Indefinite Integral

1 The Indefinite Integral: Introduction

Definition 1.1 (Antiderivative). A function F is said to be an **antiderivative** of a function f on some interval I if

$$F'(x) = f(x), \qquad \forall x \in I.$$

Example 1.1. Find antiderivatives of the following functions.

1. $f(x) = 2x + 5$
2. $f(x) = x^4 + 3x - 9$
3. $f(x) = \cos(x) + e^x$

Notice that there are many antiderivatives of a function $f(x)$ and these antiderivatives only differ by a constant. In particular, if $G'(x) = F'(x)$ for all $x \in [a, b]$, then

$$G(x) = F(x) + C. \qquad \forall x \in [a, b],$$

where C is any constant number.

Definition 1.2 (Indefinite integral). Suppose $F(x)$ is any anti-derivative of f (i.e. $F'(x) = f(x)$). The most general anti-derivative of $f(x)$ is called an **indefinite integral** and denoted,

$$\int f(x)dx = F(x) + C$$

In this definition the $\int$ is called the **integral symbol**, $f(x)$ is called the **integrand**, x is called the **integration variable** and the C is called the **constant of integration**.

Theorem 1.1 (Properties of Indefinite Integral)**.** Let $F'(x) = f(x)$ and $G'(x) = g(x)$. Then

1. $\int kf(x)dx = k\int f(x)dx$,
2. $\int [f(x) + g(x)]dx \;=\; \int f(x)dx + \int g(x)dx \;=\; F(x) + G(x) + C,$
3. $\int [f(x) - g(x)]dx \;=\; \int f(x)dx - \int g(x)dx \;=\; F(x) - G(x) + C.$

Derivatives	Integrals (Antiderivatives)	Derivatives	Integrals (Antiderivatives)
$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n,, n \neq -1$	$\int x^n dx = \frac{x^{n+1}}{n+1} + C$	$\frac{d}{dx}x = 1$	$\int dx = x + C$
$\frac{d}{dx}\sin(x) = \cos(x)$	$\int \cos(x)dx = \sin(x) + C$	$\frac{d}{dx}[-\cos(x)] = \sin(x)$	$\int \sin(x)dx = -\cos(x) + C$
$\frac{d}{dx}\tan(x) = \sec^2(x)$	$\int \sec^2(x)dx = \tan(x) + C$	$\frac{d}{dx}[-\cot(x)] = \csc^2(x)$	$\int \csc^2(x)dx = -\cot(x) + C$
$\frac{d}{dx}[e^x] = e^x$	$\int e^x dx = e^x + C$	$\frac{d}{dx}(b^x) = b^x\ln(b)$ $b > 0, b \neq 1$	$\int b^x dx = \frac{1}{\ln(b)}b^x + C$
$\frac{d}{dx}(\ln\lvert x\rvert) = \frac{1}{x}$	$\int \frac{1}{x}dx = \ln\lvert x\rvert + C$		

Example 1.2. Evaluate each of the following indefinite integrals.

1. $\int dx$
2. $\int \sqrt{x}dx$
3. $\int \left(4\cos(x) + e^x + \frac{3}{x}\right)dx$
4. $\int \frac{4x^3-1}{x}dx$

We can also use integration to solve differential equations in a certain form as shown in this example.

Example 1.3. Find a function $y = f(x)$ that passes through $(1, 2)$ for the differential equation

$$\frac{dy}{dx} = 3x^2 - 3.$$

2 Integration by the u-Substitution

In general, we may not have the integrand in the same form that we know its anti-derivative directly from the table. We can change a variable to transform the integrand first before integrating it.

Theorem 2.1 (u-Substitution Rule)**.** If $u = g(x)$ is a differentiable function where range is an interval I. Let f be a function that is continuous on I and F be an antiderivative of f on I. Then

$$\int f(g(x))g'(x)dx = \int f(u)du.$$

Guidelines for Using a u-Substitution

1. Transform the given integral in the form $\int f(g(x))g'(x)dx$ by identifying the function $g(x)$ and $g'(x)$.
2. Let $u = g(x)$ and $du = g'(x)$. Then express the integral *entirely* in terms of the symbol u. There should be no variable x left in the integral.
3. Carry out the integration with respect to the variable u.
4. Finally, re-substitute $g(x)$ for the symbol u.

Indefinite integral of a Power of a function

Consider a general form $f(x) = (g(x))^n$ for some constant $n \neq -1$. Then we will let $u = g(x)$ and $du = g'(x)dx$

$$\int [g(x)]^n g'(x)dx = \int u^n du \;=\; \frac{u^{n+1}}{n+1} + C \;=\; \frac{g(x)^{n+1}}{n+1} + C.$$

Example 2.1. Evaluate

$$\int (3x+4)^{10}dx.$$

Example 2.2. Evaluate

$$\int \frac{x}{(x^2+1)^{5/2}}\,dx.$$

Example 2.3. Evaluate

$$\int \frac{x^2}{(x^3+1)^4}\,dx.$$

Example 2.4. Evaluate

$$\int \sin^5(x)\cos(x)dx.$$

Indefinite integral of trigonometric functions

Consider a general form $f(x) = \cos(g(x))g'(x)$ or $f(x) = \sin(g(x))g'(x)$. We will use $u = g(x)$ and

$$\frac{d}{dx}\sin(u) = \cos(u)\frac{du}{dx} \qquad \text{and } \frac{d}{dx}[-\cos(u)] = \sin(u)\frac{du}{dx}$$

$$\int \cos(g(x))g'(x)dx = \int \cos(u)\frac{du}{dx}dx = \int \cos(u)du = \sin(u) + C,$$

$$\int \sin(g(x))g'(x)dx = \int \sin(u)\frac{du}{dx}dx = \int \sin(u)du = -\cos(u) + C,$$

Example 2.5. Evaluate

$$\int \sin(5x)dx.$$

Example 2.6. Evaluate
$$\int \csc^2(2-3x)dx.$$

Example 2.7. Evaluate
$$\int \frac{3}{2+e^{3x}}dx.$$

Derivatives	Integrals (Antiderivatives)	
$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n,, n \neq 1$	$\int x^n dx = \frac{x^{n+1}}{n+1} + C$	$\int u^n du = \frac{u^{n+1}}{n+1} + C$
$\frac{d}{dx} x = 1$	$\int dx = x + C$	$\int du = u + C$
$\frac{d}{dx}\sin(x) = \cos(x)$	$\int \cos(x)dx = \sin(x) + C$	$\int \cos(u)du = \sin(u) + C$
$\frac{d}{dx}[-\cos(x)] = \sin(x)$	$\int \sin(x)dx = -\cos(x) + C$	$\int \sin(u)du = -\cos(u) + C$
$\frac{d}{dx}\tan(x) = \sec^2(x)$	$\int \sec^2(x)dx = \tan(x) + C$	$\int \sec^2(u)du = \tan(u) + C$
$\frac{d}{dx}[-\cot(x)] = \csc^2(x)$	$\int \csc^2(x)dx = -\cot(x) + C$	$\int \csc^2(u)du = -\cot(u) + C$
$\frac{d}{dx}[\sec(x)] = \sec(x)\tan(x)$	$\int \sec(x)\tan(x)dx = \sec(x) + C$	$\int \sec(u)\tan(u)du = \sec(u) + C$
$\frac{d}{dx}[-\csc(x)] = \csc(x)\cot(x)$	$\int \csc(x)\cot(x)dx = -\csc(x) + C$	$\int \csc(u)\cot(u)du = -\csc(u) + C$
$\frac{d}{dx}[\sin^{-1}(x)] = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1}(x) + C$	$\int \frac{1}{\sqrt{1-u^2}}du = \sin^{-1}(u) + C$ ***
$\frac{d}{dx}[-\cos^{-1}(x)] = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}}dx = -\cos^{-1}(x) + C$	$\int \frac{1}{\sqrt{1-u^2}}du = -\cos^{-1}(u) + C$
$\frac{d}{dx}[\tan^{-1}(x)] = \frac{1}{1+x^2}$	$\int \frac{1}{1+x^2}dx = \tan^{-1}(x) + C$	$\int \frac{1}{1+u^2}du = \tan^{-1}(u) + C$ ***
$\frac{d}{dx}[-\cot^{-1}(x)] = \frac{1}{1+x^2}$	$\int \frac{1}{1+x^2}dx = -\cot^{-1}(x) + C$	$\int \frac{1}{1+u^2}du = -\cot^{-1}(u) + C$
$\frac{d}{dx}[sec^{-1}(x)] = \frac{1}{x\sqrt{x^2-1}}$	$\int \frac{1}{x\sqrt{x^2-1}}dx = \sec^{-1}(\lvert x\rvert) + C$	$\int \frac{1}{u\sqrt{u^2-1}}du = \sec^{-1}(\lvert u\rvert) + C$ ***
$\frac{d}{dx}[-\csc^{-1}(x)] = \frac{1}{x\sqrt{x^2-1}}$	$\int \frac{1}{x\sqrt{x^2-1}}dx = -\csc^{-1}(x) + C$	$\int \frac{1}{u\sqrt{u^2-1}}du = -\csc^{-1}(u) + C$
$\frac{d}{dx}\sinh(x) = \cosh(x)$	$\int \cosh(x)dx = \sinh(x) + C$	$\int \cosh(u)du = \sinh(u) + C$
$\frac{d}{dx}\cosh(x) = \sinh(x)$	$\int \sinh(x)dx = \cosh(x) + C$	$\int \sinh(u)du = \cosh(u) + C$
$\frac{d}{dx}[e^x] = e^x$	$\int e^x dx = e^x + C$	$\int e^u du = e^u + C$
$\frac{d}{dx}(b^x) = b^x\ln(b)$ $b > 0, b \neq 1$	$\int b^x dx = \frac{1}{\ln(b)}b^x + C$	$\int b^u du = \frac{1}{\ln(b)}b^u + C$
$\frac{d}{dx}(\ln\lvert x\rvert) = \frac{1}{x}$	$\int \frac{1}{x}dx = \ln\lvert x\rvert + C$	$\int \frac{1}{u}du = \ln\lvert u\rvert + C$

Alternative formulas related to inverse trigonometric functions(***) in the above table are given below.

$$\int \frac{1}{\sqrt{a^2-u^2}}du = \sin^{-1}\left(\frac{u}{a}\right) + C$$
$$\int \frac{1}{u^2+a^2}du = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$$
$$\int \frac{1}{u\sqrt{u^2-a^2}}du = \frac{1}{a}\sec^{-1}\left(\left|\frac{u}{a}\right|\right) + C$$

Special Trigonometric Integrals

$$\int \tan(u)du = -\ln(|cos(u)|) + C$$

$$\int \cot(u)du = \ln(|\sin(u)|) + C$$

$$\int \sec(u)du = \ln(|\sec(u) + \tan(u)|) + C$$

$$\int \csc(u)du = \ln(|\csc(u) - \cot(u)|) + C$$

Example 2.8. Evaluate

$$\int \frac{1}{\sqrt{9-x^2}}dx.$$

Example 2.9. Evaluate

$$\int \frac{[\tan^{-1}(x)]^2}{1+x^2}dx.$$

Example 2.10 (Exercise)**.** Evaluate the following integral.

$$\int \left(\frac{1}{x-1} + \frac{1}{2x+1} + \frac{4}{x+3} \right) \, dx$$

Ans: $\ln(|x-1|) + \frac{1}{2}\ln(|2x+1|) + 4\ln(|x+3|) + C$

3 Exercises

1. Evaluate the given indefinite integrals (these may require to use the substitution rule).

   (a) $\int \frac{(x-3)^2}{\sqrt{x}} \, dx$

   (b) $\int \left[e^x + \frac{1}{x} - \sin(x) + 3\sinh(x) + \pi^2 + \csc^2(x) - \sec(x)\tan(x)\right] \, dx$

   (c) $\int \sqrt{2+3x} \, dx$

   (d) $\int \frac{2\cos(\ln(x))}{x} \, dx$

   (e) $\int e^{5x}(1-e^{5x})^{1/3} \, dx$

   (f) $\int \frac{x}{9+4x^2} \, dx$

   (g) $\int \frac{1}{9+4x^2} \, dx$

   (h) $\int \frac{1-x}{\sqrt{3-x^2}} \, dx$

   (i) $\int \left[\sin^3(x)\cos(x) + \frac{\sec^2(x)}{\sqrt{\tan(x)}}\right] \, dx$

   (j) $\int [\tan(\pi x) + \sec(2\pi x)\ ] \, dx$

   (k) $\int \frac{1}{[1+x^2][\tan^{-1}(x)]} \, dx$

   (l) $\int 2x^3\sqrt{1+x^2} \, dx$

   (m) $\int e^{x+e^x} \, dx$

2. Determine the function $f(x)$ such that $f''(x) = (1+x)^4$, $f'(0) = 0$ and $f(0) = 0$.