EE320 (2/2013)

INTRODUCTORY MATHEMATICAL ECONOMICS

DERIVATIVES OF MORE-THAN-ONE INDEPENDENT VARIABLE FUNCTION

Roadmap after midterm

- Derivatives of more-than-one independent variable functions
- Optimization without constraint: More-than-one independent variable cases
- Optimization under equality constraint
- Integration and applications

Topics

- Partial Differentiation
  - First-order partial derivatives
  - Second-order partial derivatives
- Differentials
- Total differentials
- Total derivatives
- Implicit function and its derivative

PARTIAL DIFFERENTIATION

Partial Differentiation

- Consider $y = f(x_1, x_2, \ldots, x_n)$ where $x_1, \ldots, x_n$ are independent of one another.
- Suppose only x_1 changes by Δx_1, the corresponding change in y is Δy.
- Difference quotient:

$$\frac{\Delta y}{\Delta x_1} = \frac{f(x_1 + \Delta x_1, x_2, \ldots, x_n) - f(x_1, x_2, \ldots, x_n)}{\Delta x_1}$$

- Partial derivative of y with respect to x:

$$f_1 \equiv \frac{\partial y}{\partial x_1} \equiv \lim_{\Delta x_1 \to 0} \frac{\Delta y}{\Delta x_1}$$

- The process of taking partial derivatives is called "partial differentiation".

First-Order Partial Derivatives

- Technique of partial differentiation: allow one variable to vary while holding all other independent variables constant while.
- Example 1: $y = f(x_1, x_2) = 3x_1^2 + x_1 x_2 + 4x_2^2$. Find $f_1(1,3)$ and $f_2(1,3)$.
- Example 2: $y = f(u,v) = (u+4)(3u+2v)$. Find $f_u(2,1)$ and $f_v(2,1)$.

Geometric Interpretation of Partial Derivatives

- Suppose Q = Q(K, L)

➢ $\partial Q/\partial L$ = MPP_L

➢ $\partial Q/\partial K$ = MPP_K

Gradient Vector

- The gradient vector of the function $f(x_1, x_2, ..., x_n)$ is an n-vector of all the partial derivatives:

$$\nabla f(x_1, x_2, \ldots, x_n) = (f_1, f_2, \ldots, f_n)$$

where $f_i \equiv \frac{\partial y}{\partial x_i}$.

- Example: Find the gradient vector of the production function $Q = aK^bL^{1-b}$, where $a > 0$ and $0 < b < 1$.

Application 1: Partial Market Equilibrium

- Recall the one-commodity market model:

  Demand: $Q = a - bP$ $(a, b > 0)$

  Supply: $Q = -c + dP$ $(c, d > 0)$

- <u>Solutions</u>:

$$P^* = \frac{a+c}{b+d}$$

$$Q^* = \frac{ad-bc}{b+d}$$

- *Comparative-static derivatives* are the partial derivatives of P^* and Q^* with respect to parameters (a, b, c, and d):
  - $\partial P^*/\partial a$, $\partial P^*/\partial b$, $\partial P^*/\partial c$, $\partial P^*/\partial d$
  - $\partial Q^*/\partial a$, $\partial Q^*/\partial b$, $\partial Q^*/\partial c$, $\partial Q^*/\partial d$

Application 1: Partial Market Equilibrium

- Partial derivatives of *P** with respect to parameters *a, b, c,* and *d* are:

$$\frac{\partial P^*}{\partial a} = \frac{1}{b+d}$$

$$\frac{\partial P^*}{\partial b} = \frac{-(a+c)}{(b+d)^2}$$

$$\frac{\partial P^*}{\partial c} = \frac{1}{b+d}\left(= \frac{\partial P^*}{\partial a}\right)$$

$$\frac{\partial P^*}{\partial d} = \frac{-(a+c)}{(b+d)^2}\left(= \frac{\partial P^*}{\partial d}\right)$$

Application 1: Partial Market Equilibrium

- Graphical illustration of a change in each parameter

1. Increase in *a*

2. Increase in *b*

3. Increase in *c*

4. Increase in *d*

Application 1: Partial Market Equilibrium

- Partial derivatives of *Q** with respect to parameters *a, b, c*, and *d* are:

$$\frac{\partial Q^*}{\partial a} = \frac{d}{b+d}$$

$$\frac{\partial Q^*}{\partial b} = \frac{-(cd+ad)}{(b+d)^2}$$

$$\frac{\partial Q^*}{\partial c} = \frac{-b}{b+d}$$

$$\frac{\partial Q^*}{\partial d} = \frac{ab-bc}{(b+d)^2}$$

Application 2: Elasticities

- **Two variables**

If *z = f(x, y)*, the partial elasticity of *z* w.r.t. *x* and *y* are:

$$\varepsilon_{zx} = \frac{\partial z / z}{\partial x / x} = \frac{\partial z}{\partial x}\left(\frac{x}{z}\right) \quad \text{and} \quad \varepsilon_{zy} = \frac{\partial z / z}{\partial y / y} = \frac{\partial z}{\partial y}\left(\frac{y}{z}\right)$$

When all variables are positive, elasticities can be expressed as logarithmic derivatives:

$$\varepsilon_{zx} = \frac{\partial \ln z}{\partial \ln x} \quad \text{and} \quad \varepsilon_{zy} = \frac{\partial \ln z}{\partial \ln y}$$

- **n variables**

If $z = f(x_1, x_2, ..., x_n) = f(\mathbf{x})$, the elasticity of *z* w.r.t. x_i *when all other variables are held constant* is:

$$\varepsilon_{zi} = \frac{\partial f(x) / f(x)}{\partial x_i / x_i} = \frac{\partial \ln z}{\partial \ln x_i}$$

Application 2: Elasticities of Demand

- Given the demand function $Q_1 = a - bP_1 + cP_2 + mY$

  where Y = income, P_2 = the price of a substitute good.

- Own price elasticity of demand:

$$\varepsilon_d = \frac{\partial Q_1}{Q_1} \div \frac{\partial P_1}{P_1} = \frac{\partial Q_1}{\partial P_1}\left(\frac{P_1}{Q_1}\right)$$

- Income elasticity of demand:

$$\varepsilon_Y = \frac{\partial Q_1}{Q_1} \div \frac{\partial Y}{Y} = \frac{\partial Q_1}{\partial Y}\left(\frac{Y}{Q_1}\right)$$

- Cross-price elasticity of demand:

$$\varepsilon_c = \frac{\partial Q_1}{Q_1} \div \frac{\partial P_2}{P_2} = \frac{\partial Q_1}{\partial P_2}\left(\frac{P_2}{Q_1}\right)$$

Application 2: Elasticities of Demand

- **Example** : Given $Q_1 = 100 - P_1 + 0.75P_2 - 0.25P_3 + 0.0075Y$.

At $Y = 10{,}000$, $P_1 = 10$, $P_2 = 20$, $P_3 = 40$ and $Q_1 = 100$, find the different cross-price elasticities of demand.

- $$\varepsilon_{12} = \frac{\partial Q_1 / Q_1}{\partial P_2 / P_2} = 0.75\left(\frac{20}{100}\right) = 0.15$$

- $$\varepsilon_{13} = \frac{\partial Q_1 / Q_1}{\partial P_3 / P_3} = -0.25\left(\frac{40}{100}\right) = -0.1$$

Application 2: Output Elasticity

- Given a *linearly homogenous* Cobb-Douglas production function

$$Q = F(K, L) = AK^{\alpha}L^{\beta}$$

➢ The output elasticity of capital:

$$\varepsilon_{QK} = \frac{\partial Q / Q}{\partial K / K} = \alpha A K^{\alpha - 1} L^{\beta} \frac{K}{Q} = \alpha$$

➢ The output elasticity of labor:

$$\varepsilon_{QL} = \frac{\partial Q / Q}{\partial L / L} = \beta A K^{\alpha} L^{\beta - 1} \frac{L}{Q} = \beta$$

Application 3: Production Function

- **Example**: Given a production function

$$Q = 36KL - 2K^2 - 3L^2$$

- Marginal product of capital is: $MP_K = 36L - 4K$
- Marginal product of labor is: $MP_L = 36K - 6L$
- If the marginal revenue (MR) at K = 2 and L = 3 is \$5, the marginal revenue product (MRP) for the *third* unit of L is:

➔ $MRP_{L=3} = MR \times MP_L$

$= \$5 \times [36(2) - 6(3)]$

$= \$5 \times 54 = 270$

Application 4:
Multipliers in Macroeconomic Models

- Consider a national-income model

$$Y = C + I_0 + G_0$$
$$C = \alpha + \beta(Y - T) \qquad (\alpha > 0;\ 0 < \beta < 1)$$
$$T = \gamma + \delta Y \qquad (\gamma > 0;\ 0 < \delta < 1)$$

- Equilibrium national income (in reduced form):

$$Y^* = \frac{\alpha - \beta\gamma + I_0 + G_0}{1 - \beta + \beta\delta}$$

- Comparative-static derivatives:
  - $\frac{\partial Y^*}{\partial \alpha}; \frac{\partial Y^*}{\partial \beta}; \frac{\partial Y^*}{\partial I_0}; \frac{\partial Y^*}{\partial G_0}; \frac{\partial Y^*}{\partial \gamma}; \frac{\partial Y^*}{\partial \delta}$

Application 4:
Multipliers in Macroeconomic Models

- Comparative-static derivatives with special *policy significance*:
  - Government-expenditure multiplier:

$$\frac{\partial Y^*}{\partial G_0} = \frac{1}{1-\beta+\beta\delta} > 0$$

  - Nonincome-tax multiplier:

$$\frac{\partial Y^*}{\partial \gamma} = \frac{-\beta}{1-\beta+\beta\delta} < 0$$

  - Partial derivatives of Y* w.r.t. the income tax rate (δ):

$$\frac{\partial Y^*}{\partial \delta} = \frac{-\beta(\alpha-\beta\gamma+I_0+G_0)}{(1-\beta+\beta\delta)^2} = \frac{-\beta Y^*}{1-\beta+\beta\delta} < 0$$

Second-Order Partial Derivatives (1)

- Consider the function *z = f(x, y),* which give rise to:

$$f_x \equiv \frac{\partial z}{\partial x} \quad \text{and} \quad f_y \equiv \frac{\partial z}{\partial y}$$

- Since f_x is a function of *x* (and *y*), we can determine the rate of change of f_x with respect to *x*, while *y* is fixed, by a *second-order* partial derivative with respect to *x*:

$$f_{xx} \equiv \frac{\partial}{\partial x}(f_x) \quad \text{or} \quad \frac{\partial^2 z}{\partial x^2} \equiv \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)$$

- Similarly, the *second-order* partial derivative with respect to *y* is:

$$f_{yy} \equiv \frac{\partial}{\partial y}(f_y) \quad \text{or} \quad \frac{\partial^2 z}{\partial y^2} \equiv \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$$

Second-Order Partial Derivatives (2)

- Also, since f_x is a function of y and f_y is a function of x, *cross (or mixed) partial derivatives* can be written as:

$$f_{xy} \equiv \frac{\partial^2 z}{\partial x \partial y} \equiv \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) \quad \text{and} \quad f_{yx} \equiv \frac{\partial^2 z}{\partial y \partial x} \equiv \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$$

Young's Theorem:

Let $y = f(x_1, x_2, \ldots, x_n)$ is twice continuously differentiable (C^2). Then,

$$\frac{\partial^2 y}{\partial x_i \partial x_j} = \frac{\partial^2 y}{\partial x_j \partial x_i} ; i \neq j$$

Second-Order Partial Derivatives (3)

- Let $y = f(x_1, x_2)$. The second-order partial derivatives can be written in a matrix form called "Hessian matrix":

$$H = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}$$

- Example: Let $z = x^2e^{-y}$. Find f_x, f_y, and the Hessian matrix.

Application: Production Function

- Consider an agricultural production function

$$Q = F(K, L, T) = AK^{\alpha}L^{\beta}T^{\gamma} \qquad (A > 0;\ \alpha > 0, \beta > 0, \gamma > 0)$$

Where K = capital, L = labor, and T = land.

- Marginal product of capital is: $F_K = A\alpha K^{\alpha-1}L^{\beta}T^{\gamma}$
- Marginal product of labor is: $F_L = A\beta K^{\alpha}L^{\beta-1}T^{\gamma}$
- Marginal product of land is: $F_T = A\gamma K^{\alpha}L^{\beta}T^{\gamma-1}$

- Second-order partial derivatives:

- Cross partial derivatives:

DIFFERENTIALS & TOTAL DIFFERENTIALS

Differentials

- Recall the definition of derivatives:

$$\frac{dy}{dx} = f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

➔ $$dy = f'(x)dx$$

  - dy = differential of y
  - dx = differential of x

- Example: Let $y = 3x^2 + 7x - 5$. Find dy.

- The process of finding the differential dy from a function $y = f(x)$ is called *differentiation*. (Note: for finding dy/dx, this is called differentiation with respect to x).

Total Differentials

- Total differential of a function Y is the sum of the approximated changes from all parameters.
- The process of finding total differentials is called "total differentiation."
- Let $y = f(x_1, x_2)$

  $dy = f_1 dx_1 + f_2 dx_2$, where $f_1 = \partial y/\partial x_1$, $f_2 = \partial y/\partial x_2$

- Example: Let $z = 5x^2 + 6y + 7$. Find dz =?

Example 1: Saving Function

- Consider a saving function $S = S(Y, r)$ where Y = income, i = interest rate.
  - The total change in S is approximated by the total differential:

$$dS = \left(\frac{\partial S}{\partial Y}\right) \cdot dY + \left(\frac{\partial S}{\partial r}\right) \cdot dr$$

or $dS = S_Y dY + S_r dr$

  - For a *given change in Y*, the resulting change in S can be approximated by: $dS = \left(\frac{\partial S}{\partial Y}\right) \cdot dY$

    where $\frac{\partial S}{\partial Y}$ = marginal propensity to save.
  - For a *given change in r (dr)*, the resulting change in S can be approximated by: $dS = \left(\frac{\partial S}{\partial r}\right) \cdot dr$

Example 2: Utility Function

- General case of *n* independent variables:

  Example: $U = U(x_1, x_2, ..., x_n)$

- Total differential of U is:

$$dU = \left(\frac{\partial U}{\partial x_1}\right) \cdot dx_1 + \left(\frac{\partial U}{\partial x_2}\right) \cdot dx_2 + \ldots + \left(\frac{\partial U}{\partial x_n}\right) \cdot dx_n$$

  or

$$dU = U_1 dx_1 + U_2 dx_2 + \ldots + U_n dx_n = \sum_{i=1}^{n} U_i dx_i$$

- Example: Let $U = 100x^{0.5}y^{0.5}$. Find dU= ?

➔ $dU = (50x^{-0.5}y^{0.5})dx + (50x^{0.5}y^{-0.5})dy$

Rules of Differentials

Rule I	$dk = 0$	(k is a constant.)
Rule II	$d(cu^n) = cnu^{n-1}du$	
Rule III	$d(u \pm v) = du \pm dv$	
Rule IV	$d(u \cdot v) = v \cdot du + u \cdot dv$	
Rule V	$d(u/v) = [v \cdot du - u \cdot dv]/v^2$	
Rule VI	$d(u \pm v \pm w) = du \pm dv \pm dw$	
Rule VII	$d(uvw) = vw \cdot du + uw \cdot dv + uv \cdot dw$	

Examples: Rules of Differentials

- Example 1: $y = 3x_1^2 + x_1 x_2^2$

- Example 2: $y = \dfrac{x_1 + x_2}{2x_1^2}$

- Example 3: $y = 3x_1(2x_2 - 1)(x_3 + 5)$

Application: Utility Function

- Consider a utility function

$$U = Ax^a y^b,$$

- Marginal utility of x: $MU_x = aAx^{a-1}y^b$
- Marginal utility of y: $MU_y = bAx^a y^{b-1}$
- Marginal rate of substitution (MRS) as the slope of the indifference curve:

$$MRS_{xy} = \left|\frac{dy}{dx}\right| = \left|-\frac{MU_x}{Mu_y}\right| = \frac{ay}{bx}$$

Application: Production Function

- Consider an agricultural production function

$$Q = F(K, L) = 60K^{0.25}L^{0.75}$$

  - Marginal product of capital is:
  - Marginal product of labor is:

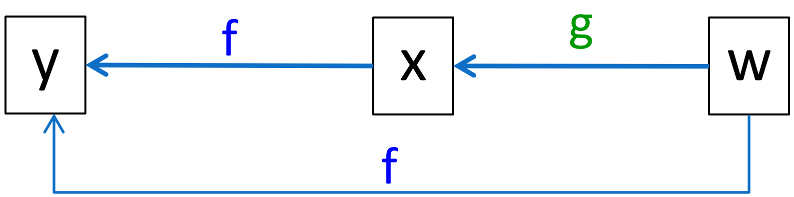
- Marginal rate of technical substitution (MRTS) as the slope of the isoquant:

$$MRTS_{KL} = \left|\frac{dK}{dL}\right| = \left|-\frac{MP_L}{MP_K}\right| = \frac{3K}{L}$$

TOTAL DERIVATIVES

Total Derivatives (1)

- Now, consider the case where independent variables are *related* to one another.
  - Example: what is the rate of change of $C(Y^*, T_0)$ w.r.t. T_0, where Y and T_0 are related?
- Consider y = f(x, w) where x = g(w)



➔ y = f[g(w), w]

➔ dy/dw = ?

Total Derivatives (2)

- From *y = f[g(w), w]*

1. Use chain rule:

$$\frac{dy}{dw} = f_x \frac{dx}{dw} + f_w = \frac{\partial y}{\partial x}\frac{dx}{dw} + \frac{\partial y}{\partial w}$$

2. Totally differentiate:

$$dy = f_x \cdot dx + f_w \cdot dw$$

$$\Rightarrow \boxed{\frac{dy}{dw} = f_x \frac{dx}{dw} + f_w}$$

- dy/dw is called the "total derivative of y w.r.t. **w**".
- ∂y/∂w (partial derivative) is a component of the total derivative.

Examples: Total Derivatives

- Example 1: y = f(x, w) = 3x – w^2, where x = g(w) = $2w^2$+w+4

$$\Rightarrow \frac{dy}{dw} = 10w + 3$$

- Example 2: U = U[c, g(c)]

$$\Rightarrow \frac{dU}{dc} = U_c + U_g \cdot \frac{dg}{dc}$$

A Variation on Total Derivatives

- Let $y = f(x_1, x_2, w)$ where $\begin{cases} x_1 = g(w) \\ x_2 = h(w) \end{cases}$

$$\Rightarrow \quad \frac{dy}{dw} = \frac{\partial y}{\partial x_1}\frac{dx_1}{dw} + \frac{\partial y}{\partial x_2}\frac{dx_2}{dw} + \frac{\partial y}{\partial w}$$

- Example: $Q = Q(K, L, t)$, where $K = K(t)$ and $L = L(t)$

$$\Rightarrow \quad \frac{dQ}{dt} = \frac{\partial Q}{\partial K}\frac{dK}{dt} + \frac{\partial Q}{\partial L}\frac{dL}{dt} + \frac{\partial Q}{\partial t}$$

Another Variation on Total Derivatives

- Let $y = f(x_1, x_2, u, v)$ where $\begin{cases} x_1 = g(u,v) \\ x_2 = h(u,v) \end{cases}$

Application: Utility Function

- Suppose $U = U(C, n)$,

where C = consumption, n = leisure = 24-L, and C = f(L).

IMPLICIT FUNCTION

Implicit Function

- Consider $y = f(x) = 3x^4$: Explicit function

  ➔ $y - 3x^4 = 0$: Implicit function

- General form: $F(y, x_1, x_2, \ldots, x_n) = 0.$
- This function may define an implicit function:

$$y = f(x_1, x_2, \ldots, x_n)$$

Derivatives of Implicit Functions

- Given $F(y, x_1, x_2, \ldots, x_n) = 0$,
- We can write $dF = d(0) = 0$,

  or $F_y dy + F_1 dx_1 + F_2 dx_2 + \ldots + F_n dx_n = 0.$ -- (*)
- Implicit function has the total differential:

  $$dy = f_1 dx_1 + f_2 dx_2 + \ldots + f_n dx_n$$
- Substituting dy in (*) gives:

  $$(F_y f_1 + F_1)dx_1 + (F_y f_2 + F_2)dx_2 + \ldots + (F_y f_n + F_n)dx_n = 0$$

  $F_y f_i + F_i = 0$ for i = 1, 2, …, n.

  ➔ $$f_i \equiv \frac{\partial y}{\partial x_i} = -\frac{F_i}{F_y}$$:Implicit Function Rule

Example: Derivatives of Implicit Functions

- Example 1: Find dy/dx for the implicit function $y - 3x^4 = 0$.

- Example 2: Find ∂y/∂x for any implicit function(s) that may be defined by $F(y, x, w) = y^3x^2 + w^3 + yxw - 3 = 0$.

- Example 3: Assume $F(Q, K, L) = 0$ implicitly defines the production function $Q = f(K,L)$. Express MPP_K and MPP_L in relation to the function F.

Application 1: Partial Market Equilibrium

- Suppose now Q_d is a function of P as well as Y_0.

$$Q_d = Q_s$$

$$Q_d = D(P, Y_0) \qquad (\partial D/\partial P < 0; \partial D/\partial Y_0 > 0)$$

$$Q_s = S(P) \qquad (\partial S/\partial P > 0)$$

- Equilibrium condition:

$$D(P, Y_0) - S(P) = 0.$$

- Equilibrium price:

$$P^* = P^*(Y_0).$$

- Equilibrium identity:

$$D(P^*, Y_0) - S(P^*) \equiv 0, \quad \text{where } D(P^*, Y_0) - S(P^*) = F(P^*, Y_0)$$

Application 1: Partial Market Equilibrium

- Comparative-static derivatives:
  - $$\frac{dP^*}{dY_0} = -\frac{\partial F / \partial Y_0}{\partial F / \partial P^*} = -\frac{\partial D / \partial Y_0}{\partial D / \partial P^* - dS / dP^*} > 0$$
  - $$\frac{dQ^*}{dY_0} = \frac{dS}{dP^*}\left(\frac{dP^*}{dY_0}\right) > 0$$

Application 2: Production Function

- **Example** : Given the equation for a production isoquant

$$F(K,L) = 16K^{0.25}L^{0.75} = 2144,$$

Use the implicit function rule to find the slope of the isoquant (|dK/dL|: marginal rate of technical substitution).

$$MRTS = \left|\frac{dK}{dL}\right| = \left|-\frac{F_L}{F_K}\right| = \frac{3K}{L}$$