Assignment 5

1. Let $X = \{1, 2, 3\}$ and $Y = \{a, b, c, d\}$. Define relations
$f : X \to Y$ by $f = \{(1, a), (2, a), (3, c)\}$,
$g : X \to Y$ by $g = \{(1, a), (3, c)\}$, and
$h : X \to Y$ by $h = \{(1, a), (2, a), (3, b), (3, c)\}$.
   (a) Draw the arrow diagrams of f, g, and h.
   (b) Show that f is a function, but g and h are not functions.
   (c) Find the domain of f, co-domain of f,and range of f.
   (d) What is the inverse image of a for the function f?
   (e) What is $f(3)$?

2. Let $A = \{-1, 1, 2, 4\}$ and $B = \{1, 2\}$ and define relations R and S from A to B as follows: For all $(x, y) \in A \times B$,
$$x \; R \; y \Leftrightarrow \; |x| = |y|.$$
and
$$x \; S \; y \; \Leftrightarrow \; x - y \text{ is even.}$$
State explicitly the sets $A \times B$, R, S, $R \cup S$, and $R \cap S$.

3. Let $A = \{1, 2, 3\}$ and $\mathbb{Z}$ be the set of all integers. Let $\mathcal{P}(A)$ be the set of all subsets of the set A, and
$$X = \{x \in \mathcal{P}(A) | \; x \cap \{1\} \neq \emptyset\}.$$
Define a relation r from X to $\mathbb{Z}$ as
$$r = \{(x, y) \in X \times \mathbb{Z} | \; y = \text{ the number of elements in } x\}.$$
   (a) List all elements in X.
   (b) Draw an arrow diagram of r .
   (c) Is r a function? If so, find the domain, co-domain, and range of r.

4. Define $H : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ as follows:
$$H(x, y) = (x + 1, 2y - 3) \text{ for all } (x, y) \in \mathbb{R} \times \mathbb{R}.$$
   (a) Is H one-to-one? Prove or give a counterexample.
   (b) Is H onto? Prove or give a counterexample.
   (c) Is H bijective? If so, find H^{-1}, the inverse function of H.

5. Define functions $f_1 : [0,2) \to \mathbb{R}$ as
$$f_1(x) = x^2$$
and define $f_2 : [2,\infty) \to \mathbb{R}$ as
$$f_2(x) = 3x - 2.$$
Let $F : [0,\infty) \to [0,\infty)$ be a function defined by using f_1 and f_2:
$F(x) = f_1(x)$, for $x \in [0,2)$, and $F(x) = f_2(x)$, for $x \in [2,\infty)$. That is,
$$F(x) = \begin{cases} x^2, & x \in [0,2) \\ 3x-2, & x \in [2,\infty). \end{cases}$$
   (a) Find the domain, co-domain, and range for each of the functions f_1, f_2, and F.
   (b) Construct the composite functions $f_1 \circ f_2$, $f_2 \circ f_1$, and $f_1 \circ F$ (if possible). Determine the domains and ranges for these composite functions.
   (c) Are f_1 and f_2 injective? Explain.
   (d) Is the function F bijective? If so, find the **inverse function** of F.

6. Let f and g be functions from $\mathbb{R}$ to $\mathbb{R}$. Find $f \circ g$, $g \circ f$, and determine whether or not $f \circ g = g \circ f$ for the given formulas for f and g. Compute $(f \circ g)(2)$ and $(g \circ f)(2)$.
   (a) $f(x) = \frac{x}{\sqrt{x^2+1}}$, $g(x) = x^3 + 1$.
   (b) $f(x) = x^5$, $g(x) = x^{1/5}$.

7. Let $f : \mathbb{R} - \{1\} \to \mathbb{R} - \{-2\}$ be a function defined by $f(x) = \frac{2x+1}{1-x}$.
   (a) Compute $f \circ f$ and determine its domain.
   (b) Determine whether f is bijective. If so, find the inverse function f^{-1} and $f \circ f^{-1}$.

Optional Problems

1. Define relations R and S on a set A as follows:

$$R = \{(x, y) \in A \times A |\ x < y\} \text{ and } \quad S = \{(x, y) \in A \times A |\ x = y\}.$$

   (a) Suppose $A = \{3, 5, 7\}$. List all elements of R and S. Draw arrow diagrams for R and S.

   (b) Suppose A is a set of all real numbers. Graph R, S, and $R \cup S$ on the Cartesian plane.

2. Define $f : \mathbb{Z} \to \mathbb{Z}$ by the rule $f(n) = 2 - 3n$, for all integers n.

   (i) Is f one-to-one? Prove or give a counterexample.

   (ii) Is f onto? Prove or give a counterexample.

3. Define $G : \mathbb{R} \to \mathbb{R}$ by the rule $G(x) = 2 - 3x$ for all real numbers x. Is G onto? Prove or give a counterexample.

4. Define $f : \mathbb{R} - \{0\} \to \mathbb{R}$ by $f(x) = \frac{x+1}{x}$, for all real numbers $x \neq 0$. Determine whether or not f is one-to-one and justify your answer.

5. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \frac{x}{x^2+1}$, for all real numbers x. Determine whether or not f is one-to-one and justify your answer.

6. Determine whether the following functions are injective. Prove or give a counterexample for each function.

   (a) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \frac{x}{2x^2+1}$

   (b) $f : \mathbb{R} - \{1/2\} \to \mathbb{R}$, $f(x) = \frac{x}{2x+1}$

7. Define $F : S \times S \to S \times S$, $F(a, b) = (2a + 1, 3b - 2)$.

   (a) If $S = \mathbb{Z}$ is the set of all integers, is F bijective? Prove or give a counterexample.

   (b) If $S = \mathbb{R}$ is the set of all real numbers, is F bijective? Prove or give a counterexample.

8. Let $f(x) = \sqrt{x - 1}$. Find the largest domain and co-domain of f such that its inverse function, f^{-1}, exists.