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1. Two individuals agree at date 0 to a forward contract that matures at date 2. The contract is written on an underlying asset that pays a dividend at date 1 equal to D_1. Let f_2 be the date 2 random payoff (profit) to the individual who is the long party in the forward contract. Also let m_{0i} be the stochastic discount factor over the period from dates 0 to i where $i = 1, 2$, and let $E_0[\cdot]$ be the expectations operator at date 0. What is the value of $E_0[m_{02} f_2]$? Explain your answer.

Let S_i be the price of the underlying asset at date i
Let D_0 be the dividend payment between date 0 and 1

Pricing using a stochastic discount factor : $S_0 = E_0[m_{01} D_1] + E_0[m_{02} S_2]$
$= D_0 + E_0[m_{02} S_2]$ —— ①

Then, let F_{02} be the forward price, the payoff to the long party : $f_2 = S_2 - F_{02}$

Under a stochastic discount factor approach: $E_0[m_{02} f_2] = E_0[m_{02}(S_2 - F_{02})]$
$= E_0[m_{02} S_2] - E_0[m_{02} F_{02}]$ —— ②

$E_0[m_{02} F_{02}] = E_0[m_{02}] F_{02} = R_f^{-2} F_{02}$; $E[m_{01}] = \frac{1}{R_f}$ —— ③

Put ① and ③ into ②; $E_0[m_{02} f_2] = S_0 - D_0 - R_f^{-2} F_{02}$

∴ According to the absence of arbitrage, forward price satisfies
$F_{02} = R_f^2 (S_0 - D_0)$; $E_0[m_{02} f_2] = 0$

2. Assume that there is an economy populated by infinitely-lived representative individuals who maximize the lifetime utility function

$$E_0\left[\sum_{t=0}^{\infty} -\delta^t e^{-ac_t}\right]$$

where c_t is consumption at date t and $a > 0$, $0 < \delta < 1$. The economy is a Lucas (1978) endowment economy having multiple risky assets paying date t dividends that total d_t per capita. Write down an expression for the equilibrium per capita price of the market portfolio in terms of the assets' future dividends.

$P_0 = E_0\left[\sum_{t=1}^{\infty} \frac{U_c(C_t,t)}{U_c(C_0,0)} d_t\right]$

$U(C_t,t) = -\delta^t e^{-aC_t}$
$U_c(C_t,t) = a\delta^t e^{-aC_t}$

Since it's an endowment economy,
$C_t = d_t$

So, $P_0 = E_0\left[\sum_{t=1}^{\infty} \delta^t e^{-a(d_t - d_0)} d_t\right]$ #

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3. For the Lucas model with labor income, show that assumptions (6.25) and (6.26) lead to the pricing relationship (6.27) and (6.28).

since $\quad c_t = d_t + y_t$

then, $\quad P_t = E_t\left[\Sigma_{j=1}^{\infty} \delta^j \left(\frac{c_{t+1}}{c_t}\right)^{\gamma-1} d_{t+j}\right]$

$$\frac{P_t}{d_t} = E_t\left[\Sigma_{j=1}^{\infty} \delta^j \left(\frac{c_{t+1}}{c_t}\right)^{\gamma-1} \frac{d_{t+j}}{d_t}\right]$$

$$= E_t\left[\Sigma_{j=1}^{\infty} \delta^j e^{(\gamma-1)\ln(c_{t+j}/c_t) + \ln(d_{t+j}/d_t)}\right]$$

Now, $\quad \ln(c_{t+j}/c_t) = \mu_c + \sigma_c u_{t+j}$

$\ln(d_{t+j}/d_t) = \mu_d + \sigma_d \varepsilon_{t+j}$

So, $\quad \frac{P_t}{d_t} = E_t\left[\Sigma_{j=1}^{\infty} \delta^j e^{(\gamma-1)(\mu_c + \sigma_c u_{t+j}) + (\mu_d + \sigma_d \varepsilon_{t+j})}\right]$

$$= \Sigma_{j=1}^{\infty} \delta^j e^{(\gamma-1)\mu_c j + 1/2(\gamma-1)^2\sigma_c^2 j + \mu_d j + 1/2\sigma_d^2 j + (1/2)(2)(\gamma-1)\sigma_c\sigma_d\rho \cdot j}$$

$$= \Sigma_{j=1}^{\infty} e^{j[\ln\delta - (1-\gamma)\mu_c + \mu_d + 1/2(1-\gamma)^2\sigma_c^2 + 1/2\sigma_d^2 - (1-\gamma)\sigma_c\sigma_d\rho}$$

$$= \frac{1}{1-\delta e^{\mu_d - (1-\gamma)\mu_c + \frac{1}{2}[(1-\gamma)^2\sigma_c^2 + \sigma_d^2] - (1-\gamma)\rho\sigma_c\sigma_d}} - 1$$

$$P_t = d_t \frac{\delta e^{\alpha}}{1-\delta e^{\alpha}}$$

where $\quad \alpha = \mu_d - (1-\gamma)\mu_c + \frac{1}{2}\left[(1-\gamma)^2\sigma_c^2 + \sigma_d^2\right] - (1-\gamma)\rho\sigma_c\sigma_d$ #

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4. Consider a special case of the model of rational speculative bubbles discussed in this chapter. Assume that infinitely-lived investors are risk-neutral and that there is an asset paying a constant, one-period risk-free return of $R_f = \delta^{-1} > 1$. There is also an infinitely-lived risky asset with price p_t at date t. The risky asset is assumed to pay a dividend of d_t which is declared at date t and paid at the end of the period, date $t+1$. Consider the price $p_t = f_t + b_t$ where

$$f_t = \sum_{i=0}^{\infty} \frac{E_t\left[d_{t+i}\right]}{{R_f}^{i+1}} \tag{1}$$

and

$$b_{t+1} = \begin{cases} \frac{R_f}{q_t} b_t + e_{t+1} & \text{with probability } q_t \\ z_{t+1} & \text{with probability } 1 - q_t \end{cases} \tag{2}$$

where $E_t\left[e_{t+1}\right] = E_t\left[z_{t+1}\right] = 0$ and where q_t is a random variable as of date $t-1$ but realized at date t and is uniformly distributed between 0 and 1.

4.a Show whether or not $p_t = f_t + b_t$ subject to the specifications in (1) and (2) is a valid solution for the price of the risky asset.

Check whether (2) satisfies $E_t[b_{t+1}] = R_f b_t$

$$E_t[b_{t+1}] = \frac{R_f}{q_t} b_t q_t + E_t[e_{t+1}]q_t + (1-q_t) E_t[z_{t+1}] = R_f b_t$$

$\therefore$ It is a valid solution.

4.b Suppose that p_t is the price of a barrel of oil. If $p_t \geq p_{solar}$, then solar energy, which is in perfectly elastic supply, becomes an economically efficient perfect substitute for oil. Can a rational speculative bubble exist for the price of oil? Explain why or why not.

Since $E_t[b_{t+1}] = R_f b_t$

$$\lim_{i \to \infty} E_t[b_{t+i}] = \begin{cases} +\infty & \text{if } b_t > 0 \\ -\infty & \text{if } b_t < 0 \end{cases} \quad \text{— } ①$$

Price cannot be negative, so we consider only $b_t > 0$

From ①, the bubble component must be expected to increase infinitely. [Irrational]
Thus, p_t cannot be exceeded p_{solar}, b_t cannot rise above $p_{solar} - p_t^*$. Therefore, a bubble path where b_t must be expected to increase to infinity cannot possibly occur. #

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4.c Suppose p_t is the price of a bond that matures at date $T < \infty$. In this context, the d_t for $t \leq T$ denotes the bond's coupon and principal payments. Can a rational speculative bubble exist for the price of this bond? Explain why or why not.

Similar to 4.b, a rational speculative bubble cannot exist for the price of a bond. Since, at maturity, p_t must be equalled to d_t and zero after date T, its price cannot rationally be expected to satisfy 1 and increase infinitely.

$\therefore$ Bubble path is invalid, and only rational price is $p_t = p_t^*$ #

5. Consider an endowment economy with representative agents who maximize the following objective function:

$$\max_{C_s, \{\omega_{is}\}, \forall s, i} E_t \left[\sum_{s=t}^{T} \delta^s u(C_s) \right]$$

where $T < \infty$. Explain why a rational speculative asset price bubble could not exist in such an economy.

Refer to $p_t = f_t + b_t$ with $b_t \neq 0$

This is because at date T, $p_T = f_T = d_T$ which is an asset's final dividend payment. Since $b_T = 0$ with certainty, then the bubble process $E_t[b_{t+1}] = \delta^{-1} b_t$

then $\quad E_{T-1}[b_T] = E_{T-1}[0] = \delta^{-1} b_{T-1}$

or $\quad b_{T-1} = 0$

$\therefore$ Similar argument implies $b_t = 0$ for all previous dates, $t < T-1$. #