OLS Matrix Approach

Scalar Notation

$$y_i = \beta_1 + \beta_2 x_{2i} + \beta_3 x_{3i} + \cdots + \beta_k x_{ki} + u_i$$

$$where \quad i = 1, 2, 3, \ldots, n$$

$$
\begin{aligned}
y_1 &= \beta_1 + \beta_2 x_{21} + \beta_3 x_{31} + \cdots + \beta_k x_{k1} + u_1 \\
y_2 &= \beta_1 + \beta_2 x_{22} + \beta_3 x_{32} + \cdots + \beta_k x_{k2} + u_2 \\
\vdots & \qquad\qquad\qquad \vdots \\
y_n &= \beta_1 + \beta_2 x_{2n} + \beta_3 x_{3n} + \cdots + \beta_k x_{kn} + u_n
\end{aligned}
$$

$$
\underset{n\times 1}{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}} = \underset{n\times k}{\begin{bmatrix} 1 & x_{21} & x_{31} & \cdots & x_{k1} \\ 1 & x_{22} & x_{32} & \cdots & x_{k2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{2n} & x_{3n} & \cdots & x_{kn} \end{bmatrix}} \cdot \underset{k\times 1}{\begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_k \end{bmatrix}} + \underset{n\times 1}{\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}}
$$

OLS Matrix Approach

$$\underset{n\times 1}{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}} = \underset{n\times k}{\begin{bmatrix} 1 & x_{21} & x_{31} & \cdots & x_{k1} \\ 1 & x_{22} & x_{32} & \cdots & x_{k2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{2n} & x_{3n} & \cdots & x_{kn} \end{bmatrix}} \cdot \underset{k\times 1}{\begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_k \end{bmatrix}} + \underset{n\times 1}{\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}}$$

Matrix Notation

$$\underset{n\times 1}{Y} = \underset{n\times k}{X}\ \underset{k\times 1}{\beta} + \underset{n\times 1}{u}$$

Least Squares

$$\hat{u}'\hat{u} = \begin{bmatrix} \hat{u}_1 & \hat{u}_2 & \cdots & \hat{u}_n \end{bmatrix} \begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \vdots \\ \hat{u}_n \end{bmatrix} = \sum_{i=1}^{n} \hat{u}_i^2$$

OLS Matrix Approach

From $$Y = X\beta + u$$

$$\hat{u} = Y - X\hat{\beta}$$

Least Squares

$$\hat{u}'\hat{u} = \left(Y - X\hat{\beta}\right)'\left(Y - X\hat{\beta}\right)$$

$$= Y'Y - 2\hat{\beta}'X'Y + \hat{\beta}'X'X\hat{\beta}$$

$$\frac{\partial \hat{u}'\hat{u}}{\partial \hat{\beta}} = -2X'Y + 2X'X\hat{\beta} = 0$$

$$\underset{k\times 1}{\hat{\beta}} = (\underset{k\times k}{X'X})^{-1}\underset{k\times n}{X'}\underset{n\times 1}{Y}$$

Assume normal distribution: $u \sim N(0, \sigma^2 I)$

Assumptions of Multiple Regression

1. Fixed regressor – X is non-stochastic
2. Random disturbances and zero mean – $E[u]=0$
3. Homoskedasticity – $E[uu']=\sigma^2 I$
4. No correlation – $E[uu']=\sigma^2 I$
5. Constant parameters
6. Linear model – $y = X\beta + u$
7. Normality – $u \sim N(0,\sigma^2 I)$

Least Squares Properties

Unbiased

$$E[\hat{\beta}] = E[\beta + (X'X)^{-1}X'u] = \beta + (X'X)^{-1}E[X'u] = \beta$$

Best Linear Unbiased Estimators (BLUE)

$$\begin{aligned}\mathrm{var}[\hat{\beta}] &= E[(\hat{\beta}-\beta)(\hat{\beta}-\beta)'] = E[(X'X)^{-1}X'uu'X(X'X)^{-1}] \\ &= \sigma^2 (X'X)^{-1}\end{aligned}$$

Gauss-Markov Theorem

$$\mathrm{var}[\hat{\beta}] \leq \mathrm{var}[b]$$

Estimate Disturbance Variance

From $E[u]=0$ and $\text{var}[u]=E[uu']=\sigma^2 I$

$$s^2=\frac{\hat{u}'\hat{u}}{n-k}$$

Coefficient of Determination: R^2

$$R^2=\frac{ESS}{TSS}=\frac{\hat{\beta}'X'NX\hat{\beta}}{y'Ny}=1-\frac{\hat{u}'\hat{u}}{y'Ny}=1-\frac{RSS}{TSS}$$

$$\underset{n\times n}{N}=I-\frac{1}{n}\iota\iota',\ \underset{n\times 1}{\iota}=\begin{bmatrix}1 & 1 & \cdots & 1\end{bmatrix}'$$

$$ESS=\hat{\beta}'X'\underset{n\times n}{N}X\hat{\beta}=\sum_{i=1}^{n}\left(\hat{y}_i-\bar{y}\right)^2,\ TSS=y'\underset{n\times n}{N}y=\sum_{i=1}^{n}\left(y_i-\bar{y}\right)^2$$

Adjusted R^2

$$\bar{R}^2=1-\frac{\hat{u}'\hat{u}/(n-k)}{y'Ny/(n-1)}=1-\frac{n-1}{n-k}(1-R^2)$$

Accuracy of Estimates

t-test $H_0 : \beta_j = 0$ and $H_a : \beta_j \neq 0$

From assumptions 1-7:

$$\hat{\beta} \sim N(\beta, \sigma^2 (X'X)^{-1})$$

Then, standardized b_j will result:

$$\frac{\hat{\beta}_j - \beta_j}{\sigma \sqrt{a_{jj}}} \sim N(0,1)$$

Since σ^2 is unknown, we use s^2 instead, then:

$$t_j = \frac{\hat{\beta}_j - \beta_j}{s_j} = \frac{\hat{\beta}_j - \beta_j}{s \sqrt{a_{jj}}} \sim t(n-k)$$

The F-test

Joint of More Than One Restriction Test

Unrestricted Model: $y = X_1\beta_1 + X_2\beta_2 + u = X\beta + u$

Restricted Model: $y = X_1\beta_R + u_R$

Hypothesis: $H_0 : \beta_2 = 0$

General Form: $F = \dfrac{(\hat{u}_R'\hat{u}_R - \hat{u}'\hat{u})/m}{\hat{u}'\hat{u}/(n-k)} \sim F(m, n-k)$

$$F = \frac{(R\hat{\beta} - r)'[R(X'X)^{-1}R']^{-1}(R\hat{\beta} - r)/m}{\hat{u}'\hat{u}/(n-k)} \sim F(m, n-k)$$

The F-test

Example: More Than One Restriction Test

Unrestricted Model:

$$y_i = \beta_1 + \beta_2 x_{2i} + \beta_3 x_{3i} + \beta_4 x_{4i} + \beta_5 x_{5i} + u_i = X\beta + u$$

Restricted Model: $y_i = \beta_1 + \beta_2 x_{2i} + \beta_3 x_{3i} + u_i$

Hypothesis: $H_0 : \beta_4 = \beta_5 = 0$

Restriction: $R = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad r = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$F = \frac{(R\hat{\beta} - r)'[R(X'X)^{-1}R']^{-1}(R\hat{\beta} - r)/m}{\hat{u}'\hat{u}/(n-k)} \sim F(m, n-k)$$