CHAPTER 6

Optimal control theory

6.1 The optimal control problem

Consider a fish stock which has some natural rate of growth and which is harvested. Too much harvesting could endanger the survival of the fish, too little and profits are forgone. Of course, harvesting takes place over time. The obvious question is: 'what is the best harvesting rate, i.e., what is the optimal harvesting?' The answer to this question requires an *optimal path or trajectory* to be identified. 'Best' itself requires us to specify a criterion by which to choose between alternative paths. Some policy implies there is a means to influence (control) the situation. If we take it that $x(t)$ represents the *state* of the situation at time t and $u(t)$ represents the *control* at time t, then the **optimal control problem** is to find a trajectory $\{x(t)\}$ by choosing a set $\{u(t)\}$ of controls so as to maximise or minimise some objective that has been set. There are a number of ways to solve such a control problem, of which the literature considers three:

(1) Calculus of variations
(2) Dynamic programming
(3) Maximum principle.

In this chapter we shall deal only with the third, which now is the dominant approach, especially in economics. This approach is based on the work of Pontryagin *et al.* (1962), and is therefore sometimes called the **Pontryagin maximum principle**.

Since minimising some objective function is the same as maximising its negative value, then we shall refer in this chapter only to maximising some objective function. Second, our control problem can either be in *continuous time* or in *discrete time*. To see the difference and to present a formal statement of the optimal control problem from the maximum principle point of view, consider table 6.1. In each case, the objective is to maximise J and so find a trajectory $\{x(t)\}$ by choosing a suitable value $\{u(t)\}$. What table 6.1 presents is the most general situation possible for both the continuous and discrete formulations of the optimal control problem under the maximisation principle. There are some special cases, the most important being the distinction between finite and infinite horizon models. In the latter case the terminal time period is at infinity. All the problems we shall discuss in this chapter involve autonomous systems, and so t does not enter explicitly into V, f or F. An important aspect of control problems is that of time preference. The

Table 6.1 The control problem

Continuous	**Discrete**
$\max\limits_{\{\mathbf{u}(t)\}} J = \int_{t_0}^{t_1} V(\mathbf{x}, \mathbf{u}, t)dt + F(\mathbf{x}^1, t)$	$\max\limits_{\{\mathbf{u}_t\}} J = \sum_{t=0}^{T-1} V(\mathbf{x}_t, \mathbf{u}_t, t) + F(\mathbf{x}^T, t)$
$\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u}, t)$	$\mathbf{x}_{t+1} - \mathbf{x}_t = f(\mathbf{x}_t, \mathbf{u}_t, t)$
$\mathbf{x}(t_0) = \mathbf{x}^0$	$\mathbf{x}_t = \mathbf{x}^0$ when $t = 0$
$\mathbf{x}(t_1) = \mathbf{x}^1$	$\mathbf{x}_t = \mathbf{x}^T$ when $t = T$
$\{\mathbf{u}(t)\} \in U$	$\mathbf{u}_t \in U$

t_0 (or $t = 0$) is initial time
t_1 (or T) is terminal time
$\mathbf{x}(t) = \{x_1(t), \ldots, x_n(t)\}$ or $\mathbf{x}_t = \{x_{1t}, \ldots, x_{nt}\}$ n-state variables
$\mathbf{x}(t_0) = \mathbf{x}^0$ or $\mathbf{x}_t = \mathbf{x}^0$ for $t = 0$ is the initial state
$\mathbf{x}(t_1) = \mathbf{x}^1$ or $\mathbf{x}_t = \mathbf{x}^T$ for $t = T$ is the final state (or terminal state)
$\mathbf{u}(t) = \{u_1(t), \ldots, u_m(t)\}$ or $\mathbf{u}_t = \{u_{1t}, \ldots, u_{mt}\}$ m-control variables
$\{\mathbf{u}(t)\}$ is a continuous control trajectory $t_0 \le t \le t_1$
$\{\mathbf{u}_t\}$ is a discrete control trajectory $0 \le t \le T$
U is the set of all admissible control trajectories
$\dot{\mathbf{x}}(t) = f(\mathbf{x}, \mathbf{u}, t)$ or $\mathbf{x}_{t+1} - \mathbf{x}_t = f(\mathbf{x}_t, \mathbf{u}_t, t)$ denote the equations of motion
J is the objective function
$V(\mathbf{x}(t), \mathbf{u}(t), t)$ or $V(\mathbf{x}_t, \mathbf{u}_t, t)$ is the intermediate function
$F(\mathbf{x}^1, t)$ or $F(\mathbf{x}^T, t)$ is the final function

simplest models involve no discounting. It is sometimes easier to consider a model with no discounting, and then to consider the more realistic case of the same model with discounting. In many models the terminal value $F(x^T)$ is zero, but this need not always be so.

A typical continuous optimal control problem incorporating the assumptions of (1) a finite time horizon, T, (2) only autonomous equations, (3) a zero function in the terminal state and, (4) only one state variable and one control variable is

$$
\begin{aligned}
&\max_{\{u(t)\}} J = \int_0^T V(x, u)dt \\
&\dot{x} = f(x, u) \\
&x(0) = x^0 \\
&x(T) = x^T
\end{aligned}
\tag{6.1}
$$

where the state variable, x and the control variable, u, are both functions of time t.

The situation is illustrated in figure 6.1. The paths u^* and u^{**} both constitute solutions to the differential equation $\dot{x} = f(x, u)$. The problem, however, is to choose one path that maximises the relation J and that satisfies the terminal condition $x(t^*) = x^T$ and $x(t^{**}) = x^T$.

6.2 The Pontryagin maximum principle: continuous model

As just pointed out, the objective is to find a control trajectory $\{u(t)\}$ that maximises J and takes the system from its present state x^0 to its terminal state x^T. What is required, therefore, is a 'set of weights' that allows a comparison of the different trajectories of alternative controls. Also note that the emphasis of this formulation

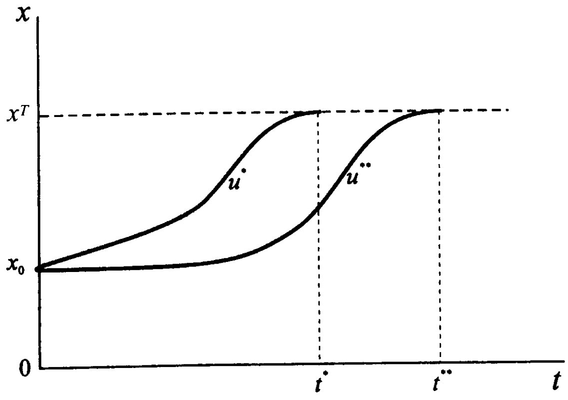


Figure 6.1.

of the control problem is to find the optimal control trajectory $\{u(t)\}$. Once this is known the optimal state trajectory $\{x(t)\}$ can be computed. The 'weights' are achieved by defining a **Hamiltonian** for the control problem (6.1).

As with Lagrangian multipliers, let $\lambda(t)$ denote the Lagrangian multiplier for the constraint $\dot{x} = f(x, u)$. This is referred to as the *costate variable* or *adjoint variable*. Then

$$
\begin{aligned}
L &= \int_0^T V(x, u)dt + \int_0^T \lambda[f(x, u) - \dot{x}]dt \\
&= \int_0^T [V(x, u) + \lambda f(x, u) - \lambda \dot{x}]dt
\end{aligned}
$$

The **Hamiltonian function** is defined as

$$H(x, u) = V(x, u) + \lambda f(x, u) \tag{6.2}$$

Hence

$$L = \int_0^T [H(x, u) - \lambda \dot{x}]dt \tag{6.3}$$

Equation (6.3) can be further transformed by noting that (see exercise 2)

$$-\int_0^T \lambda dt = \int_0^T x\dot{\lambda} dt - [\lambda(T)x(T) - \lambda(0)x(0)] \tag{6.4}$$

which allows us to express L as

$$L = \int_0^T [H(x, u) + \dot{\lambda} x]dt - [\lambda(T)x(T) - \lambda(0)x(0)] \tag{6.5}$$

Consider what happens to the state variable when the control variable changes, i.e., let $\{u(t)\}$ change to $\{u(t) + \Delta u(t)\}$ with the result on the state trajectory from $\{x(t)\}$ to $\{x(t) + \Delta x(t)\}$. Then the change in the Lagrangian, ΔL, is

$$
\begin{aligned}
\Delta L &= \int_0^T \left[\frac{\partial H}{\partial x}dx + \frac{\partial H}{\partial u}du + \dot{\lambda} dx\right]dt - \lambda(T)dx^T \\
&= \int_0^T \left[\frac{\partial H}{\partial u}du + \left(\frac{\partial H}{\partial x} + \dot{\lambda}\right)dx\right]dt - \lambda(T)dx^T
\end{aligned}
$$

For a maximum $\Delta L = 0$. This implies the necessary conditions:

$$\text{(i)} \qquad \frac{\partial H}{\partial u} = 0 \qquad 0 \le t \le T$$

$$\text{(ii)} \qquad \dot{\lambda} = -\frac{\partial H}{\partial x} \qquad 0 \le t \le T$$

$$\text{(iii)} \qquad \lambda(T) = 0 \text{ (or } x(T) = x^T \text{ if } x^T \text{ is known)}$$

Condition (i) states that the Hamiltonian function is maximised by the choice of the control variable at each point along the optimum trajectory – where we are assuming an interior solution and no constraint on the control variable. Condition (ii) is concerned with the rate of change of the costate variable, λ. It states that the rate of change of the costate variable is equal to the negative of the Hamiltonian function with respect to the corresponding state variable.[1] Condition (iii) refers to the costate variable in the terminal state, and indicates that it is zero; or if the terminal value $x(T) = x^T$ is given then $dx^T = 0$.

From the definition of the Hamiltonian function, the differential equation for the state variable can be expressed in terms of it as follows

$$\dot{x} = f(x, u) = \frac{\partial H}{\partial \lambda}$$

We therefore arrive at the following procedure. Add a costate variable λ to the problem and define a Hamiltonian function $H(x, u) = V(x, u) + \lambda f(x, u)$ and solve for trajectories $\{u(t)\}$, $\{\lambda(t)\}$, and $\{x(t)\}$ satisfying:

$$\text{(6.6)} \qquad \begin{array}{lll} \text{(i)} & \dfrac{\partial H}{\partial u} = 0 & 0 \le t \le T \\ \text{(ii)} & \dot{\lambda} = -\dfrac{\partial H}{\partial x} & 0 \le t \le T \\ \text{(iii)} & \dot{x} = \dfrac{\partial H}{\partial \lambda} = f(x, u) & \\ \text{(iv)} & x(0) = x^0 & \\ \text{(v)} & \lambda(T) = 0 & (\text{or } x(T) = x^T) \end{array}$$

These results can be generalised for $x_1(t), \ldots, x_n(t)$ state variables, $\lambda_1(t), \ldots, \lambda_n(t)$ costate variables and $u_1(t), \ldots, u_m(t)$ control variables:

$$\text{(6.7)} \qquad \begin{array}{llll} \text{(i)} & \dfrac{\partial H}{\partial u_i} = 0 & i = 1, \ldots, m & 0 \le t \le T \\ \text{(ii)} & \dot{\lambda} = -\dfrac{\partial H}{\partial x_i} & i = 1, \ldots, n & 0 \le t \le T \\ \text{(iii)} & \dot{x} = \dfrac{\partial H}{\partial \lambda_i} = f(x, u) & i = 1, \ldots, n & \end{array}$$

[1] If there were, for example, two state variables x_1 and x_2 and two corresponding costate variables λ_1 and λ_2, then

$$\dot{\lambda}_1 = -\partial H/\partial x_1 \qquad 0 \le t \le T$$
$$\dot{\lambda}_2 = -\partial H/\partial x_2 \qquad 0 \le t \le T$$

(iv) $\quad x_i(0) = x_i^0 \qquad i = 1, \ldots, n$

(v) $\quad \lambda_i(T) = 0 \qquad i = 1, \ldots, n \qquad (\text{or } x_i(T) = x_i^T \qquad i = 1, \ldots n)$

We shall now illustrate the continuous control problem by considering three examples. In each case we have the initial value and the terminal value for the state variable, i.e., $x(0) = x^0$and $x(T) = x^T$ are given, as of course is T.

Example 6.1

In this first example we consider a boundary solution. The control problem is:

$$\max_{\{u\}} \int_0^1 5x\,dx$$
$$\dot{x} = x + u$$
$$x(0) = 2, \qquad x(1) \text{ free}$$
$$u(t) \in [0, 3]$$

The Hamiltonian for this problem is

$$\begin{aligned} H(x, u) &= V(x, u) + \lambda f(x, u) \\ &= 5x + \lambda(x + u) \\ &= (5 + \lambda)x + \lambda u \end{aligned}$$

With first-order conditions:

(i) $\quad \dfrac{\partial H}{\partial u} = \lambda$

(ii) $\quad \dot{\lambda} = -\dfrac{\partial H}{\partial x} = -(5 + \lambda)$

(iii) $\quad \dot{x} = x + u$

(iv) $\quad x(0) = 2$

(v) $\quad \lambda(1) = 0$

Condition (i) is no help in determining u^*. If $\lambda > 0$ then H is a maximum at $u = 3$ the boundary, hence $u^*(t) = 3$, as shown in Figure 6.2(a).

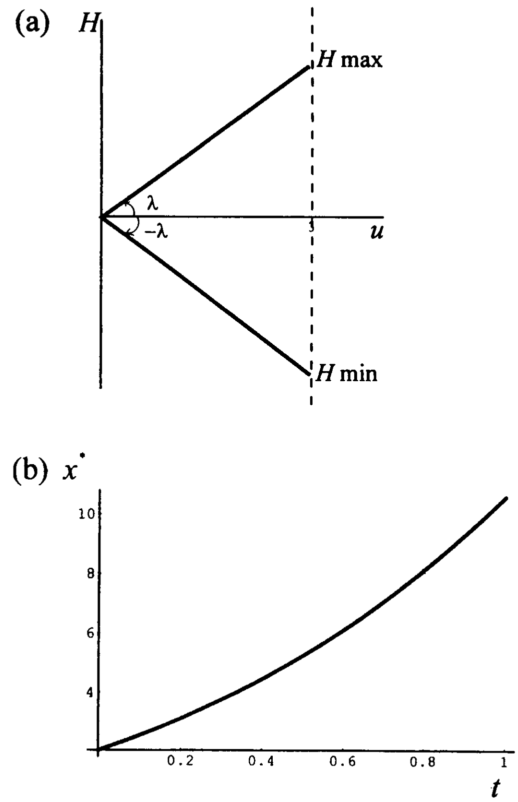
From (ii) we have

$$\dot{\lambda} = -\lambda - 5$$
$$\lambda^*(t) = ke^{-t} - 5$$
$$\lambda^*(1) = ke^{-1} - 5 = 0$$
$$k = 5e^1$$
$$\therefore \quad \lambda^*(t) = 5e^{1-t} - 5$$

Since $u^*(t) = 3$

$$\dot{x}^* = x^* + 3$$
$$x^*(t) = -3 + ke^t$$
$$x(0) = -3 + ke^0 = 2$$
$$\therefore \quad k = 5$$

Figure 6.2.



Hence

$$x^*(t) = -3 + 5e^t$$

Although the control variable remains constant throughout, the state variable increases from $x(0) = 2$, as shown in figure 6.2(b).

Example 6.2

The control problem is

$$\max_{\{u\}} \int_0^1 u^2 \, dt$$

$$\dot{x} = -u$$

$$x(0) = 1$$

$$x(1) = 0$$

The Hamiltonian for this problem is

$$\begin{aligned} H(x, u) &= V(x, u) + \lambda f(x, u) \\ &= u^2 + \lambda(-u) \end{aligned}$$

with first-order conditions:

$$\text{(i)} \quad \frac{\partial H}{\partial u} = 2u - \lambda = 0$$

$$\text{(ii)} \quad \dot{\lambda} = -\frac{\partial H}{\partial x} = 0$$

$$\text{(iii)} \quad \dot{x} = -u$$

$$\text{(iv)} \quad x(0) = 1$$

$$\text{(v)} \quad x(1) = 0$$

From (i)

$$2u = \lambda$$

$$u = \tfrac{1}{2}\lambda$$

Thus

$$\dot{x} = -\frac{\lambda}{2}$$

$$\dot{\lambda} = 0$$

Solving these with a software package we obtain

$$x(t) = c_1 - \frac{\lambda t}{2}$$

$$\lambda(t) = c_2$$

But $x(0) = 1$ so

$$1 = c_1 - \frac{0}{2} \qquad \text{or} \qquad c_1 = 1$$

Similarly $x(1) = 0$

$$x(1) = 1 - \frac{\lambda}{2} = 0$$

$$\therefore \quad \lambda = 2 \qquad \text{or} \qquad c_2 = 2$$

$$x^* = 1 - \frac{2t}{2} = 1 - t$$

$$u^* = \tfrac{1}{2}\lambda = 1$$

These optimal paths are illustrated in figure 6.3.

Example 6.3

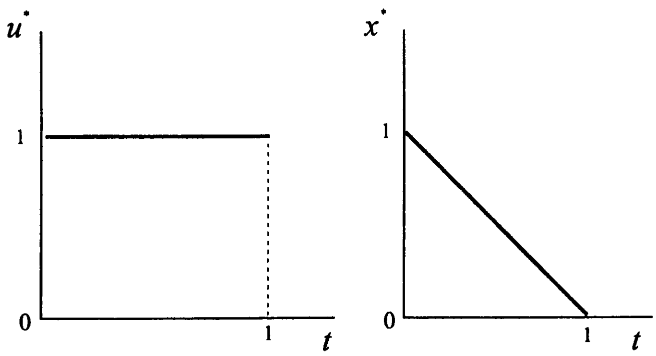
The control problem is

$$\max_{\{u\}} - \int_0^1 \tfrac{1}{4}(x^2 + u^2)dt$$

$$\dot{x} = x + u$$

$$x(0) = 2, \quad x(1) = 0$$

Figure 6.3.



Optimal path of control variable

Optimal path of state variable

The Hamiltonian for this problem is

$$\begin{aligned} H(x, u) &= V(x, u) + \lambda f(x, u) \\ &= \frac{-(x^2 + u^2)}{4} + \lambda(x + u) \end{aligned}$$

With first-order conditions

$$\text{(i)} \quad \frac{\partial H}{\partial u} = -\frac{u}{2} + \lambda = 0 \quad \text{implying} \quad u = 2\lambda$$

$$\text{(ii)} \quad \dot{\lambda} = -\frac{\partial H}{\partial x} = -\left(\frac{-x}{2} + \lambda\right) = \tfrac{1}{2}x - \lambda$$

$$\text{(iii)} \quad \dot{x} = x + u \quad \text{implying} \quad \dot{x} = x + 2\lambda$$

Substituting (i) into (iii) and eliminating u, we arrive at two differential equations in terms of x and λ

$$\begin{aligned} \dot{x} &= x + 2\lambda \\ \dot{\lambda} &= \tfrac{1}{2}x - \lambda \end{aligned}$$

Although a simple set of differential equations, the solution values are rather involved, especially when solving for the constants of integration. The general solution is[2]

$$\begin{aligned} x(t) &= c_1 e^{\sqrt{2}t} + c_2 e^{-\sqrt{2}t} \\ \lambda(t) &= \frac{c_1}{2}(\sqrt{2} - 1)e^{\sqrt{2}t} - \frac{c_2}{2}(\sqrt{2} + 1)e^{-\sqrt{2}t} \end{aligned}$$

However we can solve for c_1 and c_2 by using the conditions $x(0) = 2$ and $x(1) = 0$ as follows

$$\begin{aligned} x(0) &= c_1 + c_2 = 2 \\ x(1) &= c_1 e^{\sqrt{2}t} + c_2 e^{-\sqrt{2}t} = 0 \end{aligned}$$

[2] The software packages give, on the face of it, quite different solutions. They are, however, identical. The results provided here are a re-arrangement of those provided by *Maple*.

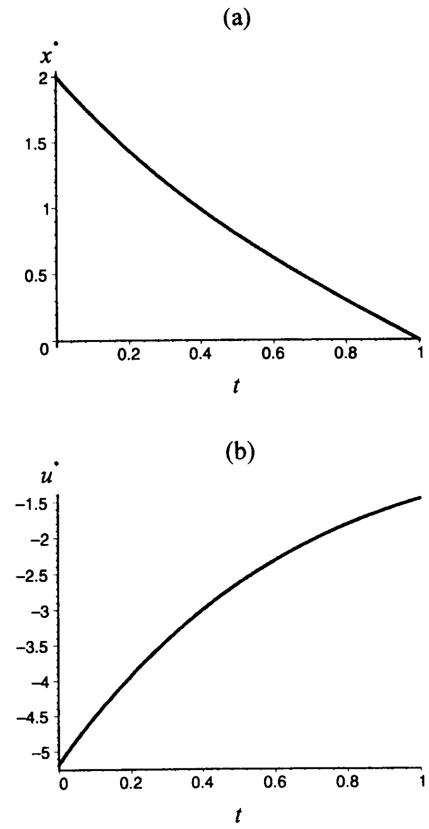


Figure 6.4.

Solving we get $c_1 = -0.1256$ and $c_2 = 2.1256$. All this can be done with the help of computer software programs, with the resulting trajectories for x^* and u^* shown in figure 6.4(a) and (b).

What these examples show is a pattern emerging for solving the control problem. The steps are:

(1) Specify the Hamiltonian and obtain the maximisation conditions
(2) Use the equation $\partial H/\partial u$ to solve for u in terms of the costate variable λ
(3) Obtain two differential equations: one for the state variable, x, and one for the costate variable, λ
(4) Solve the differential equations deriving general solutions
(5) Use the conditions on $x(0)$ and $x(T)$ to obtain values for the coefficients of integration
(6) Substitute the optimal path for λ^* into the equation for u to obtain the optimal path u^* for the control variable.

6.3 The Pontryagin maximum principle: discrete model

The discrete time control model based on the maximum principle of Pontryagin takes a similar approach to the continuous time formulation so we can be brief, although some care must be exercised in the use of time periods. Again we let x denote the only state variable, u the only control variable and λ the costate variable. Our problem amounts to:

$$\begin{aligned}
&\max_{\{u_t\}} J = \sum_{t=0}^{T-1} V(x_t, u_t) \\
&x_{t+1} - x_t = f(x_t, u_t) \qquad\qquad (6.8) \\
&\qquad x_0 = a
\end{aligned}$$

The Lagrangian is then

$$L = \sum_{t=0}^{T-1} \{V(x_t, u_t) + \lambda_{t+1}[f(x_t, u_t) - (x_{t+1} - x_t)]\} \tag{6.9}$$

Define the discrete form Hamiltonian function

$$H(x_t, u_t) = V(x_t, u_t) + \lambda_{t+1} f(x_t, u_t) \tag{6.10}$$

then

$$L = \sum_{t=0}^{T-1} [H(x_t, u_t) - \lambda_{t+1}(x_{t+1} - x_t)]$$

which can be maximised by satisfying the conditions

$$\begin{aligned}
&\frac{\partial L}{\partial u_t} = \frac{\partial H}{\partial u_t} = 0 \qquad t = 0, \ldots, T-1 \\
&\frac{\partial L}{\partial x_t} = \frac{\partial H}{\partial x_t} + \lambda_{t+1} - \lambda_t = 0 \qquad t = 1, \ldots, T-1 \\
&\frac{\partial L}{\partial \lambda_{t+1}} = \frac{\partial H}{\partial \lambda_{t+1}} - (x_{t+1} - x_t) \qquad t = 0, \ldots, T-1 \\
&\frac{\partial L}{\partial x_T} = -\lambda_T = 0
\end{aligned}$$

More succinctly:

$$\begin{aligned}
&\text{(i)} && \frac{\partial H}{\partial u_t} = 0 \qquad t = 0, \ldots, T-1 \\
&\text{(ii)} && \lambda_{t+1} - \lambda_t = -\frac{\partial H}{\partial x_t} \qquad t = 1, \ldots, T-1 \\
&\text{(iii)} && x_{t-1} - x_t = \frac{\partial H}{\partial \lambda_{t+1}} = f(x_t, u_t) \qquad t = 0, \ldots, T-1 \\
&\text{(iv)} && \lambda_T = 0 \\
&\text{(v)} && x_0 = a
\end{aligned} \tag{6.11}$$

It is useful to verify these conditions for, say, $T = 3$, most especially noting the range for t for condition (ii).

But how do we go about solving such a model? Unlike the continuous time model it is not simply solving two differential equations. It is true that in each time period we have two difference equations for the state and costate variables that require solving simultaneously. One solution method is to program the problem, as in Conrad and Clark (1987). A simpler method in the case of numerical examples is to use a spreadsheet. To illustrate the solution method by means of a spreadsheet, consider the following example.

Example 6.4[3]

Iron ore sells on the market at a constant price p per period but costs $c_t = by_t^2/x_t$, where x_t denotes the remaining reserves at the beginning of period t and y_t is the production in period t. The mine is to be shut down in period 10. What is the optimal production schedule $\{y_t^*\}$ for $t = 0, \ldots, 9$ given $p = 3$, $b = 2$ and the initial reserves $x_0 = R = 600$ tons? (Assume no discounting over the period.)

Let us first set up the model in general terms, replacing u_t by y_t. The objective function $V(x_t, y_t)$ is no more than the (undiscounted) profit, namely

$$V(x_t, y_t) = py_t - \frac{by_t^2}{x_t} = \left(p - \frac{by_t}{x_t}\right)y_t$$

Next we note that if x_t denotes the remaining reserves at the beginning of period t, then $x_{t+1} = x_t - y_t$ or $x_{t+1} - x_t = -y_t$. Thus, our Hamiltonian function is

$$H(x_t, y_t) = \left(p - \frac{by_t}{x_t}\right)y_t - \lambda_{t+1}y_t$$

Our optimality conditions are therefore:

(i) $\quad \dfrac{\partial H}{\partial y_t} = p - \dfrac{2by_t}{x_t} - \lambda_{t+1} = 0 \qquad t = 0, \ldots, 9$

(ii) $\quad \lambda_{t+1} - \lambda_t = -\dfrac{\partial H}{\partial x_t} = -\left(\dfrac{by_t^2}{x_t^2}\right) \qquad t = 1, \ldots, 9$

(iii) $\quad x_{t-1} - x_t = -y_t \qquad t = 0, \ldots, 9$

(iv) $\quad x_0 = R$

(v) $\quad \lambda_T = 0$

To solve this problem for a particular numerical example, let $p = 3$, $b = 2$ and $R = 600$. The computations are set out in detail in figure 6.5. In doing these computations we begin in period 10 and work backwards (see exercise 1 on backward solving).

Since $\lambda_{10} = 0$ then from (i) we know

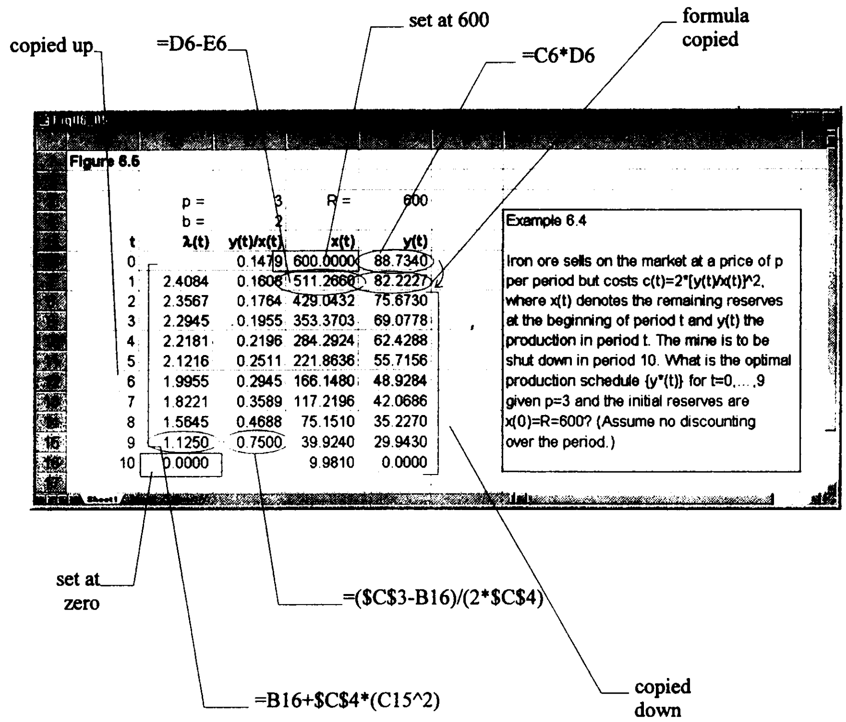
$$3 - 4\left(\frac{y_9}{x_9}\right) = 0$$

which allows us to compute y_9/x_9. Having solved for y_9/x_9 we can then use condition (ii) to solve for λ_9. We do this repeatedly back to period 0. This gives us columns 2 and 3 of the spreadsheet. Since $x_0 = R = 600$, we have the first entry in the $x(t)$ column. Then y_0 is equal to $x_0(y_0/x_0)$ and finally $x_1 = x_0 - y_0$. This allows us to complete the final two columns.

The optimal production path $\{y_t^*\}$ is therefore given by the final column in figure 6.5 and its path, along with that of the reserves, is shown in figure 6.6(a).

[3] This is adapted from Conrad and Clark (1987, p. 20).

Figure 6.5.



Given the computations the trajectory for (λ_t^*, x_t^*) can also be plotted, which is shown in figure 6.6(b), which are direct plottings from a spreadsheet.

In this example we solved the discrete optimisation problem by taking account of the first-order conditions and the constraints. We employed the spreadsheet merely as a means of carrying out some of the computations. However, spreadsheets come with nonlinear programming algorithms built in. To see this in operation, let us re-do the present example using *Excel*'s nonlinear programming algorithm, which is contained in the *Solver* add-on package.[4] The initial layout of the spreadsheet is illustrated in figure 6.7.

It is important to note that when setting out this initial spreadsheet we place in cells B7 to B16 some 'reasonable' numbers for extraction. Here we simply assume a constant rate of extraction of 60 throughout the 10 periods $t = 0$ to $t = 9$. Doing this allows us to compute columns D and E. Column D sets $\lambda_{10} = 0$ and then copies *backwards* the formula

$$\lambda_t = \lambda_{t+1} + \left(\frac{by_t^2}{x_t^2}\right)$$

for cells D16 to D8 (no value is placed in cell D7). The values in column E are the values for the objective function $V(x_t, y_t)$. The value for L, which is the sum of the values in column E for periods $t = 0$ to $t = 9$, is placed in cell E19. At the moment this stands at the value 1448.524.

[4] On using *Excel*'s Solver see Whigham (1998), Conrad (1999) and Judge (2000).

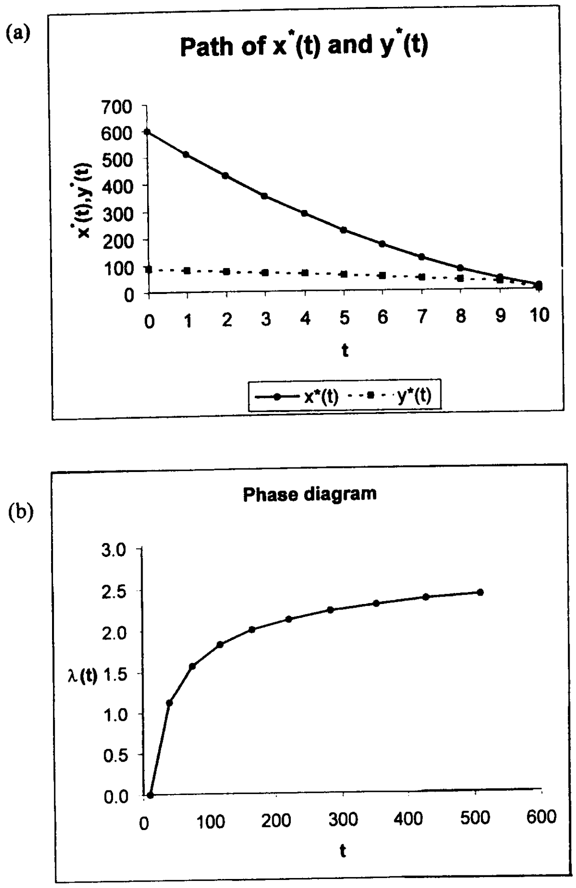


Figure 6.6.

Figure 6.7

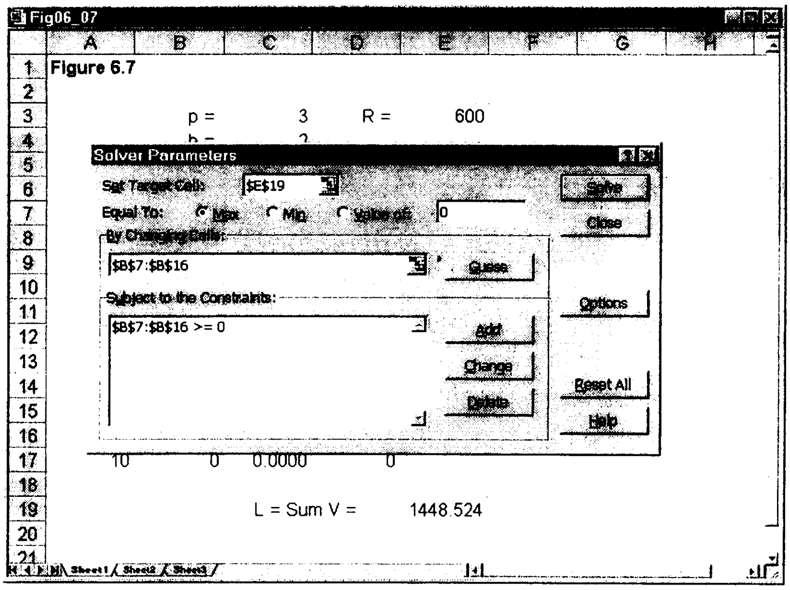
p = 3 R = 600
b = 2

t	y(t)	x(t)	λ(t)	V(x,y)
0	60.0000	600		168.0000
1	60.0000	540.0000	3.0795	166.6667
2	60.0000	480.0000	3.0548	165.0000
3	60.0000	420.0000	3.0236	162.8571
4	60.0000	360.0000	2.9828	160.0000
5	60.0000	300.0000	2.9272	156.0000
6	60.0000	240.0000	2.8472	150.0000
7	60.0000	180.0000	2.7222	140.0000
8	60.0000	120.0000	2.5000	120.0000
9	60.0000	60.0000	2.0000	60.0000
10	0	0.0000	0	

L = Sum V = 1448.524

Figure 6.7.

Figure 6.8.



Of course it would be most unlikely if L were at a maximum with such arbitrary numbers for extraction. The maximum control problem is to maximise the value in cell E19, i.e., maximise L, subject to any constraints and production flows. The constraints are already set in the spreadsheet, although we do require others on the sign of variables. First move the cursor to cell E19 and then invoke the solver. By default this is set to maximise a cell value, namely cell E19. We next need to inform the programme which is the control variable and hence which values can be changed, i.e., what cells it can change in searching for a maximum. These are cells B7 to B16. In specifying the above problem we implicitly assumed x_t and y_t were both positive. In particular, we assumed the level of production, the control variable, was positive. We need to include this additional constraint in the Solver so that any negative values are excluded from the search process. The Solver window is shown in figure 6.8.

Once all this information has been included the Solver can do its work. The result is shown in the spreadsheet in figure 6.9. As can be observed this gives more or less the same results as figure 6.5, as it should. The value of the objective function has also increased from 1448.52 to 1471.31.

It should be noted in figure 6.9 that in period 10 we have $\lambda(10) = 0$ and at this value $x(10) = 9.9811$. We have to assume that the reserves in period 10 are therefore 9.9811 and that these are simply left in the ground. In other words, $x(T)$ is free. The shadow price of a free product is zero, hence $\lambda(10) = 0$, and this implies it is not optimal to mine the remaining reserves. Hence $F(x^T) = 0$ or x^T is free.

We have spent some time on this problem because it illustrates the use of spreadsheets without having to handle algebraically the first-order conditions. It also has the advantage that it can handle corner solutions.[5] Most important of all, it provides a way of solving real-life problems.

[5] Corner solutions would require setting out the Kuhn–Tucker conditions for optimisation. See Chiang (1984), Simon and Blume (1994) and Huang and Crooke (1997).

Figure 6.9.

Figure 6.9

	p =	3	R =	600
	b =	2		

t	y(t)	x(t)	λ(t)	V(x,y)
0	88.7247	600		239.9339
1	82.2427	511.2753	2.4084	220.2694
2	75.6645	429.0325	2.3567	200.3051
3	69.0667	353.3680	2,2945	180.2015
4	62.4424	284.3013	2.2181	159.8982
5	55.7095	221.8589	2.1216	139.1509
6	48.9298	166.1493	1.9955	117.9704
7	42.0684	117.2196	1.8220	96.0097
8	35.2272	75.1512	1.5644	72.6560
9	29.9429	39.9240	1.1250	44.9145
10	0	9.9811	0	

L = Sum V = 1471.31

6.4 Optimal control with discounting

We have noted that a major feature of the control problem is to maximise the objective function $V(x, u)$. However, for many economic problems $V(x, u)$ would represent such things as profits or net benefits. The economist would not simply maximise such an income stream without first discounting it to the present. Thus, if δ were the rate of discount then the aim of the control would be to

$$\max_{\{u(t)\}} J = \int_0^T e^{-\delta t} V(x, u)dt \tag{6.12}$$

subject to various conditions which are unaffected by the discounting. Thus, the typical continuous time maximisation principle problem with discounting is the control problem

$$\begin{aligned} &\max_{\{u(t)\}} J = \int_0^T e^{-\delta t} V(x, u)dt \\ &\dot{x} = f(x, u) \\ &x(0) = x^0 \\ &x(T) = x^T \end{aligned} \tag{6.13}$$

while the discrete form is

$$\begin{aligned} &\max_{\{u_t\}} J = \sum_{t=0}^{T-1} \rho^t V(x_t, u_t) \\ &x_{t+1} - x_t = f(x_t, u_t) \\ &x_0 = a \end{aligned} \tag{6.14}$$

where $\rho = 1/(1 + \delta)$ and ρ is the discount *factor* while δ is the discount *rate*.

Let us first consider the discrete form. The Lagrangian is

(6.15) $$L = \sum_{t=0}^{T-1} \rho^t \{ V(x_t, u_t) + \rho\lambda_{t+1}[f(x_t, u_t) - (x_{t+1} - x_t)] \}$$

Notice in this expression that λ_{t+1} is discounted to period t by multiplying it by the discount factor ρ. But then the *whole* expression $\{.\}$ is discounted to the present by multiplying by the term ρ^t.

We now introduce a new concept: the *current value Hamiltonian function*, denoted $H_c(x, u)$. This is defined, for the discrete case, as

(6.16) $$H_c(x_t, u_t) = V(x_t, u_t) + \rho\lambda_{t+1} f(x_t, u_t)$$

and in all other respects the optimisation conditions are similar, i.e.

(6.17)

(i) $$\frac{\partial H_c}{\partial u_t} = 0 \qquad t = 0, \ldots, T-1$$

(ii) $$\rho\lambda_{t+1} - \lambda_t = -\frac{\partial H_c}{\partial x_t} \qquad t = 1, \ldots, T-1$$

(iii) $$x_{t-1} - x_t = \frac{\partial H_c}{\partial \rho\lambda_{t+1}} = f(x_t, u_t) \qquad t = 0, \ldots, T-1$$

(iv) $$\lambda_T = 0$$

(v) $$x_0 = a$$

We can illustrate this with the mine example (example 6.4), but now assume a discount rate of 10%. With a discount rate of 10% the discount factor $\rho = 1/(1+0.1) = 0.909091$.

Example 6.5

Given $p = 3$, $R = 600$ and $\rho = 0.909091$

$$\max_{\{y_t\}} J = \sum \rho^t \left(p - \frac{by_t}{x_t}\right) y_t$$

$$x_{t+1} - x_t = -y_t$$

$$x_0 = R$$

The current value Hamiltonian is

$$H_c(x_t, y_t) = \left(p - \frac{by_t}{x_t}\right) y_t - \rho\lambda_{t+1} y_t$$

with optimality conditions:

(i) $$\frac{\partial H_c}{\partial y_t} = p - \frac{2by_t}{x_t} - \rho\lambda_{t+1} = 0 \qquad t = 0, \ldots, 9$$

(ii) $$\rho\lambda_{t+1} - \lambda_t = -\frac{\partial H_c}{\partial x_t} = -\left(\frac{by_t^2}{x_t^2}\right) \qquad t = 1, \ldots, 9$$

(iii) $$x_{t-1} - x_t = \frac{\partial H_c}{\partial \rho\lambda_{t+1}} = -y_t \qquad t = 0, \ldots, 9$$

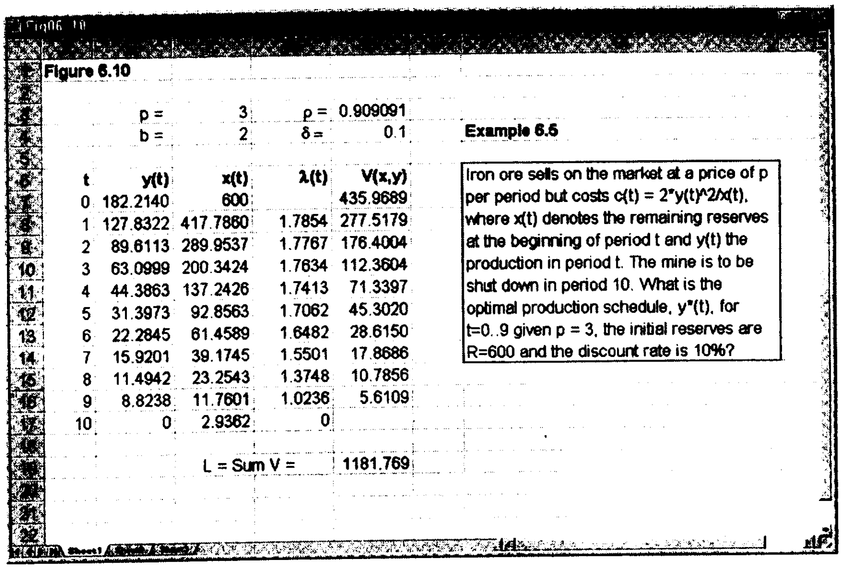


Figure 6.10

	p =	3	ρ =	0.909091
	b =	2	δ =	0.1
t	y(t)	x(t)	λ(t)	V(x,y)
0	182.2140	600		435.9689
1	127.8322	417.7860	1.7854	277.5179
2	89.6113	289.9537	1.7767	176.4004
3	63.0999	200.3424	1.7634	112.3604
4	44.3863	137.2426	1.7413	71.3397
5	31.3973	92.8563	1.7062	45.3020
6	22.2845	61.4589	1.6482	28.6150
7	15.9201	39.1745	1.5501	17.8686
8	11.4942	23.2543	1.3748	10.7856
9	8.8238	11.7601	1.0236	5.6109
10	0	2.9362	0	
		L = Sum V =		1181.769

Example 6.5

Iron ore sells on the market at a price of p per period but costs c(t) = 2*y(t)^2/x(t), where x(t) denotes the remaining reserves at the beginning of period t and y(t) the production in period t. The mine is to be shut down in period 10. What is the optimal production schedule, y*(t), for t=0..9 given p = 3, the initial reserves are R=600 and the discount rate is 10%?

Figure 6.10.

(iv) $\lambda_T = 0$

(v) $x_0 = R = 600$

It should be noted that the only difference between this and the undiscounted conditions is in terms of condition (i), where λ_{t+1} is multiplied by the discount factor. Once again we use *Excel*'s *Solver* to handle the computations of this problem, with the results shown in figure 6.10, which should be compared with figure 6.9.

Notice once again that in period 10 we have $\lambda(10) = 0$ and that at this value $x(10) = 2.9362$. This level of reserves in period 10 is simply left in the ground, $x(T)$ is free. The shadow price of a free good is zero, hence $\lambda(10) = 0$, and this implies it is not optimal to mine the remaining reserves. Put another way, it is cheaper to leave the remaining reserves unmined than incur the costs of mining them.

What these computations show is a similar trajectory for optimal production but starting from a much higher level of production. This is understandable. The future in a discounting model is weighted less significantly than the present. The comparison is shown in figure 6.11.

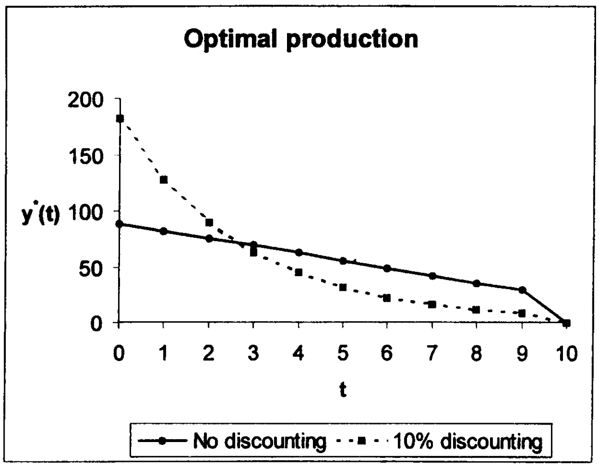
Consider now discounting under a continuous time model. Consider the control problem

$$\begin{aligned}
&\max_{\{u(t)\}} J = \int_0^T e^{-\delta t} V(x, u)dt \\
&\dot{x} = f(x, u) \\
&x(0) = 0 \\
&x(T) = x^T
\end{aligned} \tag{6.18}$$

The Lagrangian is

$$L = \int_0^T \{e^{-\delta t} V(x, u) + \lambda[f(x, u) - \dot{x}]\}dt$$

Figure 6.11.



and the Hamiltonian is

$$H(x, u) = e^{-\delta t}V(x, u) + \lambda f(x, u)$$

Define the *current value Hamiltonian function*, H_c, by

$$H_c(x, u) = V(x, u) + \mu f(x, u)$$

then

$$H_c = He^{\delta t} \quad \text{or} \quad H = H_c e^{-\delta t}$$
$$\mu = \lambda e^{\delta t} \quad \text{or} \quad \lambda = \mu e^{-\delta t}$$

Now reconsider our five optimality conditions. Since $e^{\delta t}$ is a constant for a change in the control variable, then condition (i) is simply $\partial H_c/\partial u = 0$. The second condition is less straightforward. We have

$$\dot{\lambda} = -\frac{\partial H}{\partial x} = -\frac{\partial H_c}{\partial x}e^{-\delta t}$$

From $\lambda = \mu e^{-\delta t}$

$$\dot{\lambda} = \dot{\mu}e^{-\delta t} - \delta\mu e^{-\delta t}$$

Equating these we have

$$-\frac{\partial H_c}{\partial x}e^{-\delta t} = \dot{\mu}e^{-\delta t} - \delta\mu e^{-\delta t}$$

$$\text{or} \quad \dot{\mu} = -\frac{\partial H_c}{\partial x} + \delta\mu$$

Condition (iii) is

$$\dot{x} = \frac{\partial H}{\partial \lambda} = \frac{\partial H_c}{\partial \lambda}e^{-\delta t} = \frac{\partial H_c}{\partial \mu} = f(x, u)$$

while condition (iv) becomes

$$\lambda(T) = \mu(T)e^{-\delta t} = 0$$

and condition (v) remains unchanged.

To summarise, define the current value Hamiltonian and current value Lagrangian multiplier, i.e.

$$H_c(x, u) = H(x, u)e^{\delta t} = V(x, u) + \mu f(x, u)$$

where $\lambda = \mu e^{-\delta t}$. Then the optimality conditions are:

$$\begin{aligned}
&\text{(i)} && \frac{\partial H_c}{\partial u} = 0 \qquad 0 \le t \le T \\
&\text{(ii)} && \dot{\mu} = -\frac{\partial H_c}{\partial x} + \delta\mu \qquad 0 \le t \le T \\
&\text{(iii)} && \dot{x} = \frac{\partial H_c}{\partial \mu} = f(x, u) \\
&\text{(iv)} && x(0) = x^0 \\
&\text{(v)} && \mu(T)e^{-\delta t} = 0 \qquad (\text{or } x(T) = x^T)
\end{aligned} \tag{6.19}$$

These optimality conditions allow us to eliminate the control variable u using condition (i) and to obtain two differential equations: one for the state variable, x, and the other for the current value costate variable, μ.

Example 6.6

$$\max_{\{u\}} J = -\int_0^{10} u^2 e^{-0.1t} dt$$

$$\dot{x} = u$$
$$x(0) = 0$$
$$x(10) = 1000$$

and find the optimal path $x^*(t)$.

The current value Hamiltonian is

$$H_c = -u^2 + \mu u$$

with optimality conditions

$$\begin{aligned}
&\text{(i)} && \frac{\partial H_c}{\partial u} = -2u + \mu = 0 \\
&\text{(ii)} && \dot{\mu} = 0 + 0.1\mu \\
&\text{(iii)} && \dot{x} = u
\end{aligned}$$

From (i) we have $u = 0.5\mu$, which when substituted into (iii) gives $\dot{x} = 0.5\mu$. Thus we have two differential equations

$$\dot{x} = 0.5\mu$$
$$\dot{\mu} = 0.1\mu$$

Solving we obtain

$$x(t) = c_1 + c_2 e^{0.1t}$$

$$\mu(t) = 0.2c_2 e^{0.1t}$$

Given $x(0) = 0$ and $x(10) = 1000$ then we can solve for c_1 and c_2 by solving

$$c_1 + c_2 = 0$$
$$0.2c_2e = 1000$$

which gives $c_1 = -581.9767$ and $c_2 = 581.9767$. Hence

$$\begin{aligned} x^*(t) &= -581.9767 + 581.9767e^{0.1t} \\ &= 581.9767(e^{0.1t} - 1) \end{aligned}$$

6.5 The phase diagram approach to continuous time control models

First let us reconsider examples 6.1–6.3.

Example 6.1 (cont.)

In example 6.1 we derived the two differential equations

$$\dot{x} = x + u$$
$$\dot{\lambda} = -(5 + \lambda)$$

In this instance, $\partial H/\partial u = \lambda$ which is of no help in eliminating u. We did, however, establish that H is a maximum when $u = 3$ and that this variable remains constant throughout. Therefore,

$$\dot{x} = x + 3$$
$$\dot{\lambda} = -5 - \lambda$$

and so we have two isoclines. The x-isocline at $x = -3$ and the λ-isocline at $\lambda = -5$. Furthermore,

$$\begin{aligned} \dot{x} > 0 &\quad \text{implies} \quad x > -3 \\ \dot{\lambda} < 0 &\quad \text{implies} \quad \lambda > -5 \end{aligned}$$

so we know that the optimal trajectory starting from $x(0) = 2$ will lead to a rise in the state variable x and a fall in the costate variable λ. This is verified in figure 6.12. The system begins from point $(x(0), \lambda(0)) = (2, 8.5914)$, satisfying the initial condition on the state variable x; and has a terminal point $(x(1), \lambda(1)) = (10.5914, 0)$, which satisfies the terminal condition on the costate variable, λ. Of all possible trajectories in the phase plane, this is the optimal trajectory.

Example 6.2 (cont.)

In example 6.2 we derived the following two differential equations

$$\dot{x} = -\tfrac{1}{2}\lambda$$
$$\dot{\lambda} = 0$$

There is only one isocline for this problem. When $\dot{x} = 0$ then $\lambda = 0$ and so the x-isocline coincides with the x-axis. Our initial point is $(x(0), \lambda(0)) = (1, 2)$ and

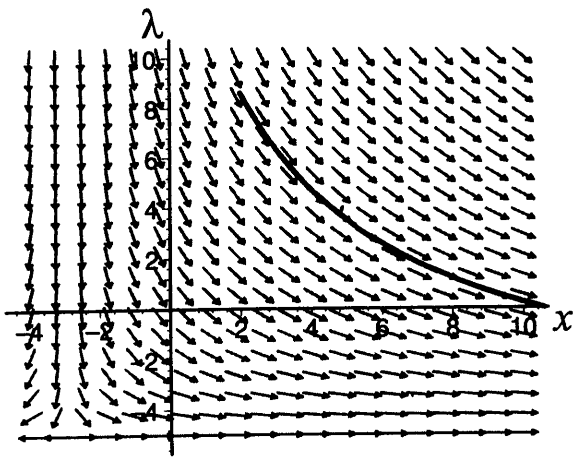


Figure 6.12.

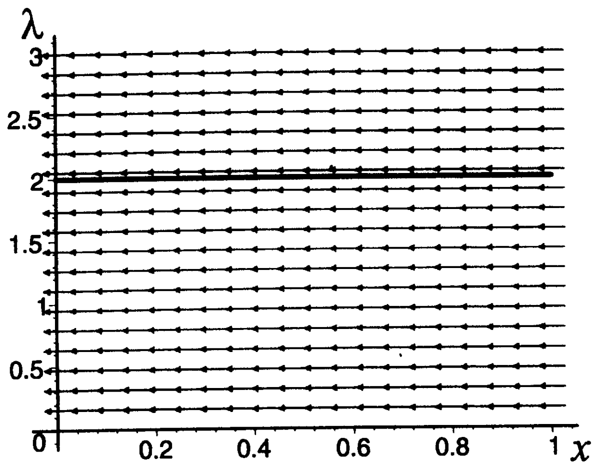


Figure 6.13.

for $\lambda > 0$ we have $\dot{x} < 0$ and so the trajectory is moving to the left. Earlier we demonstrated that λ remains at the value of 2 throughout the trajectory. When $t = 1$ then $x(1) = 0$, which satisfies the condition on the terminal point, which in the phase plane is the point $(x(1), \lambda(1)) = (0, 2)$. As can be seen in terms of figure 6.13, the optimal trajectory in the phase plane is the horizontal line pointing to the left.

Example 6.3 (cont.)

The two differential equations we derived for example 6.3 were

$$\dot{x} = x + 2\lambda$$
$$\dot{\lambda} = \tfrac{1}{2}x - \lambda$$

When $\dot{x} = 0$ then $\lambda = -\frac{1}{2}x$ and when $\dot{\lambda} = 0$ then $\lambda = \frac{1}{2}x$. We have therefore two distinct isoclines in this example. Furthermore,

$$\text{if } \dot{x} > 0 \text{ then } x + 2\lambda > 0 \text{ implying } \lambda > -\tfrac{1}{2}x$$

Hence, above the x-isocline, x is rising while below it is falling. Similarly,

$$\text{if } \dot{\lambda} > 0 \text{ then } \tfrac{1}{2}x - \lambda > 0 \text{ implying } \lambda < \tfrac{1}{2}x$$

Hence, below the λ-isocline, λ is rising while above it is falling. This suggests that we have a saddle-point solution.

This is also readily verified by considering the eigenvalues of the system. The matrix of the system is

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ \frac{1}{2} & -1 \end{bmatrix}$$

with eigenvalues $r = \sqrt{2}$ and $s = -\sqrt{2}$. Since these are real and of opposite sign, then we have a saddle point solution.

When $t = 0$ we already have $x(0) = 2$ but we need to solve for $\lambda(0)$. But

$$\lambda(0) = \frac{c_1}{2}(\sqrt{2} - 1) - \frac{c_2}{2}(\sqrt{2} + 1)$$

and we know that $c_1 = -0.1256$ and $c_2 = 2.1256$. Substituting these values we get $\lambda(0) = -2.6$. The initial point $(x(0), \lambda(0)) = (2, -2.6)$ therefore begins below the x-isocline, and so the vector forces are directing the system up and to the left. The optimal trajectory is shown in figure 6.14.

It is apparent from example 6.3 and 6.6 that the maximisation approach of Pontryagin gives us first-order conditions in terms of the Hamiltonian which, in the present simple models, leads to two differential equations in terms of the state variable x and the costate variable λ (or μ). Control problems, however, pose two difficulties:

(1) the differential equations are often nonlinear
(2) in economics functional forms are often unspecified.[6]

Even the most simple control problem can lead to nonlinear differential equations, and although we have developed techniques elsewhere for dealing with these,[7] until the advent of the computer they were largely left to the mathematician. When the functional forms are not even specified then there are no explicit differential equations to solve. However, the qualitative properties of the fixed points can still be investigated by considering the system's qualitative properties in the phase plane.

First consider a simple example for which we have an explicit solution.

[6] What we often know are certain properties. Thus we may have a production function $y = f(k)$ where $f(k)$ is unspecified other than being continuous, differentiable and where $f'(k) > 0$ and $f''(k) < 0$.
[7] See sections 2.7 and 3.9.

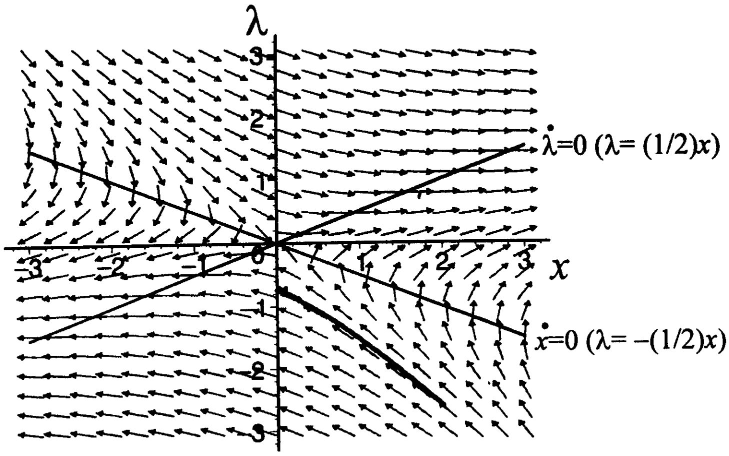


Figure 6.14.

Example 6.7[8]

Our problem is

$$\max_{\{u\}} J = \int_0^{\infty} (20 \ln x - 0.1u^2)dt$$

$$\dot{x} = u - 0.1x$$

$$x(0) = 80$$

The Hamiltonian for this problem is

$$H = 20 \ \ln x - 0.1u^2 + \lambda(u - 0.1x)$$

with first-order conditions

$$\frac{\partial H}{\partial u} = -0.2u + \lambda = 0$$

$$\dot{\lambda} = -\frac{\partial H}{\partial x} = -\left(\frac{20}{x} - 0.1\lambda\right)$$

$$\dot{x} = u - 0.1x$$

which can be reduced to two differential equations in terms of x and λ

$$\dot{x} = -0.1x + 5\lambda$$

$$\dot{\lambda} = \frac{-20}{x} + 0.1\lambda$$

The fixed point of this system is readily found by setting $\dot{x} = 0$ and $\dot{\lambda} = 0$, giving $x^* = 100$ and $\lambda^* = 2$. Furthermore, the two isoclines are readily found to be

$$\lambda = 0.02x \qquad (\dot{x} = 0)$$

$$\lambda = \frac{200}{x} \qquad (\dot{\lambda} = 0)$$

and illustrated in figure 6.15.

[8] Adapted from Conrad and Clark (1987, pp. 46–8).

Figure 6.15 also shows the vector of forces in the four quadrants, which readily indicate a saddle point solution. This can be verified by considering a linearisation about the fixed point $(x^*, \lambda^*) = (100, 2)$. This gives the linear equations

$$\dot{x} = -0.1(x - x^*) + 5(\lambda - \lambda^*)$$
$$\dot{\lambda} = 0.002(x - x^*) + 0.1(\lambda - \lambda^*)$$

The resulting matrix of the linear system is

$$\mathbf{A} = \begin{bmatrix} -0.1 & 5 \\ 0.002 & 0.1 \end{bmatrix}$$

with eigenvalues $r = 0.14142$ and $s = -0.14142$, confirming a saddle point solution.

To establish the equations of the arms of the saddle point solution, take first the eigenvalue $r = 0.14142$. Then

$$(\mathbf{A} - r\mathbf{I})\mathbf{v}^r = \mathbf{0}$$

i.e.

$$\left(\begin{bmatrix} -0.1 & 5 \\ 0.002 & 0.1 \end{bmatrix} - 0.14142\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1^r \\ v_2^r \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

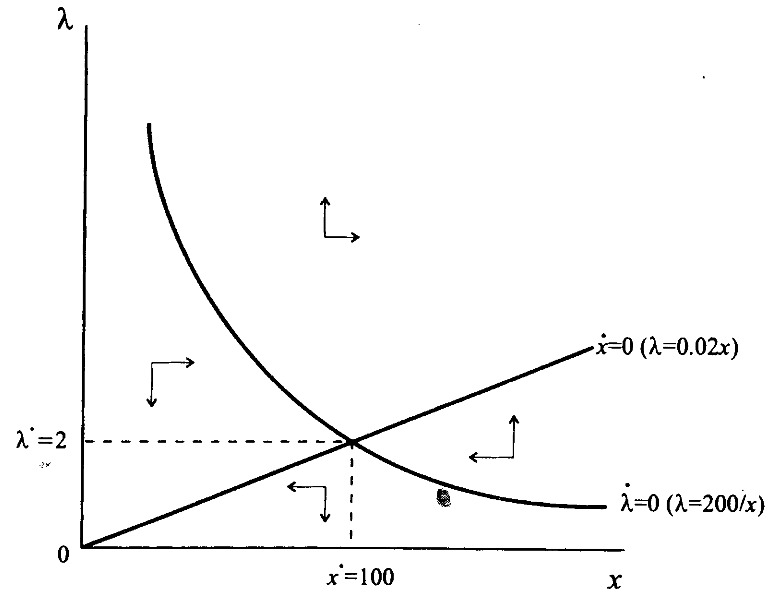
or

$$\begin{bmatrix} -0.24142 & 5 \\ 0.002 & -0.04142 \end{bmatrix}\begin{bmatrix} v_1^r \\ v_2^r \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Using the first equation,

$$-0.24142v_1^r + 5v_2^r = 0$$

Figure 6.15.



Let $v_1^r = 1$, then

$$5v_2^r = 0.24142$$

$$v_2^r = 0.048284$$

Therefore,

$$(\lambda - \lambda^*) = 0.048284(x - x^*)$$

$$(\lambda - 2) = 0.048284(x - 100)$$

i.e.

$$\lambda = -2.8284 + 0.48284x$$

and since this is positively sloped it represents the equation of the unstable arm.

Now consider the second eigenvalue, $s = -0.14142$

$$(\mathbf{A} - r\mathbf{I})\mathbf{v}^s = \left(\begin{bmatrix} -0.1 & 5 \\ 0.002 & 0.1 \end{bmatrix} + 0.14142\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1^s \\ v_2^s \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

i.e.

$$\begin{bmatrix} 0.04142 & 5 \\ 0.002 & 0.24142 \end{bmatrix}\begin{bmatrix} v_1^s \\ v_2^s \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Using again the first equation, then

$$0.04142v_1^s + 5v_2^s = 0$$

Let $v_1^s = 1$, then

$$5v_2^s = -0.04142$$

$$v_2^s = -0.008284$$

Therefore,

$$(\lambda - \lambda^*) = -0.008284(x - x^*)$$

$$(\lambda - 2) = -0.008284(x - 100)$$

i.e.

$$\lambda = 2.8284 - 0.00828284x$$

and since this is negatively sloped it represents the equation of the stable arm.

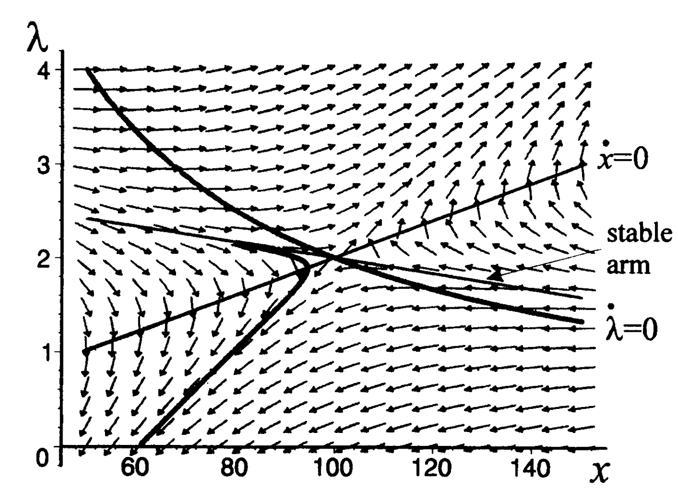
If $x(0) = 80$ then the value of λ on the stable arm is $\lambda(0) = 2.16568$. The trajectory, along with isoclines and the stable arm, are shown in figure 6.16. Although the point begins on the stable arm, it gets pulled away before it reaches the equilibrium! What this diagram reveals is that this system is *very sensitive to initial conditions*. But the direction field does show a clear saddle point equilibrium.

Example 6.8 (Ramsey growth model)

In this example we shall consider the Ramsey growth model,[9] which is the basis of much of the optimal growth theory literature. We shall consider the model in terms

[9] Ramsey (1928). See also Burmeister and Dobell (1970), Barro and Sala-i-Martin (1995) and Romer (2001).

Figure 6.16.



of continuous time. We begin with simple definitions of income and investment, namely

(6.20)
$$Y(t) = C(t) + I(t)$$
$$I(t) = \dot{K}(t) + \delta K(t)$$

Hence

$$\frac{Y(t)}{L(t)} = \frac{C(t)}{L(t)} + \frac{\dot{K}(t)}{L(t)} + \frac{\delta K(t)}{L(t)}$$

$$\text{i.e.} \quad y(t) = c(t) + \frac{\dot{K}(t)}{L(t)} + \delta k(t)$$

But

$$\dot{k} = \frac{d}{dt}\left(\frac{K}{L}\right) = \frac{L\dot{K} - K\dot{L}}{L^2} = \frac{\dot{K}}{L} - \left(\frac{K}{L}\right)\frac{\dot{L}}{L}$$
$$= \frac{\dot{K}}{L} - k\frac{\dot{L}}{L}$$

We assume population grows at a constant rate n, so that $\dot{L}/L = n$ hence

$$\frac{\dot{K}}{L} = \dot{k} + kn$$

and

$$y(t) = c(t) + \dot{k}(t) + (n + \delta)k(t)$$

If we have a homogeneous of degree one production function then we can express output, y, as a function of k. Thus, $y = f(k)$. Dropping the time variable for convenience, we therefore have the condition

(6.21)
$$\dot{k} = f(k) - (n + \delta)k - c$$

In order to consider the optimal growth path we require to specify an objective. Suppose $U(c)$ denotes utility as a function of consumption per head. The aim is to maximise the discounted value of utility subject to the equation we have just derived, i.e.

$$\max_{\{c\}} J = \int_0^{\infty} e^{-\beta t} U(c) dt$$

$$\dot{k} = f(k) - (n + \delta)k - c$$

$$k(0) = k_0$$

$$0 \leq c \leq f(k)$$

The current value Hamiltonian function is

$$H_c = U(c) + \mu[f(k) - (n + \delta)k - c]$$

with first-order conditions:

$$\text{(i)} \quad \frac{\partial H_c}{\partial c} = U'(c) - \mu = 0$$

$$\text{(ii)} \quad \dot{\mu} = -\mu f'(k) + \mu(n + \delta) + \beta\mu$$

$$\text{(iii)} \quad \dot{k} = f(k) - (n + \delta)k - c$$

or

$$\text{(i)} \quad U'(c) = \mu$$

$$\text{(ii)} \quad \dot{\mu} = -\mu f'(k) + (n + \delta + \beta)\mu$$

$$\text{(iii)} \quad \dot{k} = f(k) - (n + \delta)k - c$$

As they stand these equations are not easy to interpret or solve. We can, however, with some rearrangement, derive two differential equations in terms of the state variable k and the control variable c.

From (i) differentiate with respect to time. Then

$$\frac{d[U'(c)]}{dt} = \dot{\mu}$$

$$U''(c)\frac{dc}{dt} = \dot{\mu} = -\mu f'(k) + (n + \delta + \beta)\mu$$

$$\text{i.e.} \quad U''(c)\dot{c} = -\mu[f'(k) - (n + \delta + \beta)]$$

or

$$-\frac{U''(c)}{U'(c)}\dot{c} = f'(k) - (n + \delta + \beta) \quad (\text{since } \mu = U'(c))$$

Now define Pratt's measure of relative risk aversion[10]

$$\sigma(c) = -\frac{cU''(c)}{U'(c)}$$

then

$$\frac{\sigma(c)}{c}\dot{c} = f'(k) - (n + \delta + \beta)$$

[10] Pratt (1964), see also Shone (1981, application 2, section A2.4).

or

$$\dot{c} = \frac{1}{\sigma(c)}[f'(k) - (n + \delta + \beta)]c$$

We therefore have two differential equations

$$\dot{c} = \frac{1}{\sigma(c)}[f'(k) - (n + \delta + \beta)]c$$
$$\dot{k} = f(k) - (n + \delta)k - c$$

If $\dot{c} = 0$ then $f'(k^*) = n + \delta + \beta$. On the other hand, if $\dot{k} = 0$ then $c^* = f(k^*) - (n + \delta)k^*$. Furthermore, if $\dot{c} > 0$ then $f(k^*) > (n + \delta + \beta)$ which implies $k < k^*$ as seen in terms of the upper diagram of figure 6.17. Hence, to the left of the $\dot{c} = 0$ isocline, c is rising; to the right of $\dot{c} = 0$, then c is falling. Similarly, if $\dot{k} > 0$ then $f(k^*) - (n + \delta)k^* > c$. Thus below the $\dot{k} = 0$ isocline k is rising, while above the $\dot{k} = 0$ isocline k is falling. The vector forces clearly indicate that (k^*, c^*) is a saddle point solution. The only optimal trajectory is that on the stable arm. For any k_0 the only viable level of consumption is that represented by the associated point on the stable arm. Given the initial point on the stable arm, the system is directed towards the equilibrium. Notice that in equilibrium k is constant and so capital is growing at the same rate as the labour force. Furthermore, since k is constant in equilibrium then so is y, and hence Y is also growing at the same rate as the labour force. We have, therefore, a balanced-growth equilibrium.

Example 6.9 (Ramsey growth model: a numerical example)

Consider the optimal growth problem

$$\max_{\{c\}} J = \int_0^{\infty} e^{-\beta t} U(c) dt$$
$$\dot{k} = f(k) - (n + \delta)k - c$$
$$k(0) = k_0$$
$$0 \leq c \leq f(k)$$

where

$$\beta = 0.02, f(k) = k^{0.25}, n = 0.01, \delta = 0.05, k(0) = 2$$

and

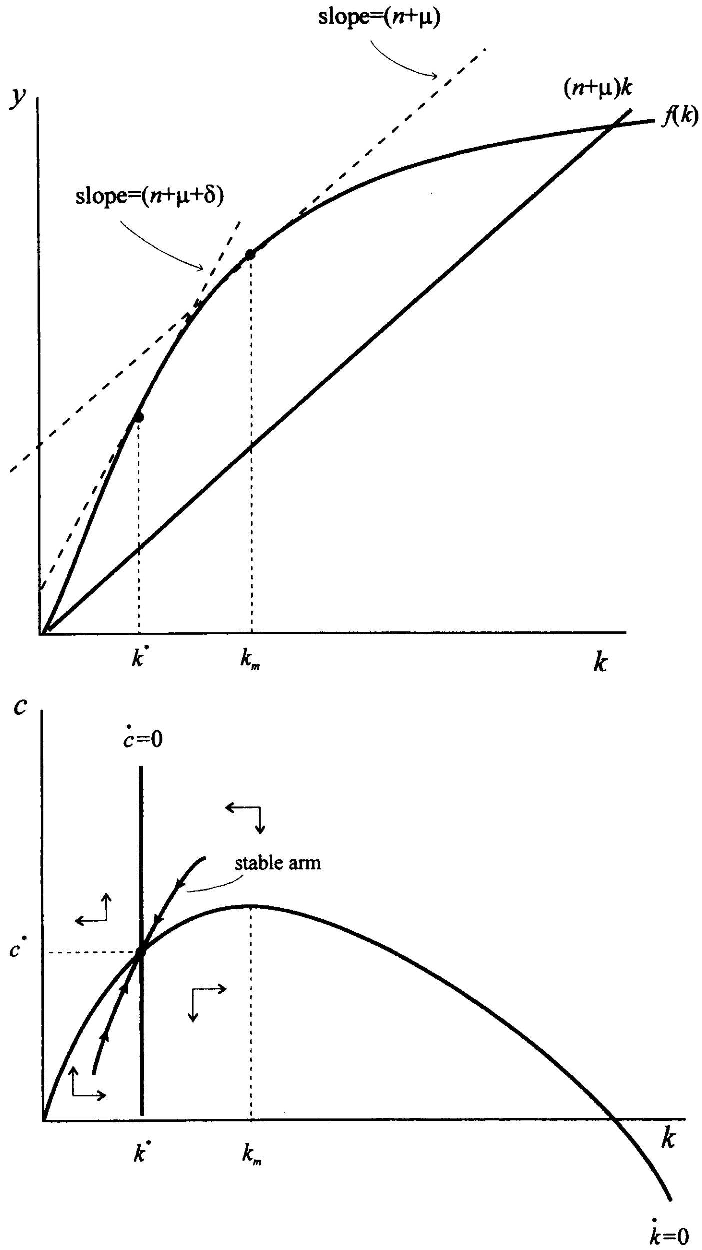
$$U(c) = \frac{c^{1-\theta}}{1 - \theta}$$

If $\theta = \frac{1}{2}$, then $U(c) = 2\sqrt{c}$.[11] Then our maximisation problem is

$$\max_{\{c\}} J = \int_0^{\infty} e^{-0.02t} 2\sqrt{c}\, dt$$
$$\dot{k} = k^{0.25} - 0.06k - c$$
$$k(0) = 2$$

[11] Notice that this utility function has a relative measure of risk aversion equal to θ.

Figure 6.17.



The current value Hamiltonian is

$$H_c = 2\sqrt{c} + \mu(k^{0.25} - 0.06k - c)$$

with first-order conditions

(i) $$\frac{\partial H_c}{\partial c} = 2\left(\tfrac{1}{2}\right)c^{-1/2} - \mu = 0$$

(ii) $$\dot{\mu} = -\mu(0.25)k^{-0.75} + 0.08\mu$$

(iii) $$\dot{k} = k^{0.25} - 0.06k - c$$

From the first condition we have $c^{-1/2} = \mu$. Differentiating this with respect to time, then

$$-\tfrac{1}{2}c^{-3/2}\dot{c} = \dot{\mu}$$

Using condition (ii) we have

$$-\tfrac{1}{2}c^{-3/2}\dot{c} = -\mu(0.25)k^{-0.75} + 0.08c^{-1/2}$$

But $\mu = c^{-1/2}$ hence

$$-\tfrac{1}{2}c^{-3/2}\dot{c} = -c^{-1/2}(0.25)k^{-0.75} + 0.08c^{-1/2}$$

Dividing throughout by $c^{-1/2}$ we obtain

$$-\tfrac{1}{2}c^{-1}\dot{c} = -(0.25)k^{-0.75} + 0.08$$

i.e.

$$\begin{aligned}\dot{c} &= 2c(0.25)k^{-0.75} - 2(0.08)c \\ &= (0.5k^{-0.75} - 0.16)c\end{aligned}$$

We now have two differential equations for the state variables c and k, which are

$$\begin{aligned}\dot{c} &= (0.5k^{-0.75} - 0.16)c \\ \dot{k} &= k^{0.25} - 0.06k - c\end{aligned}$$

The first thing to note about these equations is that they are nonlinear and therefore not easy to solve without some software.[12] Using either *Mathematica* or *Maple* (or *Excel* as indicated in n. 12), the following equilibrium values are obtained

$$k^* = 4.5688, \quad c^* = 1.1879$$

Second we note that the consumption-isocline is given by the formula $c = k^{0.75} - 0.06k$. Differentiating this with respect to k and setting this equal to zero allows us to solve for the value of k at which consumption is at a maximum

$$\begin{aligned}c &= k^{0.25} - 0.06k \\ \frac{dc}{dk} &= 0.25k^{-0.75} - 0.06 = 0 \\ k_{\max} &= 6.7048\end{aligned}$$

At this value of k then consumption takes the value $c_{\max} = 1.2069$.[13]

[12] If you do not have a software package like *Mathematica* or *Maple*, you can use *Excel*'s *Solver* to solve for the equilibrium values. Place an arbitrary value of k in one cell; say, our starting value of 2. Suppose this is cell C3. Now place the formula

```
= (0.5*$C$3^(−0.75) − 0.16)*($C$3^0.25 − 0.06*$C$3)
```

in the target cell. In the Solver window declare the cell where the formula is located as the target cell and set this to have a value of zero; allow cell \$C\$3 to be the cell whose values are changed. In order to avoid the problem of a zero solution, place a constraint on \$C\$3 that it should be greater than or equal to unity. Having calculated the equilibrium value of k in this manner, it is a simple matter then to solve for the equilibrium value of consumption, c.

[13] Note that the c-isocline cuts the k-axis at 0 and the value 42.5727.

To establish the properties of the equilibrium, we can linearise the system around the point $(k^*, c^*) = (4.5688, 1.1879)$. Let

$$\dot{c} = f(c, k) = (0.5k^{-0.75} - 0.16)c$$
$$\dot{k} = g(c, k) = k^{0.25} - 0.06k - c$$

The system can then be written in the linearised form

$$\dot{c} = f_c(c^*, k^*)(c - c^*) + f_k(c^*, k^*)(k - k^*)$$
$$\dot{k} = g_c(c^*, k^*)(c - c^*) + g_k(c^*, k^*)(k - k^*)$$

with

$$f_c(c^*, k^*) = 0, \qquad f_k(c^*, k^*) = -0.0312$$
$$g_c(c^*, k^*) = -1, \qquad g_k(c^*, k^*) = 0.02$$

and so the matrix of the system is

$$\mathbf{A} = \begin{bmatrix} 0 & -0.0312 \\ -1 & 0.02 \end{bmatrix}$$

with eigenvalues $r = 0.1869$ and $s = -0.1669$. Since these are opposite in sign, then the equilibrium is a saddle point solution.

Given that we are dealing with a numerical example then we can approximate the saddle path equations utilising the linear approximation to the system. First take the eigenvalue $r = 0.1869$

$$(\mathbf{A} - r\mathbf{I})\mathbf{v}^r = \mathbf{0}$$

i.e.

$$\begin{bmatrix} -0.1869 & -0.0312 \\ -1 & -0.1669 \end{bmatrix} \begin{bmatrix} v_1^r \\ v_2^r \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

then

$$-v_1^r - 0.1669v_2^r = 0$$
$$v_1^r = -0.1669v_2^r$$

Let $v_2^r = 1$ then $v_1^r = -0.1669$. This saddle path is therefore negatively sloped and denotes the unstable arm. Turn next to the eigenvalue $s = -0.1669$ then

$$\begin{bmatrix} 0.1669 & -0.0312 \\ -1 & 0.1869 \end{bmatrix} \begin{bmatrix} v_1^s \\ v_2^s \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

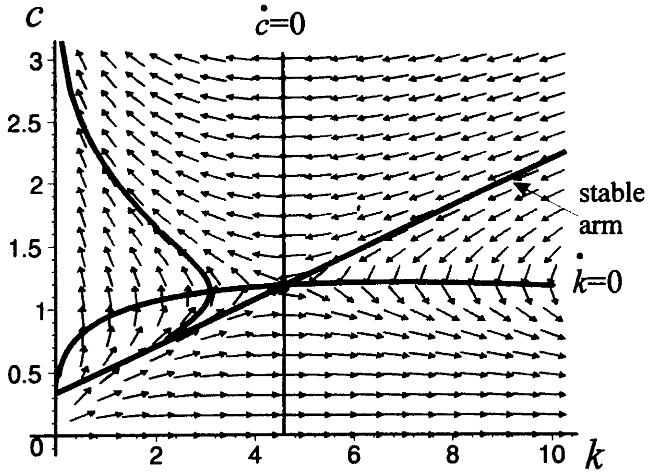
and

$$-v_1^s + 0.1869v_2^s = 0$$
$$v_1^s = 0.1869v_2^s$$

Let $v_2^s = 1$ then $v_1^s = 0.1869$. This saddle path is positively sloped and represents the stable arm. The equation of the stable arm can be found from

$$c - c^* = 0.1869(k - k^*)$$
$$c - 1.1879 = 0.1869(k - 4.5688)$$
$$c = -0.33399 + 0.1869k$$

Figure 6.18.



All these results are illustrated in figure 6.18 in the neighbourhood of the fixed point. As can be seen from figure 6.18, however, although the trajectory does begin on the saddle path, the system is extremely sensitive to initial conditions, and given the initial point $(k(0), c(0)) = (2, 0.70779)$, the trajectory begins to move away from the saddle path before it reaches the equilibrium! The direction field shown in figure 6.18 does, however, illustrate the existence of the stable arm with trajectories tending to the balanced-growth path equilibrium.

What these examples show is that we can eliminate the control variable using the first-order conditions and derive two differential equations, one for the state variable and another for the costate variable. These can more generally be expressed

$$\begin{aligned} \dot{x} &= R(x, \lambda) \\ \dot{\lambda} &= S(x, \lambda) \end{aligned}$$

We can then define two isoclines, one for $\dot{x} = 0$ (or $R = 0$), and another for $\dot{\lambda} = 0$ (or $S = 0$). As these examples illustrate, however, such isoclines do not always exist. When they both exist, the state space is separated into four quadrants. Each quadrant exerts different dynamic forces on any trajectory beginning in it. In most of these examples we know the initial point and terminal point. The derived dynamic equations maximise the objective function, satisfy the equation of motion and satisfy initial and terminal states. So we know the optimal trajectory.

When we have two state variables, as in example 6.8 (and its numerical version, example 6.9), then we sketch the state-space only and the optimal trajectory $\{\mathbf{x}(t)\}$.

If we have a discrete system then

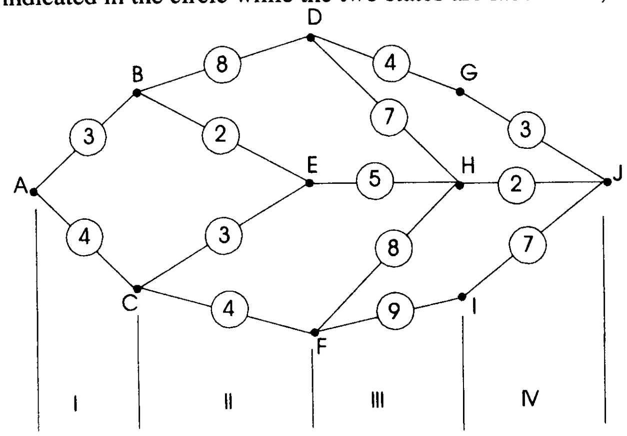
$$\begin{aligned} x_{t+1} - x_t &= R(x_t, \lambda_t) \\ \lambda_{t+1} - \lambda_t &= S(x_t, \lambda_t) \end{aligned}$$

and the isoclines remain $R = 0$ $(\Delta x_{t+1} = 0)$ and $S = 0$ $(\Delta\lambda_{t+1} = 0)$.

For many problems we do not have specific functional forms either for the objective function or for the equations of motion. This was the situation in the general Ramsey growth model (example 6.8). It is in such circumstances that deriving isoclines and establishing properties of the state space provides *qualitative insight into the optimal path*. For instance, example 6.8 illustrates that the trajectory of the economy is along the stable arm eventually resulting in a balanced-growth equilibrium. Such a path will maximise discounted consumption over the infinite time horizon.

Exercises

1. Given the following stages of production labelled I, II, III and IV, a variety of possible processes can be followed, as shown in the accompanying figure, where the cost of transforming from one stage into another is indicated in the circle while the two states are labelled A, B, C, etc.



   (i) Compute all possible solution paths and show which is the minimum.
   (ii) 'Back solve' by starting at the terminal state J and minimise at each node arrived at. Is this the same solution path you derived in (i) when then looked at forward?
   (iii) Why is it sensible to 'back solve' but not to 'forward solve'?

2. Prove that

$$-\int_{t_0}^{t_1} \lambda \dot{x} dt = \int_{t_0}^{t_1} x \dot{\lambda} dt - [\lambda(t_1)x(t_1) - \lambda(t_0)x(t_0)]$$

3. (i) For the model in section 6.2 derive the optimal path for production under both no discounting and discounting at 10% under the following alternative assumptions:
   (a) $p = 5$
   (b) $b = 3$
   (c) $R = 800$

   (ii) What conclusions do you draw?

4. Consider the model in example 6.5. Suppose the manager is unsure of the discount rate. He decides to choose three rates: 5%, 10% and 15%. Compare the optimal production schedules under each assumption.
5. Consider examples 6.4 and 6.5 under the assumption that price is expected to rise at 5% per period (i.e. inflation is 5%), but costs are not subject to any rise; derive the optimal production schedule in each case assuming $p_0 = 3$.
6. Consider example 6.4 under the assumption that price is expected to rise at 10% per period and costs are expected to rise at 15% per period. Derive the optimal production schedule assuming $p_0 = 3$ and $b_0 = 2$.
7. Set up the following problem as a control problem.
A government has an objective function that indicates it wants to maximise votes, v, by pursing policies towards unemployment, u, and inflation π. The party is constrained in its behaviour by the existence of an augmented Phillips curve of the form

$$\pi = -\alpha(u - u_n) + \pi^e \quad \alpha > 0$$

and expectations take the form of adaptive expectations, i.e.

$$\dot{\pi}^e = \beta(\pi - \pi^e) \qquad \beta > 0$$

The government has just won the election at $t = 0$ and the next election is in 5 years' time. It assumes that voters have poor memories, and weight more heavily the economic situation the closer it is to the election. It accordingly assumes a weighting factor of $e^{0.05t}$.
8. Solve the following control problem

$$\max_{\{u\}} - \int_0^1 \left(\frac{x^2}{4} - \frac{u^2}{9} \right) dt$$
$$\dot{x} = -x + u$$
$$x(0) = 5, \quad x(1) = 10$$

Plot the trajectory in (x, λ)-space.
9. Solve the following control problem

$$\max_{\{u\}} \int_0^1 (3x^2 - u^2)dt$$
$$\dot{x} = 2x + u$$
$$x(0) = 10, \quad x(1) = 15$$

Plot the trajectory in (x, λ)-space
10. Solve the equilibrium for the following Ramsey model. Linearise the system about the equilibrium and establish its stability properties

$$\max_{\{c\}} J = \int_0^\infty e^{-0.03} U(c)dt$$

$$\dot{k} = k^{0.3} - (n + \delta)k - c$$

$$k(0) = 1$$

where $U(c) = 4c^{1/4}$, $n = 0.02$, $\delta = 0.03$

Additional reading

Beavis and Dobbs (1990), Blackburn (1987), Bryson, Jr. and Ho (1975), Burmeister and Dobell (1970), Chiang (1992), Conrad (1999), Conrad and Clark (1987), Fryer and Greenman (1987), Intriligator (1971), Kirk (1970), Léonard and Long (1992), Pontryagin *et al.* (1962), Ramsey (1928), Romer (2001) and Takayama (1994).