4 Testing Hypotheses about a Single Linear Combination of the Parameter

Consider

$$\log(wage) = \beta_0 + \beta_1 jc + \beta_2 univ + \beta_3 \exp er + u$$

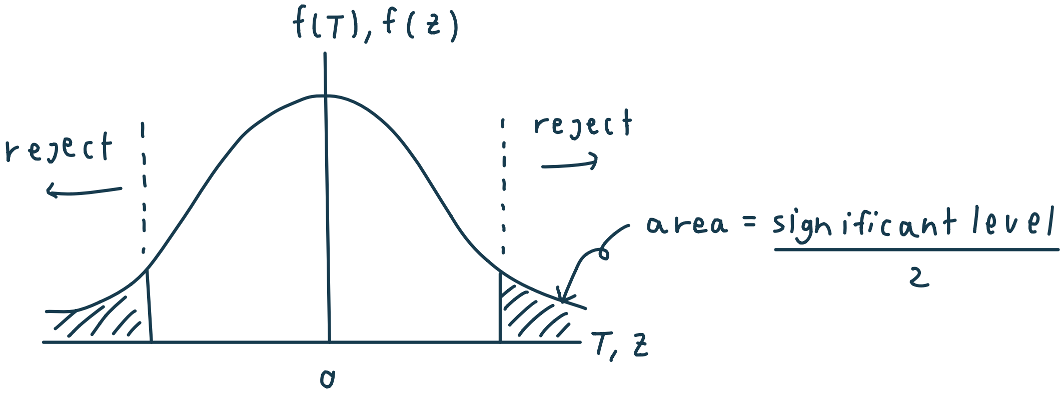
where jc = number of years attending a two-year college
$univ$ = number of years at a four-year college
$\exp er$ = months in the workforce.
We want to test whether $\beta_1 = \beta_2$. → if the return from 1 more year of education at a junior college is the same as that of the university

$H_0: \beta_1 = \beta_2 \Rightarrow H_0: \beta_1 - \beta_2 = 0$

against

$H_a: \beta_1 \neq \beta_2 \Rightarrow H_a: \beta_1 - \beta_2 \neq 0$

2-tailed test



$$t = \frac{(\hat{\beta}_1 - \hat{\beta}_2) - 0}{s.e.(\hat{\beta}_1 - \hat{\beta}_2)}$$

we compute this t-statistic and compare with the critical value

where $s.e.(\hat{\beta}_1 - \hat{\beta}_2) = \sqrt{\widehat{var}(\hat{\beta}_1 - \hat{\beta}_2)}$

$$= \sqrt{\widehat{var}(\hat{\beta}_1) + \widehat{var}(\hat{\beta}_2) - 2\widehat{cov}(\hat{\beta}_1, \hat{\beta}_2)}$$

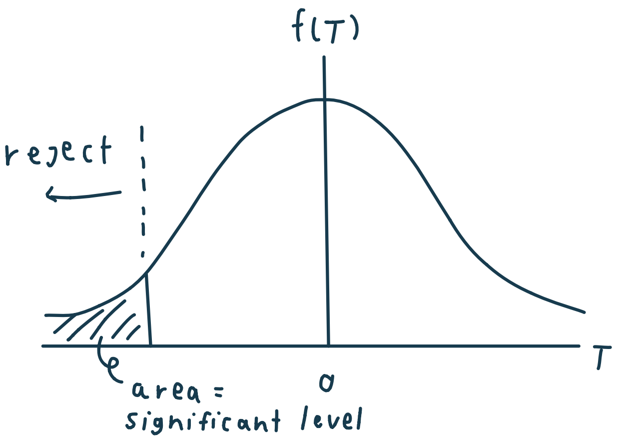
not very straight forward to calculate
→ we use a variable transformation trick → see notes

another possible hypothesis test (one-tailed alternative)

$H_0 : \beta_1 = \beta_2 \Rightarrow H_0 : \beta_1 - \beta_2 = 0$

$H_a : \beta_1 < \beta_2 \Rightarrow H_a : \beta_1 - \beta_2 < 0$

- It is assumed that β_1 would not be more than β_2 (returns to a 2-year college would never be more than returns to university education).

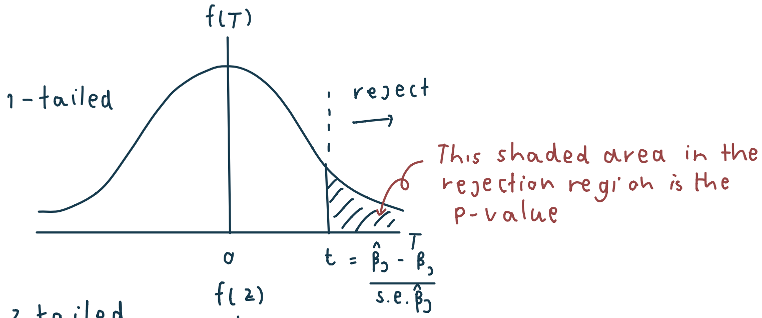


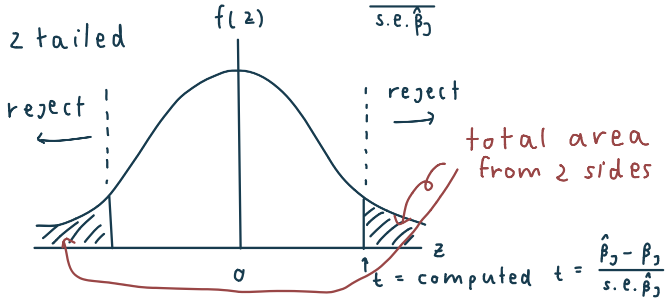
$$t = \frac{(\hat{\beta}_1 - \hat{\beta}_2) - 0}{s.e.(\hat{\beta}_1 - \hat{\beta}_2)}$$

* Then go the extra note

5 Computing p-Values for t-Tests

- What is the significance level given the computed t-statistics?





- p-value : $P(|T| > |t|)$

T = t-distributed random variable with d.f. $= n-k-1$

t = computed t-statistic

$\rightsquigarrow$ p-value = probability that a random T value will be greater (in the $|\ |$ term) than our t in the H_0 test.

EXTRA NOTES

In-class exercise

consider the multiple regression model, assume

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + u$$

you would like to test $H_0 : \beta_1 - 3\beta_2 = 1$
H_a : otherwise is true

1st) write the t-statistic for testing H_0

$$t = \frac{(\hat{\beta}_1 - 3\hat{\beta}_2) - 1}{s.e.(\hat{\beta}_1 - 3\hat{\beta}_2)}$$

2nd) Define $\theta = \hat{\beta}_1 - 3\hat{\beta}_2 \Rightarrow H_0 : \theta_1 = 1,\ H_a : \theta_1 \neq 1$

$$t = \frac{\hat{\theta}_1 - 1}{s.e.(\hat{\theta}_1)}$$

$\Rightarrow$ we need our regression to have θ_1 in it. So, STATA or OLS estimation will automatically give $\hat{\theta}_1$ & s.e. $\hat{\theta}_1$

Now, $\hat{\beta}_1 = \hat{\theta}_1 + 3\hat{\beta}_2$

$\beta_1 = \theta_1 + 3\beta_2$

substitute in the main regression and get

$$y = \beta_0 + (\theta_1 + 3\beta_2)x_1 + \beta_2 x_2 + \beta_3 x_3 + u$$

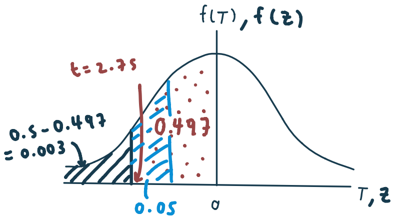
$$= \beta_0 + \theta_1 x_1 + 3\beta_2 x_2 + \beta_2 x_2 + \beta_3 x_3 + u$$

$$= \beta_0 + \theta_1 x_1 + \beta_2 (x_2 + 3x_1) + \beta_3 x_3 + u$$

* Now, the explanatory variables are going to be x_1, $x_2 + 3x_1$, and x_3

- we can calculate $t = \dfrac{\hat{\theta}_1 - 1}{s.e.\ \hat{\theta}_1}$

Example 1: $H_0 : \beta_j \geq 0$, $H_a : \beta_j < 0$, d.f.= 140. → z- table



→ p-value = what should be the significant level, given the critical value of -2.75 ??

⇒ find the shaded area!

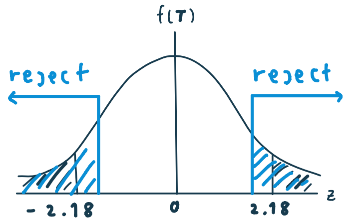
suppose the calculated $t_{\hat{\beta}_j} = -2.75$

↳ $t_{\hat{\beta}_j} = (\hat{\beta}_j - \beta_j) / s.e.(\hat{\beta}_j)$

- From the z-table, the value -2.75 corresponds to area = 0.003.
- Thus, p-value = 0.003.
- Would we reject H_0 if we use the significance level = 5%? Yes

* RULE! we reject H_0 if p-value < sig value

Example 2: $H_0 : \beta_j = a_j$, $H_a : \beta_j \neq a_j$, d.f.= 18. ← use t table



suppose the calculated $t_{\hat{\beta}_j} = -2.18$

- From the t-table, the value -2.18 corresponds to area = 0.02 to 0.05.
- Thus, p-value = is between 0.02 - 0.05.
- Would we reject H_0 if we use the significance level = 5%?

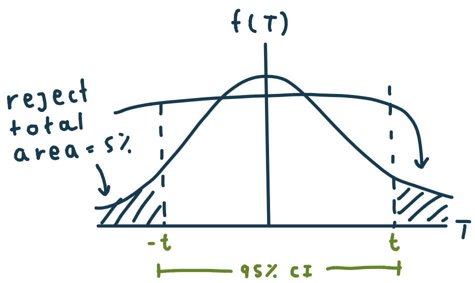
yes, reject H_0 because the area is less than 0.05 or p-value < 0.45

6 Confidence Intervals (CI)

- Confidence Intervales for the POPULATION PARAMETER (β_j)

↳ The range of values that would capture the true β_j at a 95% chance

- A 95% CI of β_j is given by



CI ⇒ $\hat{\beta}_j \pm c \times s.e.(\hat{\beta}_j)$

c is the $\boxed{97.5}$ percentile in the t-distribution with n-k-1 d.f.

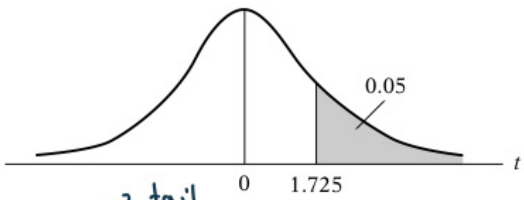
TABLE D.2 PERCENTAGE POINTS OF THE t DISTRIBUTION

Example

$\Pr(t > 2.086) = 0.025$

$\Pr(t > 1.725) = 0.05$ for df $= 20$

$\Pr(|t| > 1.725) = 0.10$



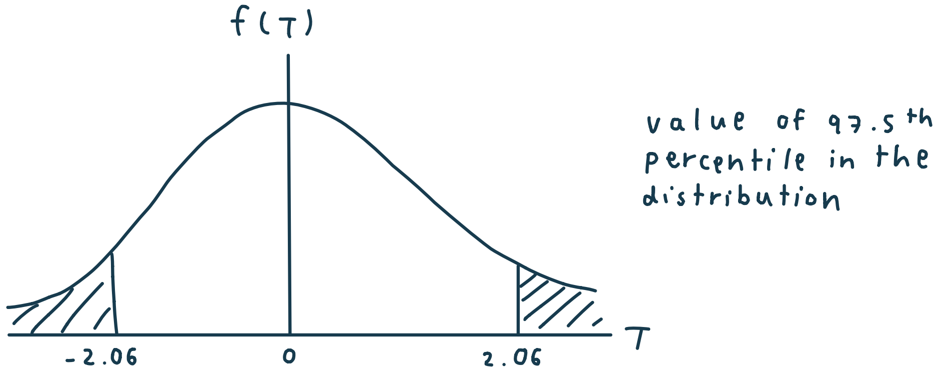
2 tail
1 tail

df \ Pr	0.25 0.50	0.10 0.20	0.05 0.10	0.025 0.05	0.01 0.02	0.005 0.010	0.001 0.002
1	1.000	3.078	6.314	12.706	31.821	63.657	318.31
2	0.816	1.886	2.920	4.303	6.965	9.925	22.327
3	0.765	1.638	2.353	3.182	4.541	5.841	10.214
4	0.741	1.533	2.132	2.776	3.747	4.604	7.173
5	0.727	1.476	2.015	2.571	3.365	4.032	5.893
6	0.718	1.440	1.943	2.447	3.143	3.707	5.208
7	0.711	1.415	1.895	2.365	2.998	3.499	4.785
8	0.706	1.397	1.860	2.306	2.896	3.355	4.501
9	0.703	1.383	1.833	2.262	2.821	3.250	4.297
10	0.700	1.372	1.812	2.228	2.764	3.169	4.144
11	0.697	1.363	1.796	2.201	2.718	3.106	4.025
12	0.695	1.356	1.782	2.179	2.681	3.055	3.930
13	0.694	1.350	1.771	2.160	2.650	3.012	3.852
14	0.692	1.345	1.761	2.145	2.624	2.977	3.787
15	0.691	1.341	1.753	2.131	2.602	2.947	3.733
16	0.690	1.337	1.746	2.120	2.583	2.921	3.686
17	0.689	1.333	1.740	2.110	2.567	2.898	3.646
18	0.688	1.330	1.734	$\boxed{2.101}$	$\boxed{2.552}$	2.878	3.610
19	0.688	1.328	1.729	2.093	2.539	2.861	3.579
20	0.687	1.325	1.725	2.086	2.528	2.845	3.552
21	0.686	1.323	1.721	2.080	2.518	2.831	3.527
22	0.686	1.321	1.717	2.074	2.508	2.819	3.505
23	0.685	1.319	1.714	2.069	2.500	2.807	3.485
24	0.685	1.318	1.711	2.064	2.492	2.797	3.467
25	0.684	1.316	1.708	2.060	2.485	2.787	3.450
26	0.684	1.315	1.706	2.056	2.479	2.779	3.435
27	0.684	1.314	1.703	2.052	2.473	2.771	3.421
28	0.683	1.313	1.701	2.048	2.467	2.763	3.408
29	0.683	1.311	1.699	2.045	2.462	2.756	3.396
30	0.683	1.310	1.697	2.042	2.457	2.750	3.385
40	0.681	1.303	1.684	2.021	2.423	2.704	3.307
60	0.679	1.296	1.671	2.000	2.390	2.660	3.232
120	0.677	1.289	1.658	1.980	2.358	2.617	3.160
∞	0.674	1.282	1.645	1.960	2.326	2.576	3.090

Note: The smaller probability shown at the head of each column is the area in one tail; the larger probability is the area in both tails.

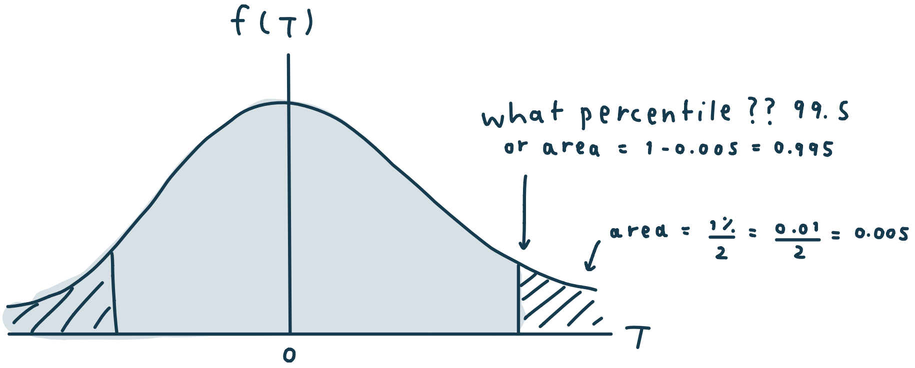
Source: From E. S. Pearson and H. O. Hartley, eds., *Biometrika Tables for Statisticians,* vol. 1, 3d ed., table 12, Cambridge University Press, New York, 1966. Reproduced by permission of the editors and trustees of *Biometrika.*

Example 1: **95% CI** df = 25



The 95% CI for $\hat{\beta}_j = [\hat{\beta}_j - 2.06 \cdot s.e.(\hat{\beta}_j), \hat{\beta}_j + 2.06 \cdot s.e.(\hat{\beta}_j)]$

Example 2: **99% CI** d.f 25



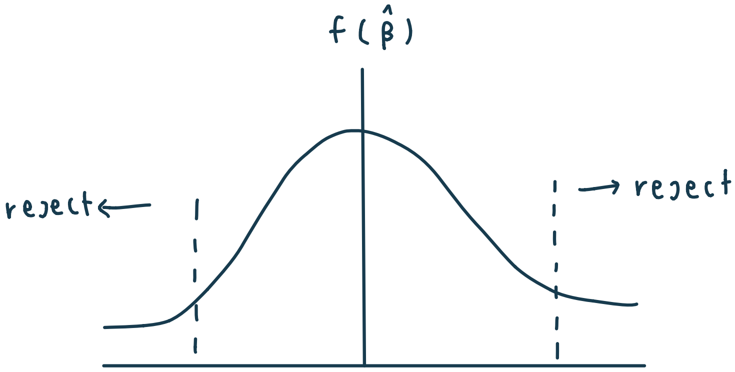
The 99% CI for $\hat{\beta}_j = [\hat{\beta}_j - 2.787 \cdot s.e.\ \hat{\beta}_j, \hat{\beta}_j + 2.787 \cdot s.e.\ \hat{\beta}_j]$

Inference → Hypothesis testing about "β" the true parameter

$wage = \beta_0 + \beta_1 educ + \beta_2 expt \ldots + u$

we want to test about the true impact (β) of each x variables (educ, experience) on the dependent variable (y).

BUT we don't know what the true β are. So, we use $\hat{\beta}$ (estimator) and s.e.($\hat{\beta}$) to test the hypothesis.



β = a hypothesized value

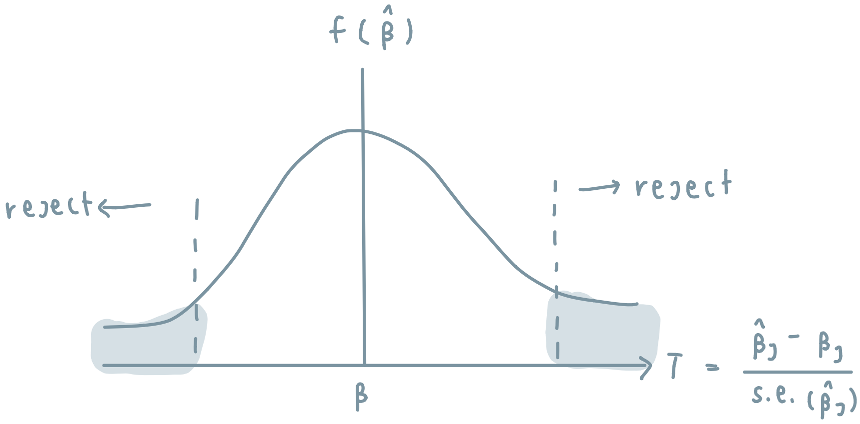
ex. $\beta = 0$ or $\beta_1 = 1$ etc.

1) Test if β = same number

e.g. $\beta_j = 0 \to x_j$ has no impact on y

$\beta_j = 1 \to$ 1 unit ↑ in x_j correspond to 1 unit ↑ y

⇓



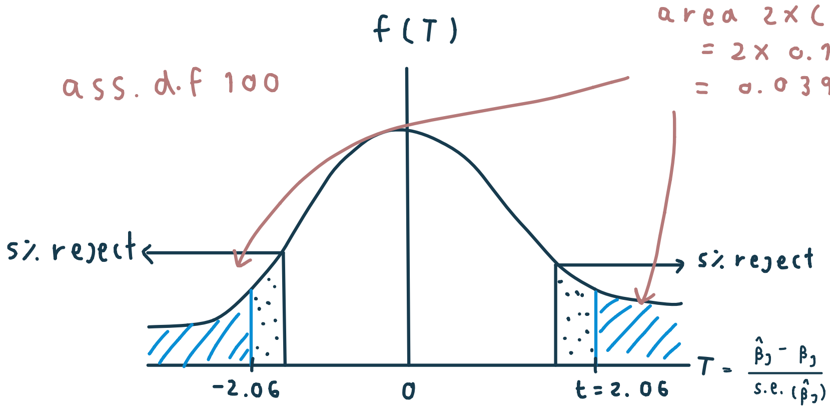
⇒ t-test How?

$$\frac{\hat{\beta}_j - \beta_j}{s.e.(\hat{\beta}_j)} \sim t_{df.}$$

$\boxed{\text{significant level = total area in the rejection region}}$

ass. d.f 100

area $2 \times (0.5 - 0.4803)$
$= 2 \times 0.197$
$= 0.0394$ = p-value



- suppose, we calculate a t-statistic = $\dfrac{\hat{\beta}_j - \beta_j}{s.e.(\hat{\beta}_j)} = 5.78$
- suppose, we are testing $H_0 : \beta_j = 0$, $H_a : \beta_j \neq 0$ — 2 tailed test
- p-value = total shaded area

$\boxed{\text{p-value = significant level which we will reject the } H_0 \text{ or prob that we will reject } H_0.}$

* if p-value < significant level ⇒ reject H_0 !!

F-test motivation

$\Rightarrow$ we want to test the significance of a group of hypotheses (multiple hypotheses)

$$Grade_{325} = \beta_0 + \beta_1 \#times\_front + \beta_2 \#times\_back + \beta_3 \, hr\_study + \beta_4 \, past\_GPA + \beta_5 \, gender + u$$

H_0 : seat position doesn't have impact on GPA

$\beta_1 = 0$ and $\beta_2 = 0$ $\Rightarrow$ $\beta_1 = \beta_2 = 0$

H_a : seat position matters

$$\left.\begin{array}{ll} & \beta_1 \neq 0 \text{ and } \beta_2 \neq 0 \\ \text{or} & \beta_1 \neq 0 \text{ and } \beta_2 = 0 \\ \text{or} & \beta_1 = 0 \text{ and } \beta_2 \neq 0 \end{array}\right\} \text{at least one of the } \beta_1, \beta_2 \neq 0$$

7 Testing Multiple Linear Restrictions: The F-test

Suppose the model is specified by

$$
\begin{aligned}
y &= \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + u \\
H_0 &: \beta_2 = 0 \text{ and } \beta_3 = 0 \\
H_a,\ H_1 &: H_0 \text{ is not true}
\end{aligned}
$$

→ want to test if x_1 and x_2 BOTH have no impact on Y.

We can use the F-test to test this type of "multiple hypotheses".

1. Our full model is called the "unrestricted" model (ur). Suppose it can be expressed as:

   Big model

   $Y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + u$ is True $\Rightarrow$ reject H_0

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + ... + \beta_k x_k + u$$

2. The model which takes out x (which we think its associated $\beta = 0$) is called the restricted model (r).

   small model

$Y = \beta_0 + \beta_1 x_1 + u$ is true $\Rightarrow$ do not reject H_0

- suppose there are "q" number of β that we would like to perform a joint-test of $= 0$

e.g. In this model $q = 2$

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + ... + \beta_{k-q} X_{k-q} + u$$

$$H_0 : \beta_{k-q+1} = \beta_{k-q+2} = ... = \beta_k = 0$$

(the last q $\beta_s = 0$)

H_a : H_0 is not true

$$\underbrace{\underbrace{Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + ... + \beta_{k-q} X_{k-q}}_{(r)} + \beta_{k-q+1} X_{k-q+1} + \beta_{k-q+2} X_{k-q+2} + ... + \beta_k X_k + u}_{ur}$$

$$F \equiv \frac{(SSR_r - SSR_{ur})/q}{SSR_{ur}/(n-k-1)}$$

more x : smaller

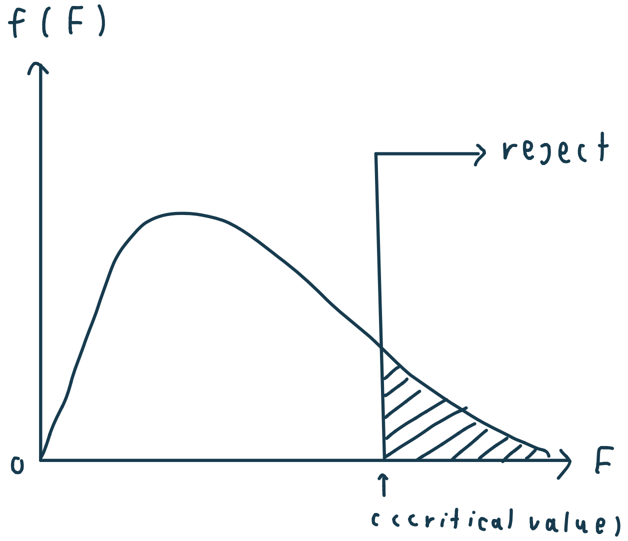
This is always positive b/c $SSR_{ur} < SSR_r$

Every time you add 1 more x, the model will be better explained

d.f. of the "ur" model

- So, if everytime you add 1 more x variable, the SSR ↓ and R^2 ↑, why don't we keep the additional x in the model ??

⇒ Because everytime we add 1 more x, $var(\hat{\beta}_s)$ will increase, making the prediction of β less precise. So, we only keep the additional x_s if it / they can improve the model enough → can ↓ SSR (↑ R^2) enough. can significantly ↓ SSR and ↑ R^2



$H_0: \beta_2 = \beta_3 = \ldots = 0$

H_a : H_0 not true

$F \sim F_{q,\, n-k-1}$

# of joint Hypotheses being tested

d.f. of unrestricted model.

we reject H_0 of jointly no effect if $F > c$

3. Some useful facts

① $R^2_{ur} > R^2_r$ because any additional x would increase R^2 (improve fit).

$\Rightarrow SSR_{ur} < SSR_r$

② By including more x, the model is certainly better explained. However, we would like to reject H_0 if the inclusion of extra variables doesn't improve the model enough.

4. Other ways to calculate the F-statistics:

$\Rightarrow$ From $R^2 = 1 - \frac{SSR}{SST}$ RSS TSS

We have $F \equiv \dfrac{(R^2_{ur} - R^2_r)/q}{(1-R^2_{ur})/(n-k-1)}$

# of β that are set to "0"

intercept

# of slope β

# of observation

$\Rightarrow$ If we want to test the overall significance of the model

$H_0: \beta_1 = \beta_2 = \beta_3 = \ldots = \beta_k = 0$, H_a = otherwise

$F \equiv \dfrac{R^2/k}{(1-R^2)/(n-k-1)}$

R^2 of the model $\approx$ ur

the "r" model has no X at all.

Example: Suppose we are interested in understanding the determinant of a baseball player's salary.

y	*salary*	= season salary
r, ur	*years*	= years in major leagues
r, ur	*gamesyr*	= games per year in the league
ur	*bavg*	= career batting average
ur	*hrunsyr*	= homeruns per year
ur	*rbisyr*	= runs batted in per year

If we want to test whether performance has any impact on salary

$H_0: \beta_{bavg} = \beta_{hrunsyr} = \beta_{rbisyr} = 0$

H_a : otherwise is true

- the unrestricted model (ur) is defined by ur model

```
. regress log_salary years gamesyr bavg hrunsyr rbisyr

      Source |       SS           df       MS      Number of obs   =       353
-------------+----------------------------------   F(5, 347)       =    117.06
       Model |  308.989208         5  61.7978416   Prob > F        =    0.0000
    Residual |  183.186327       347  .527914487   R-squared       =    0.6278
-------------+----------------------------------   Adj R-squared   =    0.6224
       Total |  492.175535       352  1.39822595   Root MSE        =    .72658

------------------------------------------------------------------------------
  log_salary |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
       years |   .0688626   .0121145     5.68   0.000     .0450355    .0926898
     gamesyr |   .0125521   .0026468     4.74   0.000     .0073464    .0177578
        bavg |   .0009786   .0011035     0.89   0.376    -.0011918     .003149
     hrunsyr |   .0144295    .016057     0.90   0.369    -.0171518    .0460107
      rbisyr |   .0107657    .007175     1.50   0.134    -.0033462    .0248776
       _cons |   11.19242   .2888229    38.75   0.000     10.62435    11.76048
------------------------------------------------------------------------------
```

$q = 3$ (bavg, hrunsyr, rbisyr); k (years 1, gamesyr 2, bavg 3, hrunsyr 4, rbisyr 5)

- when considering each of the performance x one-by-one, none of them has a significant impact at 5%

- the restricted model (r) is defined by

```
. regress log_salary years gamesyr

      Source |       SS           df       MS      Number of obs   =       353
-------------+----------------------------------   F(2, 350)       =    259.32
   SSE Model |  293.864058         2  146.932029   Prob > F        =    0.0000
SSR Residual |  198.311477       350  .566604221   R-squared       =    0.5971
-------------+----------------------------------   Adj R-squared   =    0.5948
   SST Total |  492.175535       352  1.39822595   Root MSE        =    .75273

------------------------------------------------------------------------------
  log_salary |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
       years |    .071318    .012505     5.70   0.000     .0467236    .0959124
     gamesyr |   .0201745   .0013429    15.02   0.000     .0175334    .0228156
       _cons |    11.2238    .108312   103.62   0.000     11.01078    11.43683
------------------------------------------------------------------------------
```

- But when performing an F-test, performance have joint impact.

Now, our H_0 and H_a becomes

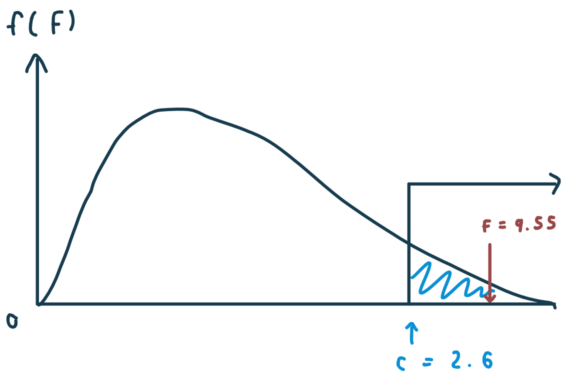
$$F \equiv \frac{(SSR_r - SSR_{ur})/q}{SSR_{ur}/(n-k-1)}$$

$$\equiv \frac{(198.311 - 183.186)/3}{183.186/(353-5-1)} \approx 9.55$$

HW:

$$F \equiv \frac{(R^2/q)}{(1-R^2)/n-k-1}$$

$$\equiv ?$$



Reject at 5%. Let's use 5% level of sig

since $F = 9.55 > 2.6$, we reject H_0 at 5% level and conclude that performances have joint effects on salary.

TABLE D.3 UPPER PERCENTAGE POINTS OF THE *F* DISTRIBUTION

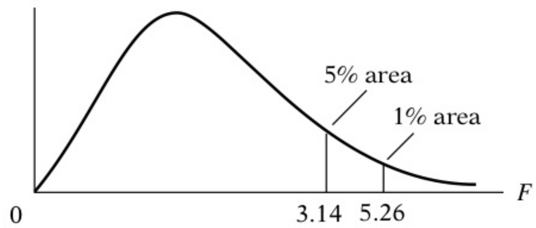
Example

$\Pr(F > 1.59) = 0.25$

$\Pr(F > 2.42) = 0.10$ for df $N_1 = 10$

$\Pr(F > 3.14) = 0.05$ and $N_2 = 9$

$\Pr(F > 5.26) = 0.01$



df for denominator (n-k-1) → N_2	Pr	df for numerator N_1 → or the # of joint hypotheses tested (q) 1	2	3	4	5	6	7	8	9	10	11	12
sig level 25%	.25	5.83	7.50	8.20	8.58	8.82	8.98	9.10	9.19	9.26	9.32	9.36	9.41
1 10%	.10	39.9	49.5	53.6	55.8	57.2	58.2	58.9	59.4	59.9	60.2	60.5	60.7
5%	.05	161	200	216	225	230	234	237	239	241	242	243	244
	.25	2.57	3.00	3.15	3.23	3.28	3.31	3.34	3.35	3.37	3.38	3.39	3.39
2	.10	8.53	9.00	9.16	9.24	9.29	9.33	9.35	9.37	9.38	9.39	9.40	9.41
	.05	18.5	19.0	19.2	19.2	19.3	19.3	19.4	19.4	19.4	19.4	19.4	19.4
	.01	98.5	99.0	99.2	99.2	99.3	99.3	99.4	99.4	99.4	99.4	99.4	99.4
	.25	2.02	2.28	2.36	2.39	2.41	2.42	2.43	2.44	2.44	2.44	2.45	2.45
3	.10	5.54	5.46	5.39	5.34	5.31	5.28	5.27	5.25	5.24	5.23	5.22	5.22
	.05	10.1	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79	8.76	8.74
	.01	34.1	30.8	29.5	28.7	28.2	27.9	27.7	27.5	27.3	27.2	27.1	27.1
	.25	1.81	2.00	2.05	2.06	2.07	2.08	2.08	2.08	2.08	2.08	2.08	2.08
4	.10	4.54	4.32	4.19	4.11	4.05	4.01	3.98	3.95	3.94	3.92	3.91	3.90
	.05	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.94	5.91
	.01	21.2	18.0	16.7	16.0	15.5	15.2	15.0	14.8	14.7	14.5	14.4	14.4
	.25	1.69	1.85	1.88	1.89	1.89	1.89	1.89	1.89	1.89	1.89	1.89	1.89
5	.10	4.06	3.78	3.62	3.52	3.45	3.40	3.37	3.34	3.32	3.30	3.28	3.27
	.05	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77	4.74	4.71	4.68
	.01	16.3	13.3	12.1	11.4	11.0	10.7	10.5	10.3	10.2	10.1	9.96	9.89
	.25	1.62	1.76	1.78	1.79	1.79	1.78	1.78	1.78	1.77	1.77	1.77	1.77
6	.10	3.78	3.46	3.29	3.18	3.11	3.05	3.01	2.98	2.96	2.94	2.92	2.90
	.05	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	4.03	4.00
	.01	13.7	10.9	9.78	9.15	8.75	8.47	8.26	8.10	7.98	7.87	7.79	7.72
	.25	1.57	1.70	1.72	1.72	1.71	1.71	1.70	1.70	1.69	1.69	1.69	1.68
7	.10	3.59	3.26	3.07	2.96	2.88	2.83	2.78	2.75	2.72	2.70	2.68	2.67
	.05	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.60	3.57
	.01	12.2	9.55	8.45	7.85	7.46	7.19	6.99	6.84	6.72	6.62	6.54	6.47
	.25	1.54	1.66	1.67	1.66	1.66	1.65	1.64	1.64	1.63	1.63	1.63	1.62
8	.10	3.46	3.11	2.92	2.81	2.73	2.67	2.62	2.59	2.56	2.54	2.52	2.50
	.05	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.31	3.28
	.01	11.3	8.65	7.59	7.01	6.63	6.37	6.18	6.03	5.91	5.81	5.73	5.67
	.25	1.51	1.62	1.63	1.63	1.62	1.61	1.60	1.60	1.59	1.59	1.58	1.58
9	.10	3.36	3.01	2.81	2.69	2.61	2.55	2.51	2.47	2.44	2.42	2.40	2.38
	.05	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	3.10	3.07
	.01	10.6	8.02	6.99	6.42	6.06	5.80	5.61	5.47	5.35	5.26	5.18	5.11

Source: From E. S. Pearson and H. O. Hartley, eds., *Biometrika Tables for Statisticians,* vol. 1, 3d ed., table 18, Cambridge University Press, New York, 1966. Reproduced by permission of the editors and trustees of *Biometrika.*

TABLE D.3 UPPER PERCENTAGE POINTS OF THE *F* DISTRIBUTION (*Continued*)

df for denominator N_2	df for numerator N_1												
	Pr	1	2	3	4	5	6	7	8	9	10	11	12
22	.25	1.40	1.48	1.47	1.45	1.44	1.42	1.41	1.40	1.39	1.39	1.38	1.37
	.10	2.95	2.56	2.35	2.22	2.13	2.06	2.01	1.97	1.93	1.90	1.88	1.86
	.05	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34	2.30	2.26	2.23
	.01	7.95	5.72	4.82	4.31	3.99	3.76	3.59	3.45	3.35	3.26	3.18	3.12
24	.25	1.39	1.47	1.46	1.44	1.43	1.41	1.40	1.39	1.38	1.38	1.37	1.36
	.10	2.93	2.54	2.33	2.19	2.10	2.04	1.98	1.94	1.91	1.88	1.85	1.83
	.05	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30	2.25	2.21	2.18
	.01	7.82	5.61	4.72	4.22	3.90	3.67	3.50	3.36	3.26	3.17	3.09	3.03
26	.25	1.38	1.46	1.45	1.44	1.42	1.41	1.39	1.38	1.37	1.37	1.36	1.35
	.10	2.91	2.52	2.31	2.17	2.08	2.01	1.96	1.92	1.88	1.86	1.84	1.81
	.05	4.23	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.27	2.22	2.18	2.15
	.01	7.72	5.53	4.64	4.14	3.82	3.59	3.42	3.29	3.18	3.09	3.02	2.96
28	.25	1.38	1.46	1.45	1.43	1.41	1.40	1.39	1.38	1.37	1.36	1.35	1.34
	.10	2.89	2.50	2.29	2.16	2.06	2.00	1.94	1.90	1.87	1.84	1.81	1.79
	.05	4.20	3.34	2.95	2.71	2.56	2.45	2.36	2.29	2.24	2.19	2.15	2.12
	.01	7.64	5.45	4.57	4.07	3.75	3.53	3.36	3.23	3.12	3.03	2.96	2.90
30	.25	1.38	1.45	1.44	1.42	1.41	1.39	1.38	1.37	1.36	1.35	1.35	1.34
	.10	2.88	2.49	2.28	2.14	2.05	1.98	1.93	1.88	1.85	1.82	1.79	1.77
	.05	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21	2.16	2.13	2.09
	.01	7.56	5.39	4.51	4.02	3.70	3.47	3.30	3.17	3.07	2.98	2.91	2.84
40	.25	1.36	1.44	1.42	1.40	1.39	1.37	1.36	1.35	1.34	1.33	1.32	1.31
	.10	2.84	2.44	2.23	2.09	2.00	1.93	1.87	1.83	1.79	1.76	1.73	1.71
	.05	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12	2.08	2.04	2.00
	.01	7.31	5.18	4.31	3.83	3.51	3.29	3.12	2.99	2.89	2.80	2.73	2.66
60	.25	1.35	1.42	1.41	1.38	1.37	1.35	1.33	1.32	1.31	1.30	1.29	1.29
	.10	2.79	2.39	2.18	2.04	1.95	1.87	1.82	1.77	1.74	1.71	1.68	1.66
	.05	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04	1.99	1.95	1.92
	.01	7.08	4.98	4.13	3.65	3.34	3.12	2.95	2.82	2.72	2.63	2.56	2.50
120	.25	1.34	1.40	1.39	1.37	1.35	1.33	1.31	1.30	1.29	1.28	1.27	1.26
	.10	2.75	2.35	2.13	1.99	1.90	1.82	1.77	1.72	1.68	1.65	1.62	1.60
	.05	3.92	3.07	2.68	2.45	2.29	2.17	2.09	2.02	1.96	1.91	1.87	1.83
	.01	6.85	4.79	3.95	3.48	3.17	2.96	2.79	2.66	2.56	2.47	2.40	2.34
200	.25	1.33	1.39	1.38	1.36	1.34	1.32	1.31	1.29	1.28	1.27	1.26	1.25
	.10	2.73	2.33	2.11	1.97	1.88	1.80	1.75	1.70	1.66	1.63	1.60	1.57
	.05	3.89	3.04	2.65	2.42	2.26	2.14	2.06	1.98	1.93	1.88	1.84	1.80
	.01	6.76	4.71	3.88	3.41	3.11	2.89	2.73	2.60	2.50	2.41	2.34	2.27
∞	.25	1.32	1.39	1.37	1.35	1.33	1.31	1.29	1.28	1.27	1.25	1.24	1.24
	.10	2.71	2.30	2.08	1.94	1.85	1.77	1.72	1.67	1.63	1.60	1.57	1.55
	.05	3.84	3.00	2.60	2.37	2.21	2.10	2.01	1.94	1.88	1.83	1.79	1.75
	.01	6.63	4.61	3.78	3.32	3.02	2.80	2.64	2.51	2.41	2.32	2.25	2.18