

EE320 (1/2013)

INTRODUCTORY MATHEMATICAL ECONOMICS

OPTIMIZATION WITHOUT CONSTRAINTS:

ONE-INDEPENDENT-VARIABLE CASE

Topics

- Optimal Values and Extreme Values (**Maxima, Minima, inflection points**)
 - First-derivative test for relative extremum
 - Second-derivative Test for relative extremum
- Convexity and Concavity
- Profit maximization
 - Competitive market
 - Monopoly
- Effects of taxes
 - Lump-sum tax
 - Profit tax
 - Excise
- Tax Revenue Maximization

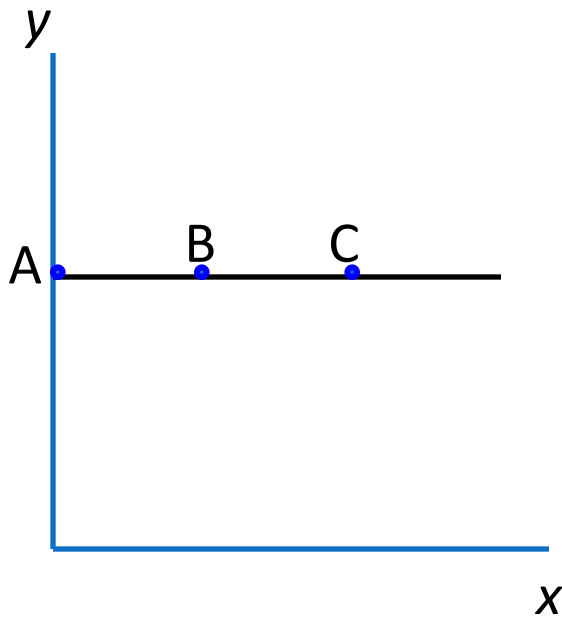
OPTIMAL VALUES AND EXTREME VALUES

Optimal Values and Extreme Values

- Most common criterion of choice among alternatives in economics is to find an *extreme* value – *maximum* or *minimum*.
- *Optimization* problems could be *maximizing* something (e.g. maximizing profit) or *minimizing* something (e.g. minimizing costs).
 - *Objective function* → dependent variable
 - *Choice* → independent variable
- Example: Firm's objective is to maximize profit:
$$\pi(Q) = R(Q) - C(Q).$$
 - π is the object of maximization
 - Q is the choice variable.
- This lecture will focus on the general objective function of one variable: $y = f(x)$.

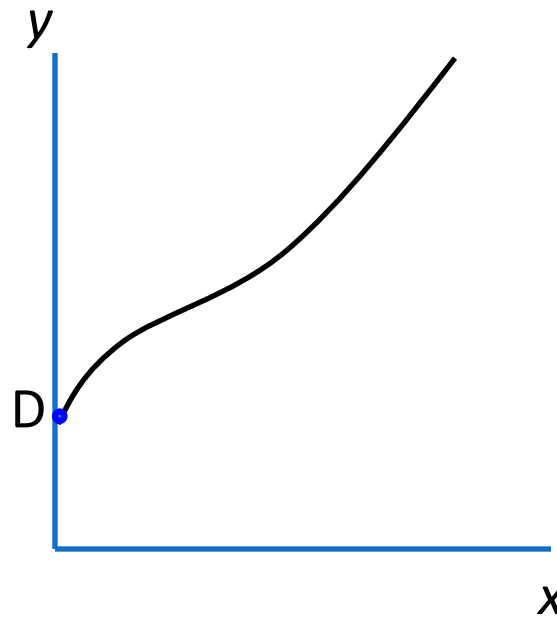
Relative VS. Absolute Extremum

(a)



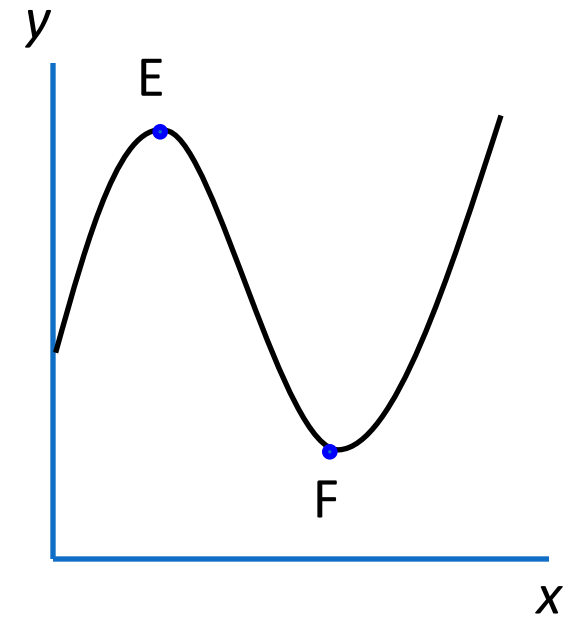
No extremum

(b)



Absolute (or global)
minimum

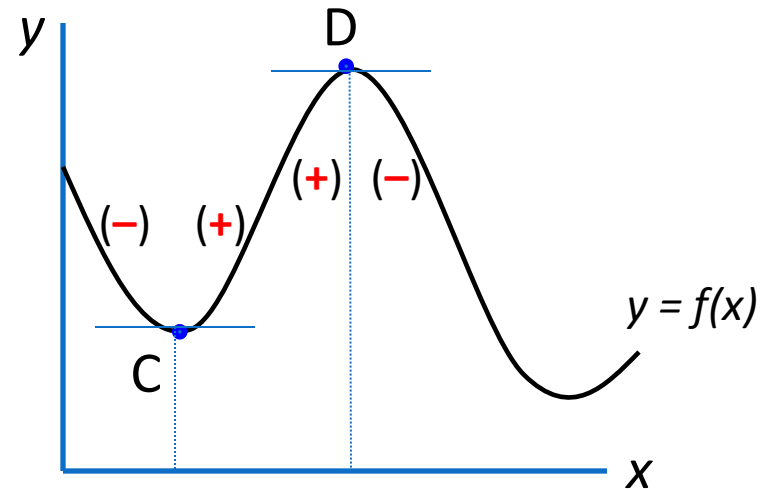
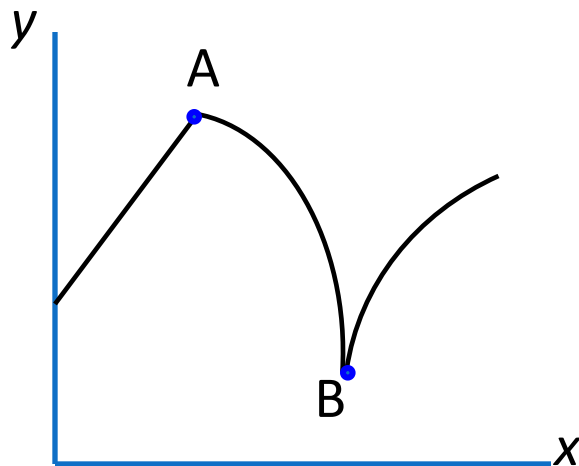
(c)



Relative (or local)
extremums

First-Derivative Test (1)

- 2 cases if a relative extremum of $f(x)$ occurs at $f(x_0)$:
 1. $f'(x_0)$ does not exist.
 2. $f'(x_0) = 0$.



For smooth functions, the *necessary* condition to determine relative extreme values (local max/local min) is $f'(x) = 0$.

First-Derivative Test (2)

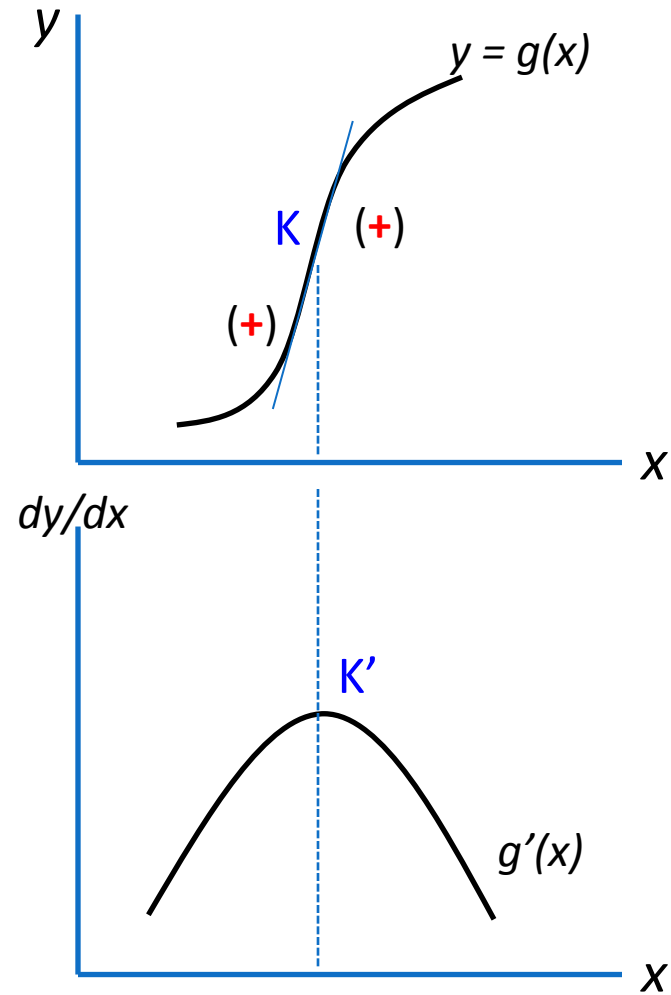
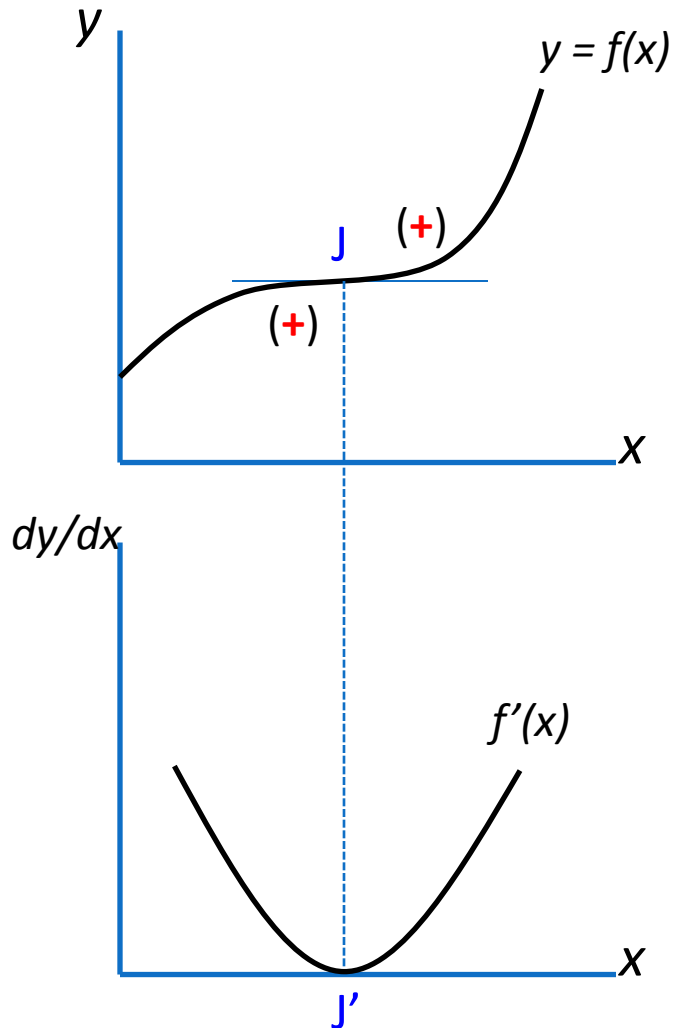
First-Derivative Test for Relative Extremum:

If the first derivative of a function $f(x)$ at $x = x_0$ is $f'(x_0) = 0$, then the value of $f(x_0)$ will be

- a. A relative *maximum* if $f'(x)$ changes its sign from $+$ to $-$ from the immediate left of x_0 to is immediate right.
- b. A relative *minimum* if $f'(x)$ changes its sign from $-$ to $+$ from the immediate left of x_0 to is immediate right.
- c. *Neither* a relative maximum nor a relative minimum if $f'(x)$ has the same sign on both the immediate left and the immediate right of x_0 .

→ x_0 where $f'(x_0) = 0$ is called the critical value, and $f(x_0)$ is the stationary value of the function $f(x)$.

Inflection Points



Example

- Find the relative minimum of the average cost function

$$AC(Q) = Q^2 - 5Q + 8$$

→ Let $f(Q) = AC(Q)$

$$f'(Q) = 2Q - 5$$

Necessary condition for an extremum: $2Q - 5 = 0$

→ $Q^* = 2.5$

Check: if $Q = 2$, $f'(x) = -1$,

if $Q = 3$, $f'(x) = 1$.

Thus, $Q^* = 2.5$ is a minimum.

Second and Higher Derivatives

- The derivative of a function f is called the **first derivative** of f .
- If f' is also differentiable, we can differentiate f' to get the **second derivative** of f :

- **Definition:**

$$f''(x) \equiv \lim_{h \rightarrow 0} \frac{f'(x+h) - f'(x)}{h}$$

- **Alternative notations:** $\frac{d^2 f(x)}{dx^2}$, $\frac{d^2 y}{dx^2}$, y''

- As long as the differentiability condition is met, higher-order derivatives can be written similarly as:

$$f'''(x), f^{(4)}(x), \dots, f^{(n)}(x) \quad \text{or} \quad \frac{d^3 y}{dx^3}, \frac{d^4 y}{dx^4}, \dots, \frac{d^n y}{dx^n}$$

Interpretation of the Second Derivative

- **First derivative:**

$f'(x_0) > 0$: the value of the function tends to *increase*.

$f'(x_0) < 0$: the value of the function tends to *decrease*.

- **Second derivative:**

$f''(x_0) > 0$: the slope of the curve tends to *increase*.

$f''(x_0) < 0$: the slope of the curve tends to *decrease*.

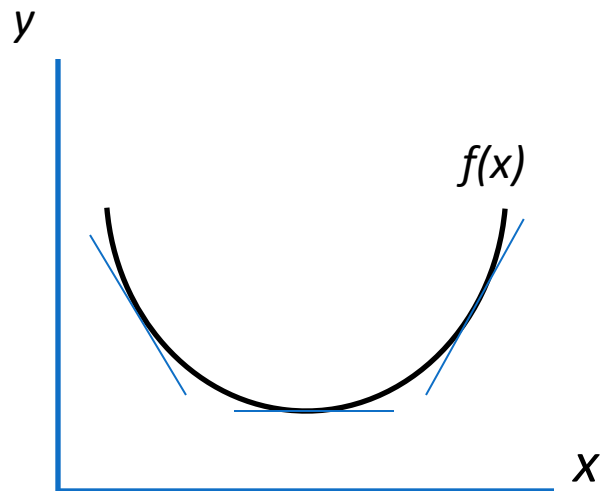
- **Possible cases:**

- $f'(x) > 0$ and $f''(x) > 0$: the slope is *positive and increasing* as x increases.
- $f'(x) < 0$ and $f''(x) > 0$: the slope is *negative and increasing* as x increases.
- $f'(x_0) > 0$ and $f''(x) < 0$: the slope is *positive and decreasing* as x increases.
- $f'(x_0) < 0$ and $f''(x) < 0$: the slope is *negative and decreasing* as x increases.

Curvature of a Graph

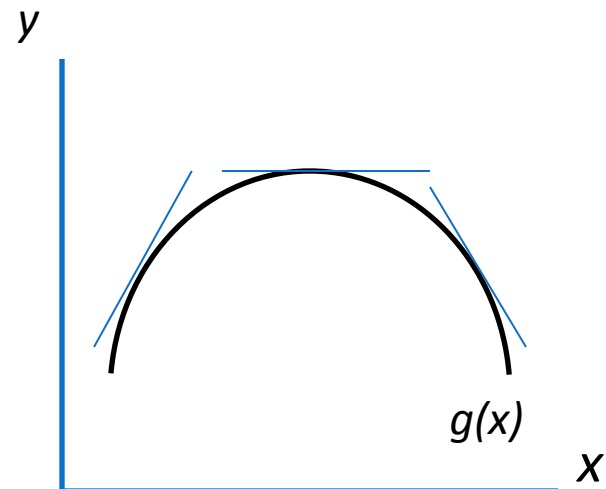
$$f''(x) > 0:$$

Slope of the tangent increases as x increases (i.e. $f'(x)$ is increasing).



$$g''(x) < 0:$$

Slope of the tangent decreases as x increases (i.e. $g'(x)$ is decreasing).



Convexity and Concavity (1)

Definition: Strictly Convex

A function $f(x)$ is *strictly convex* if we pick any pair of points M and N on its curve and join them by a straight line, the line segment MN must lie entirely *above* the curve, excepts at points M and N .

Definition: Strictly Concave

A function $f(x)$ is *strictly concave* if we pick any pair of points M and N on its curve and join them by a straight line, the line segment MN must lie entirely *below* the curve, excepts at points M and N .

Note: When the segment MN *either* lies above (below) *or* lies along the curve, the function is *convex* (*concave*).

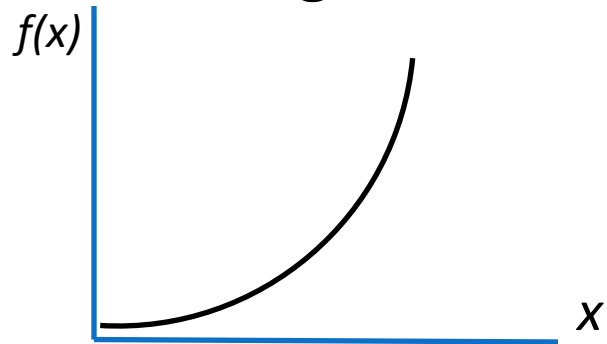
Convexity and Concavity (2)

- **Definition:** Assume that f is continuous and twice differentiable.
If $f''(x) > 0$ for all x , then $f(x)$ is a *strictly convex* function.
If $f''(x) < 0$ for all x , then $f(x)$ is a *strictly concave* function.
- **Example 1:** $f(x) = x^2 - 2x + 2$
- **Example 2:** $f(x) = ax^2 + bx + c$

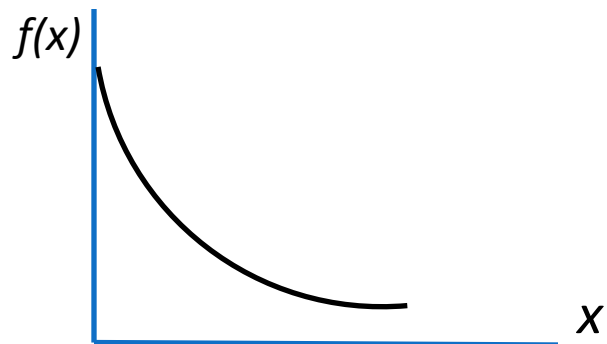
Note: In certain cases, $f''(x)$ may have a zero value at the stationary point. Thus, the second derivative is a **sufficient but not necessary condition**. (Example: $f(x) = x^4$).

Convexity and Concavity (3)

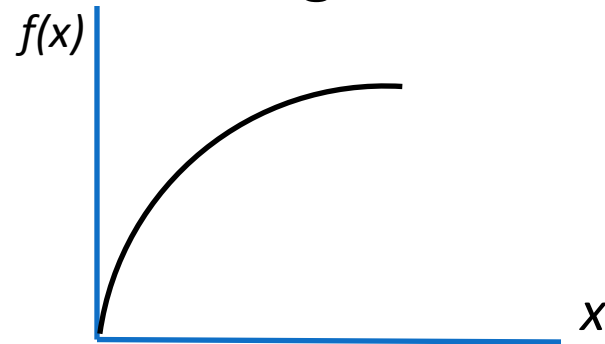
- Increasing Convex



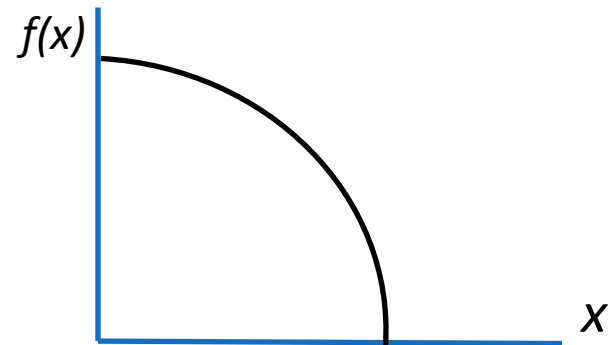
- Decreasing Convex



- Increasing Concave

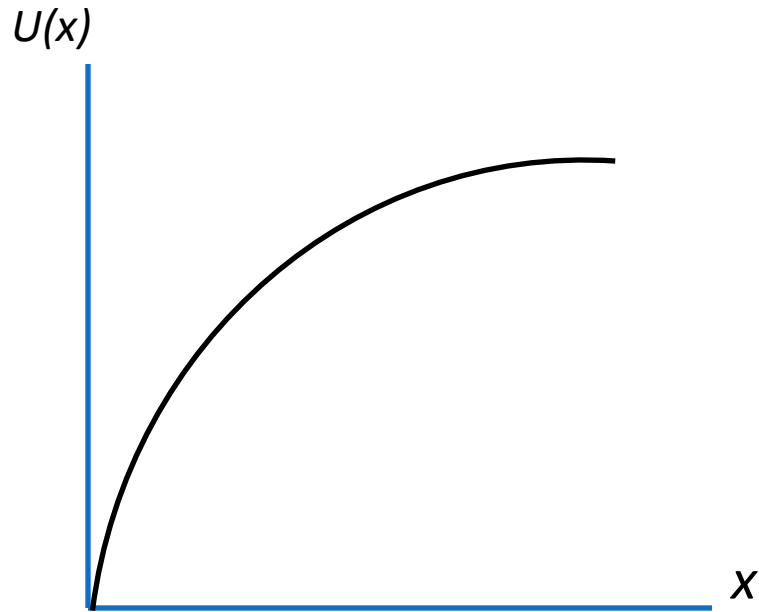


- Decreasing Concave

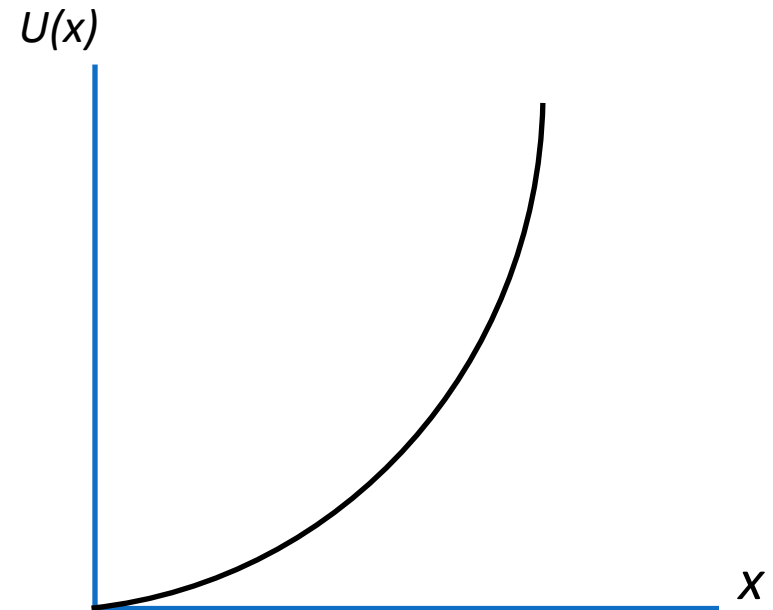


Example: Attitudes toward Risk

Risk-Averse Preference



Risk-Loving Preference



Second-Derivative Test

Second-Derivative Test for Relative Extremum:

If $f'(x_0) = 0$, then the value of $f(x_0)$ will be

- a. A relative *maximum* if $f''(x) < 0$.
- b. A relative *minimum* if $f''(x) > 0$.

Example 1: Determine the extremum of $f(x) = x^3 - 3x^2 + 2$

Example 2: Determine the extremum of $g(x) = \frac{x^4}{4} - \frac{3}{2}x^2$

Necessary VS. Sufficient Conditions for Relative Extremum

Condition	Maximum	Minimum
First-order necessary	$f'(x) = 0$	$f'(x) = 0$
Second-order necessary	$f''(x) \leq 0$	$f''(x) \geq 0$
Second-order sufficient	$f''(x) < 0$	$f''(x) > 0$

Example: Determine the extremum of $f(x) = x^4$

- FOC: $f'(x) = 4x^3 = 0$ when $x^* = 0$.
- SOC: $f''(x) = 12x^2 \geq 0$ is a second-order necessary condition for a minimum.

Optimization without Constraints

- Main application of finding **derivatives** is to solve optimization problems.
- Recall: The essence of the **optimization problem** is to choose the **best alternative** available.
- Two types of optimization problems:
 - Optimization without constraints**
 - Profit maximization
 - Tax revenue maximization
 - Optimization with constraints (**after midterm**)

OPTIMIZATION PROBLEMS

Profit maximization

Profit Maximization

- Profit function: $\pi(Q) = R(Q) - C(Q)$

- First-order necessary condition:

$$\pi'(Q) = R'(Q) - C'(Q) = 0$$

$$\Leftrightarrow MR(Q) = MC(Q)$$

(i.e. slope of revenue function = slope of cost function)

- Second-order sufficient condition:

$$\pi''(Q^*) < 0$$

$$\Leftrightarrow MR'(Q) < MC'(Q)$$

- Profit-maximizing output:

Q^* such that $\pi'(Q^*) = 0$.

Profit Maximization: Perfect Competition

- Let the profit function be $\pi = TR - TC$ where $TR=R(Q)$; $TC=C(Q)$
- Objective function: $\max_Q \pi = R(Q) - C(Q)$
- Perfectly competitive market: P is constant.

→ Total revenue: $TR = P \times Q$

- First-order necessary condition:

$$\pi'(Q) = R'(Q) - C'(Q) = 0$$

$$\Leftrightarrow P = MC(Q)$$

- Second-order sufficient condition:

$$R''(Q) = 0 \text{ \& } C''(Q) > 0$$

$$\Leftrightarrow \pi''(Q^*) < 0$$

Graph: π -max in perfect competition

Example: π -max in perfect competition

- Let $P = 30$; $TC = 100 + 19Q - 5Q^2 + (1/3)Q^3$. Find Q^* which maximizes profit.

Profit Maximization: Monopoly

- **Monopoly**: Producer/seller can set the price in the market.

- Let $P = f(Q) = a - bQ$

- Total revenue:

$$TR = P \times Q = [a - bQ]Q = aQ - bQ^2$$

- Marginal revenue:

$$MR = d(TR)/dQ = a - 2bQ$$

- **Profit maximization**:

- Objective function:

$$\max_Q \pi = TR(Q) - TC(Q)$$

Graph: π -max in monopoly market

Example: Profit Maximization

Let $R(Q) = 4000Q - 33Q^2$

$$C(Q) = 2Q^3 - 3Q^2 + 400Q + 5000$$

Determine the profit maximizing output level.

FONC: $\pi'(Q) = R'(Q) - C'(Q) = 0$

$$\Leftrightarrow (4000Q - 33Q^2) - (2Q^3 - 3Q^2 + 400Q + 5000) = 0$$

$$\Leftrightarrow Q^* = 20$$

SOSC: $\pi''(20) = -300 < 0$.

Thus, $Q^* = 20$ is the profit-maximizing output level.

Example: π -max in monopoly market

- Let $P = 48 - 0.5Q$; $TC = 2 + 60Q - 8Q^2 + Q^3$. Find Q^* which maximizes profit.

Effect of Taxes

- Given that $\pi = TR(Q) - TC(Q)$, what will happen to profit-maximizing quantity and price if the government imposes:
 - Lump-sum tax
 - Fixed tax amount, e.g. $T = t_0$.
 - Profit tax
 - Tax changes according to profit: $T = t\pi$, where $0 < t < 1$.
 - Specific (or excise) tax
 - Tax varies with quantity: $T = tQ$, where $0 < t < 1$.

Effect of Taxes: Lump-Sum Tax

- Net profit after lump-sum tax: $\pi_N = TR - TC - t_0$

- Objective function:

$$\max_Q \pi_N = TR(Q) - TC(Q) - t_0$$

- Profit-maximizing condition:

- Effect of lump-sum tax on profit-max quantity:

Effect of Taxes: Profit Tax

- Net profit after profit tax: $\pi_N = TR - TC - t\pi = (1-t)[TR - TC]$

- Objective function:

$$\max_Q \pi_N = (1-t)[TR(Q) - TC(Q)]$$

- Profit-maximizing condition:

- Effect of profit tax on profit-max quantity:

Effect of Taxes: Excise (or Specific) Tax

- Net profit after tax: $\pi_N = TR - TC - tQ$

- Objective function:

$$\max_Q \pi_N = TR(Q) - TC(Q) - tQ$$

- Profit-maximizing condition:

- Effect of specific tax on profit-max quantity:

OPTIMIZATION PROBLEMS

Tax Revenue Maximization

Tax Revenue Maximization (1)

- Let $P = a - bQ$, so that $TR = aQ - bQ^2$.
- Let $TC = c_0 + c_1Q + c_2Q^2$.
- Suppose the government imposes a specific tax t baht per unit.
 - After-tax total cost is:
 - After-tax profit:
 - Profit-maximizing condition:

 - *Total* tax revenue:

Tax Revenue Maximization (2)

- Tax-revenue-maximizing condition:
- Example: Let $P = 40 - 0.5Q$, and $TC = 2 - 5Q + 7Q^2$. Find the specific tax rate t that maximizes the government tax revenue.