and 8) are different, then these statements are not equivalent. ■

   (c) Using the order of connective operations gives $p \to (q \vee r)$, $(p \wedge \sim q) \to r$, $(p \wedge \sim r) \to q$.
   Truth table:

| $p$ | $q$ | $r$ | $\sim q$ | $\sim r$ | $q \vee r$ | $p \wedge \sim q$ | $p \wedge \sim r$ | $p \to (q \vee r)$ | $(p \wedge \sim q) \to r$ | $(p \wedge \sim r) \to q$ |
|---|---|---|---|---|---|---|---|---|---|---|
| T | T | T | F | F | T | F | F | T | T | T |
| T | T | F | F | T | T | F | T | T | T | T |
| T | F | T | T | F | T | T | F | T | T | T |
| T | F | F | T | T | F | T | T | F | F | F |
| F | T | T | F | F | T | F | F | T | T | T |
| F | T | F | F | T | T | F | F | T | T | T |
| F | F | T | T | F | T | F | F | T | T | T |
| F | F | F | T | T | F | F | F | T | T | T |

Since the last three columns of $p \to (q \vee r)$, $(p \wedge \sim q) \to r$, $(p \wedge \sim r) \to q$ have the same truth values for all possible cases in the truth table, then these statement forms are logically equivalent. ■

2. Determine whether or not the statement $\sim q \ \wedge p \ \to \sim q$ is a tautology.

**Answer:** The order of the connective operators has to be used here:

$$\sim q \ \wedge p \ \to \sim q \equiv ((\sim q) \ \wedge p) \ \to \ (\sim q).$$

Truth table:

| $p$ | $q$ | $\sim q$ | $(\sim q) \ \wedge p$ | $((\sim q) \ \wedge p) \ \to \ (\sim q)$ |
|---|---|---|---|---|
| T | T | F | F | T |
| T | F | T | T | T |
| F | T | F | F | T |
| F | F | T | F | T |

Notice from the table that the truth values of $\sim q \ \wedge p \ \to \sim q$ are true for all possible cases of $p$ and $q$, then this statement form is a tautology. ■

3. Let $p$ and $q$ be statements such that $p \leftrightarrow q$ is true. Find the truth values of each of the followings statement forms and provide some justifications.
(a) $p \to q$ (b) $\sim p \to \sim q$ (c) $\sim p \wedge q$ (d) $p \vee \sim q$ (e) $\sim p \leftrightarrow q$

**Answer:**
Given that $p \leftrightarrow q$ is true, we know that there are two possible cases for the truth values of $p$ and $q$:

- both $p$ and $q$ are true,
- both $p$ and $q$ are false.

Consider the truth values for (a)-(e) for these cases.

| $p$ | $q$ | $\sim p$ | $\sim q$ | (a) $p \to q$ | (b) $\sim p \to \sim q$ | (c) $\sim p \wedge q$ | (d) $p \vee \sim q$ | (e) $\sim p \leftrightarrow q$ |
|---|---|---|---|---|---|---|---|---|
| T | T | F | F | T | T | F | T | F |
| F | F | T | T | T | T | F | T | F |

Notice that the truth values for each of (a)-(e) are the same for both cases. That is, we can conclude that:

(a) $p \to q$ is true (b) $\sim p \to \sim q$ is true (c) $\sim p \wedge q$ is false (d) $p \vee \sim q$ is true
(e) $\sim p \leftrightarrow q$ is false.

■

4. Consider the following statement.

*If its color is green and it is edible, then it is a vegetable or it is a fruit.*

(a) Write the **negation** of the above statement.

(b) Write the **contrapositive**, **inverse**, and **converse** of the above statement.

**Answer:**
Let $p$ be "its color is green,"
$q$ be "it is edible,"
$r$ be "it is a vegetable."
$s$ be "it is a fruit."
Then the statement can be written as

$$(p \wedge q) \to (r \vee s).$$

(a) The **negation :**

$$\sim [(p \wedge q) \to (r \vee s)] \equiv (p \wedge q) \wedge \sim (r \vee s) \equiv (p \wedge q) \wedge (\sim r \wedge \sim s)$$

*"Its color is green and it is edible, but(and) it is not a vegetable and it is not a fruit."*

(b) **Contrapositive:**

$$\sim (r \vee s) \to \sim (p \wedge q) \equiv (\sim r \wedge \sim s) \to (\sim p \vee \sim q)$$

*"If it is not a vegetable and it is not a fruit, then its color is not green or it is not edible."*

**Inverse:**

$$\sim (p \wedge q) \to \sim (r \vee s) \equiv (\sim p \vee \sim q) \to (\sim r \wedge \sim s)$$

*"If its color is not green or it is not edible, then it is not a vegetable and it is not a fruit."*

**Converse**

$$(r \vee s) \to (p \wedge q)$$

*"If it is a vegetable or it is a fruit, then its color is green and it is edible."*

5. Use truth tables to determine whether the argument forms are valid. Indicate which columns represent the premises and which represent the conclusion, and include a sentence explaining how the truth table supports your answer.

(a)
$$\begin{array}{ll} & p \to q \\ & q \to p \\ \therefore & p \vee q \end{array}$$

(b)
$$\begin{array}{ll} & p \\ & p \to q \\ & \sim q \vee r \\ \therefore & r \end{array}$$

(c)
If it rains, then I stay home.
If it does not rain, then I go shopping.
∴ I stay home or I go shopping.

(d)
If it rains, then I stay home.
If it does not rain, then I go shopping.
I do not go shopping.
∴ It rains.

**Answer:**

(a) Truth table:

. Premises Conclusion

| $p$ | $q$ | $p \to q$ | $q \to p$ | $p \lor q$ | |
|---|---|---|---|---|---|
| T | T | T | T | T | ←-- critical row |
| T | F | F | T | | |
| F | T | T | F | | |
| F | F | T | T | F | ←-- critical row: This has true premises but false conclusion. |

Notice that the columns 3 and 4 consist of the truth values of premises and rows 1 and 4 are critical rows. Since the forth row (which is a critical row) has true premises with false conclusion, then this argument is **invalid**.

(b)
$p$
$p \to q$
$\sim q \lor r$
$\therefore$ $r$

Truth table:

Premises Conclusion

| $p$ | $q$ | $r$ | $\sim q$ | $p$ | $p \to q$ | $\sim q \lor r$ | $r$ | |
|---|---|---|---|---|---|---|---|---|
| T | T | T | F | T | T | T | T | ←-- critical row: This has true premises and true conclusion. |
| T | T | F | F | T | T | F | | |
| T | F | T | T | T | F | T | | |
| T | F | F | T | T | F | T | | |
| F | T | T | F | F | T | T | | |
| F | T | F | F | F | T | F | | |
| F | F | T | T | F | T | T | | |
| F | F | F | T | F | T | T | | |

Notice that the columns 5, 6, and 7 consist of the truth values of premises and row 1 is the only critical row. Since this critical row has true premises with true conclusion, then this argument is **valid**. ■

(c) Let $p$ be "If it rains, then I stay home"; $q$ be "I stay home." and $r$ be "I go shopping." Then we can transform the given argument as follows.

| | | | |
|---|---|---|---|
| | If it rains, then I stay home. | | $p \to q$ |
| | If it does not rain, then I go shopping. | $\equiv$ | $\sim p \to r$ Truth table: |
| $\therefore$ | I stay home or I go shopping. | $\therefore$ | $q \vee r$ |

Premises Conclusion

| $p$ | $q$ | $r$ | $\sim p$ | $p \to q$ | $\sim p \to r$ | $q \vee r$ | |
|---|---|---|---|---|---|---|---|
| T | T | T | F | T | T | T | ←-- critical row |
| T | T | F | F | T | T | T | ←-- critical row |
| T | F | T | F | F | T | T | |
| T | F | F | F | F | T | F | |
| F | T | T | T | T | T | T | ←-- critical row |
| F | T | F | T | T | F | T | |
| F | F | T | T | T | T | T | ←-- critical row |
| F | F | F | T | T | F | F | |

Notice that the columns 5 and 6 consist of the truth values of premises and rows 1,2,5,7 are the critical rows. Since each of these critical rows has all true premises with true conclusion, then this argument is **valid**. ■

(d) Let $p$ be "If it rains, then I stay home"; $q$ be "I stay home." and $r$ be "I go shopping." Then we can transform the given argument as follows.

| | | | |
|---|---|---|---|
| | If it rains, then I stay home. | | $p \to q$ |
| | If it does not rain, then I go shopping. | $\equiv$ | $\sim p \to r$ Truth table: |
| | I do not go shopping. | | $\sim r$ |
| $\therefore$ | It rains. | $\therefore$ | $p$ |

Premises Conclusion

| $p$ | $q$ | $r$ | $\sim p$ | $p \to q$ | $\sim p \to r$ | $\sim r$ | $p$ | |
|---|---|---|---|---|---|---|---|---|
| T | T | T | F | T | T | F | T | |
| T | T | F | F | T | T | T | T | ←-- critical row |
| T | F | T | F | F | T | F | T | |
| T | F | F | F | F | T | T | T | |
| F | T | T | T | T | T | F | F | |
| F | T | F | T | T | F | T | F | |
| F | F | T | T | T | T | F | F | |
| F | F | F | T | T | F | T | F | |

Notice that the columns 5, 6 and 7 consist of the truth values of premises and row 2 is the only critical row. Since this critical row has true premises with true conclusion, then this argument is **valid**. ■

6. Suppose that you discover a note written by a pirate. In the note he wrote that he had hidden treasure somewhere on the property. He listed five true statements (a-e below) and challenged the reader to use them to figure out the location of the treasure.

a. If this house is next to a lake, then the treasure is not in the kitchen.
b. If the tree in the front yard is a lemon tree, then the treasure is in the kitchen.
c. This house is next to a lake.
d. The tree in the front yard is a lemon tree or the treasure is buried under the flagpole.
e. If the tree in the back yard is a coconut tree, then the treasure is in the garage.

Where is the treasure hidden?

Answer: Define the following statement variables.
HL = This house is next to a lake.
TK = The treasure is in the kitchen.
YL = The tree in the front yard is a lemon tree.
TP = The treasure is buried under the flagpole.
YC = The tree in the back yard is a coconut tree.
TG = The treasure is in the garage.

(1) HL $\rightarrow \sim$ TK by (a)
HL by (c)
$\therefore \sim$ TK
(2) YL $\rightarrow$ TK by (b)
$\sim$ TK by by the conclusion of (1)
$\therefore \sim$ YL by "modus tollens"
(2) YL $\vee$ TP by (b)
$\sim$ YL by the conclusion of (2)
$\therefore$ TP by "elimination"

Therefore, we conclude that the treasure is buried under the flagpole.