Chapter 3: Linear Space (Vector Spaces)

This chapter we will consider a general **vector space** or **linear space**.

1 Real Vector Spaces

We will study structures with two operations, an addition and a scalar multiplication, that are subject to some simple conditions.

Definition 1.1. A vector space (over $\mathbb{R}$) consists of a set V along with two operations:
an addition, '+', and a scalar multiplication, '·',
subject to the conditions that for all vectors bv, $\mathbf{w}$, $\mathbf{u} \in V$ and all scalars $r, s \in \mathbb{R}$:

(1) the set V is closed under vector addition, that is, $\mathbf{v} + \mathbf{w} \in V$

(2) vector addition is commutative,$\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$

(3) vector addition is associative, $(\mathbf{v} + \mathbf{w}) + \mathbf{u} = \mathbf{v} + (\mathbf{w} + \mathbf{u})$

(4) there is a zero vector $0 \in V$ such that $\mathbf{v} + \underline{0} = \mathbf{v}$ for all $\mathbf{v} \in V$

(5) each $\mathbf{v} \in V$ has an additive inverse $\mathbf{w} \in \mathbf{V}$ such that $\mathbf{w} + \mathbf{v} = 0$

(6) the set V is closed under scalar multiplication, that is, $r \cdot \mathbf{v}V$

(7) addition of scalars distributes over scalar multiplication, $(r + s) \cdot \mathbf{v} = r \cdot \mathbf{v} + s \cdot \mathbf{v}$

(8) scalar multiplication distributes over vector addition, $r \cdot (\mathbf{v} + \mathbf{w}) = r \cdot \mathbf{v} + r \cdot \mathbf{w}$

(9) ordinary multiplication of scalars associates with scalar multiplication, $(rs) \cdot \mathbf{v} = r \cdot (s \cdot \mathbf{v})$

(10) multiplication by the scalar 1 is the identity operation, $1 \cdot \mathbf{v} = \mathbf{v}$.

Example 1.1. Verify that the following set L, which is a subset of $\mathbb{R}^2$

$$L = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \mid \ y = 3x \right\}$$

is a vector space under the usual meaning of '+' and '·'

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix}, \qquad r \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} rx \\ ry \end{bmatrix}.$$

(Cont')

We shall verify that it is a a vector space, under the usual meaning of '+' and '·'.

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix} \qquad r \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} rx \\ ry \end{pmatrix}$$

These operations are just the ones of $\mathbb{R}^2$, reused on its subset L. We say that L *inherits* these operations from $\mathbb{R}^3$.

We shall check all ten conditions. The paragraph having to do with addition has five conditions. For condition (1), closure under addition, suppose that we start with two vectors from the line L,

$$\vec{v}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

so that they satisfy the restrictions that $y_1 = 3x_1$ and $y_2 = 3x_2$. Their sum

$$\vec{v}_1 + \vec{v}_2 = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix}$$

is also a member of the line L because the fact that its second component is three times its first $y_1 + y_2 = 3(x_1 + x_2)$ follows from the restrictions on $\vec{v}_1$ and $\vec{v}_2$. For (2), that addition of vectors commutes, just compare

$$\vec{v}_1 + \vec{v}_2 = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix} \quad \vec{v}_2 + \vec{v}_1 = \begin{pmatrix} x_2 + x_1 \\ y_2 + y_1 \end{pmatrix}$$

and note that they are equal since their entries are real numbers and real numbers commute. (That the vectors satisfy the restriction of lying in the line is not relevant for this condition; they commute just because all vectors in the plane commute.) Condition (3), associativity of vector addition, is similar.

$$\begin{aligned} (\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}) + \begin{pmatrix} x_3 \\ y_3 \end{pmatrix} &= \begin{pmatrix} (x_1 + x_2) + x_3 \\ (y_1 + y_2) + y_3 \end{pmatrix} \\ &= \begin{pmatrix} x_1 + (x_2 + x_3) \\ y_1 + (y_2 + y_3) \end{pmatrix} \\ &= \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + (\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} + \begin{pmatrix} x_3 \\ y_3 \end{pmatrix}) \end{aligned}$$

(Cont')

For the fourth condition we must produce a vector that acts as the zero element. The vector of zeroes will do.

$$\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

Note that $\vec{0} \in L$ as its second component is triple its first. For (5), that given any $\vec{v} \in L$ we can produce an additive inverse, we have

$$\begin{pmatrix} -x \\ -y \end{pmatrix} + \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and so the vector $-\vec{v}$ is the desired inverse. As for the prior condition, observe here that if $\vec{v} \in L$, so that $y = 3x$, then $-\vec{v} \in L$ also, since $-y = 3(-x)$.

The checks for the five conditions having to do with scalar multiplication are similar. For (6), closure under scalar multiplication, suppose that $r \in \mathbb{R}$ and $\vec{v} \in L$

$$\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix}$$

so that it satisfies the restriction $y = 3x$. Then

$$r \cdot \vec{v} = r \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} rx \\ ry \end{pmatrix}$$

is also a member of L: the fact that its second component is three times its first $ry = 3(rx)$ follows from the restriction on $\vec{v}$. Next, this checks (7).

$$(r+s) \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} (r+s)x \\ (r+s)y \end{pmatrix} = \begin{pmatrix} rx + sx \\ ry + sy \end{pmatrix} = r \cdot \begin{pmatrix} x \\ y \end{pmatrix} + s \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

For (8) we have this.

$$r \cdot (\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}) = \begin{pmatrix} r(x_1 + x_2) \\ r(y_1 + y_2) \end{pmatrix} = \begin{pmatrix} rx_1 + rx_2 \\ ry_1 + ry_2 \end{pmatrix} = r \cdot \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + r \cdot \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

The ninth

$$(rs) \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} (rs)x \\ (rs)y \end{pmatrix} = \begin{pmatrix} r(sx) \\ r(sy) \end{pmatrix} = r \cdot (s \cdot \begin{pmatrix} x \\ y \end{pmatrix})$$

and tenth conditions are also straightforward.

$$1 \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1x \\ 1y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

Remark: In a similar way, each $\mathbb{R}^n$ is a vector space with the usual operations of vector addition and scalar multiplication

Example 1.2. (Exercise) Verify that the following set of one element zero vector

$$\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

is a vector space under the usual meaning of '+' and '$\cdot$' in $\mathbb{R}^3$.

Remark: In general, a vector space must have at least one element, its zero vector. Thus a one-element vector space is the smallest possible.

Definition 1.2. A one-element vector space is called a **trivial space**.

Example 1.3. (Exercise) Consider the set of polynomials of degree three or less:

$$\mathcal{P}_3 = \{a_0 + a_1x + a_2x^2 + a_3x^3 \ |a_0, a_1, a_2, a_3 \in \mathbb{R}\}.$$

Show that $\mathcal{P}_3$ under the following addition and scalar multiplication

$$\left(a_0 + a_1x + a_2x^2 + a_3x^3\right) + \left(b_0 + b_1x + b_2x^2 + b_3x^3\right) + (a_0+b_0) + (a_1+b_1)x + (a_2+b_2)x^2 + (a_3+b_3)x^3$$

and

$$r \cdot \left(a_0 + a_1x + a_2x^2 + a_3x^3\right) = ra_0 + ra_1x + ra_2x^2 + ra_3x^3$$

Note: Although this space is not a subset of any $\mathbb{R}^n$, there is a sense in which we can think of $\mathcal{P}_3$ as "the same" as $\mathbb{R}^4$. If we identify these two space's elements in this way:

$$a_0 + a_1x + a_2x^2 + a_3x^3 \qquad \Leftrightarrow \qquad \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

In general, we write $\mathcal{P}_n$ for the vector space of polynomials of degree n or less

$$\mathcal{P}_n = \{a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \ |a_0, a_1, a_2, ..., a_n \in \mathbb{R}\}.$$

under the operations of the usual polynomial addition and scalar multiplication.

Example 1.4. (Exercise) The set $\mathcal{M}_{2\times 2}$ of 2×2 matrices with real number entries is a vector space under the natural entry-by-entry operations.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} a+w & b+x \\ c+y & d+z \end{bmatrix}, \qquad r\cdot\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ra & rb \\ rc & rd \end{bmatrix}$$

is a vector space.

As in the prior example, we can think of this space as "the same" as $\mathbb{R}^4$.

In general, we write $\mathcal{M}_{n\times n}$ for the vector space of $n\times m$ matrices under the natural operations of matrix addition and scalar multiplication.

Example 1.5. (exercise) Determine if the following set $\{f|f:\mathbb{N}\to\mathbb{R}\}$ of all real-valued functions of one natural number variable is a vector space under the operations:

$$(f_1+f_2)(n)=f_1(n)+f_2(n), \qquad (r\cdot f)(n)=rf(n).$$

Example 1.6. Determine if the following set $V=\{f|f:\mathbb{R}\to\mathbb{R}\}$ of all real-valued functions of one real variable is a vector space under the operations:

$$(f_1+f_2)(x)=f_1(x)+f_2(x), \qquad (r\cdot f)(x)=rf(x).$$

Note: In general, the set $\{f|f:\mathbb{R}\to\mathbb{R}\}$ with above operations is denoted by the symbol $F(-\infty,\infty)$.

Example 1.7. Determine if the set $V = \mathbb{R}^+$ is a vector space under the operations:

$$u + v = uv, \qquad ku = u^k, \qquad u, v \in V.$$

Example 1.8. Determine if the set $V = \mathbb{R}^2$ is a vector space under the operations:

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix}, \qquad r\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} rx \\ 0 \end{bmatrix}.$$

Theorem 1.1. In any vector space V, for any $v \in V$ and $r \in \mathbb{R}$, we have

(1) $0 \cdot \mathbf{v} = \mathbf{0}$

(2) $(-1 \cdot \mathbf{v}) + \mathbf{v} = \mathbf{0}$ or $(-1) \cdot \mathbf{v} = -\mathbf{v}$

(3) $r \cdot \mathbf{0} = \mathbf{0}$

2 Subspaces and Spanning Sets

Definition 2.1. A subset W of a vector space V is called a subspace of V if W is itself a vector space under the addition and scalar multiplication defined on V.

Remark:
In general, to show that a nonempty set W with two operations is a vector space one must verify the ten vector space axioms. However, if W is a subspace of a known vector space V, then certain axioms need not be verified because they are inherited from V. In fact, we can only need to show that W is closed under addition and scalar multiplication, as shown in the next theorem.

Theorem 2.1. If W is a set of one or more vectors in a vector space V, then W is a subspace of V if and only if the following conditions are satisfied.

(a) If $\mathbf{u}$ and $\mathbf{v}$ are vectors in W, then $\mathbf{u}+\mathbf{v}$ is in W.

(b) If k is a scalar and $\mathbf{u}$ is a vector in W, then $k\mathbf{u}$ is in W.

Example 2.1. If V is any vector space, and if $W = \{0\}$ is the subset of V that consists of the zero vector only, then W is closed under addition and scalar multiplication since

$$0 + 0 = 0 \quad \text{and} \quad k0 = 0$$

for any scalar k. We call W the zero subspace of V.

Example 2.2. Consider the plane through the origin

$$P = \left\{ [x, y, z]^T \in \mathbb{R}^3, x + y + z = 0 \right\}.$$

Then, P is a subspace of $\mathbb{R}^3$.

Example 2.3. Let $F(-\infty, \infty)$ be the space of all functions defined from $\mathbb{R}$ to $\mathbb{R}$ under the operations:

$$(f_1 + f_2)(x) = f_1(x) + f_2(x), \qquad (r \cdot f)(x) = rf(x).$$

- The set of continuous functions on $(-\infty, \infty)$ together with the above operations is denoted by $C(-\infty, \infty)$.
- The set of functions with a continuous derivative on $(-\infty, \infty)$, i.e. set of continuously differentiable functions together with the above operations is denoted by $C^1(-\infty, \infty)$.
- The set of functions with m continuous derivative on $(-\infty, \infty)$ together with the above operations is denoted by $C^m(-\infty, \infty)$.
- The set of functions with derivatives of all orders on $(-\infty, \infty)$ together with the above operations is denoted by $C^\infty(-\infty, \infty)$.
- The set of polynomials together with the above operations is denoted by P_∞.
- The set of polynomials of degree n *or less* together with the above operations is denoted by P_n.

Then,
(1) $C(-\infty, -\infty)$ is a subspace of $F(-\infty, \infty)$ because a sum of continuous functions is continuous and a constant times a continuous function is continuous.

(2) $C^m(-\infty, -\infty)$ and $C^\infty(-\infty, -\infty)$ are subspaces of $F(-\infty, \infty)$ because the sum of two continuously differentiable functions is continuously differentiable and that a constant times a continuously differentiable function is continuously differentiable.

(3) P_∞ and P_n are subspaces of $F(-\infty, \infty)$ sum of two polynomials is a polynomial and that a constant times a polynomial is a polynomial (with the same degree or less).

Example 2.4. From the previous section, we have that $\mathcal{M}_{n\times n}$ is a vector space. The followings are subspaces of $\mathcal{M}_{n\times n}$ under the inherited addition and scalar multiplication operations from $\mathcal{M}_{n\times n}$:

- The set of symmetric,
- The sets of upper triangular matrices,
- The sets of lower triangular matrices,
- The sets of diagonal matrices.

Example 2.5. Let W be the set of all invertible matrix of size 2×2
Show that the subset W of $\mathcal{M}_{2\times 2}$ is not a subspace of $\mathcal{M}_{2\times 2}$ under the inherited addition and scalar multiplication operations from $\mathcal{M}_{n\times n}$.

Definition 2.2. If $\mathbf{w}$ is a vector in a vector space V, then $\mathbf{w}$ is said to be a **linear combination** of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ in V if w can be expressed in the form

$$\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r$$

where $c_1, c_2, \ldots, c_r$ are scalars. These scalars are called the **coefficients** of the linear combination.

Theorem 2.2. For a nonempty subset S of a vector space V, under the inherited operations the following are equivalent statements.

(1) S is a subspace of that vector space.

(2) S is closed under linear combinations of pairs of vectors, i.e. for any vectors $\mathbf{s}_1, \mathbf{s}_2 \in S$ and scalars r_1, r_2 the vector $r_1\mathbf{s}_1 + r_2\mathbf{s}_2$ is in S.

(3) S is closed under linear combinations of any number of vectors: for any vectors $\mathbf{s}_1, ..., \mathbf{s}_n \in S$ and scalars $r_1, ..., r_n$ the vector $r_1\mathbf{s}_1 + \cdots + r_n\mathbf{s}_n$ is an element of S.

The previous theorem suggests that a good way to think of a vector space is as a collection of unrestricted linear combinations.

Example 2.6. show that this plane through the origin subset of $\mathbb{R}^3$

$$S = \{[x, y, z]^T \mid x - 2y + z = 0\}$$

is a subspace under the usual addition and scalar multiplication operations of column vectors.
- This can be done by checking that it is nonempty and closed under linear combinations of two vectors.
- Alternatively,

Definition 2.3. The **span** (or linear closure) of a nonempty subset S of a vector space is the set of all linear combinations of vectors from S.

$$\text{span}(S) = \{c_1\mathbf{s}_1 + \cdots c_n\mathbf{s}_n \mid c_1, \ldots, c_n \in \mathbb{R}\}$$

The span of the empty subset of a vector space is its trivial subspace.
Notations for span of S: $\text{span}(S)$, $\text{sp}(S)$, $[S]$

Theorem 2.3. In a vector space, the span of any subset is a subspace.

Remarks:

- The converse of Theorem 2.3 holds: any subspace is the span of some set, because a subspace is obviously the span of itself, the set of all of its members. Thus a subset of a vector space is a subspace if and only if it is a span. This fits the intuition that a good way to think of a vector space is as a collection in which linear combinations are sensible.
- Taken together, Theorem 2.2 and Theorem 2.3 show that **the span of a subset S of a vector space is the *smallest* subspace containing all of the members of S.**

Example 2.7. Show that the span of the set $\{[1,1]^T, [1,-1]^T\}$ is all of $\mathbb{R}^2$.

Example 2.8. Show that the standard unit vectors in $\mathbb{R}^n$ which are

$$\mathbf{e}_1 = [1,0,0,...,0]^T, \mathbf{e}_2 = [0,1,0,...,0]^T, ..., \mathbf{e}_n = [0,0,0,...,1]^T,$$

span $\mathbb{R}^n$.

Example 2.9. Show that the polynomials $1, x, x^2, ..., x^n$ span the vector space P_n

Example 2.10. Consider the vectors $\mathbf{u} = [1, 2, -1]^T$ and $\mathbf{v} = [6, 4, 2]^T$ in $\mathbb{R}^3$. Show that $\mathbf{w} = [9, 2, 7]^T$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$ and that $\mathbf{z} = [4, -1, 8]^T$ is not a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

Example 2.11. Determine whether the vectors $\mathbf{v}_1 = [1, 1, 2]^T, \mathbf{v}_2 = [1, 0, 1]$, and $\mathbf{v}_3 = [2, 1, 3]^T$ span the vector space $\mathbb{R}^3$.

Theorem 2.4. The solution set of a homogeneous linear system $\mathbf{Ax} = 0$ of m equations in n unknowns is a subspace of Rn

Note: We will generally refer to the solution set of a homogeneous system as the solution space of the system, since it is a subspace of $\mathbb{R}^n$.

Example 2.12. Find the solution space for each of the following linear systems.

(a) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & -2 & 3 \\ 2 & -4 & 6 \\ 3 & -6 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$