

# EE320 (1/2015)

## INTRODUCTORY MATHEMATICAL ECONOMICS

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DERIVATIVES OF MORE-THAN-ONE INDEPENDENT  
VARIABLE FUNCTION

# Roadmap after midterm

- Derivatives of more-than-one independent variable functions
- Optimization without constraint: More-than-one independent variable cases
- Optimization under equality constraint
- Integration and applications

# Topics

- Partial Differentiation
  - First-order partial derivatives
  - Second-order partial derivatives
- Differentials
- Total differentials
- Total derivatives
- Implicit function and its derivative

# Introduction

- This lecture illustrates calculus tools used to deal with functions with more than one independent variables. Consider the following functions:

$$Q_d = (P_1, P_2, Y)$$

$$Q(K, L) = AK^\alpha L^{1-\alpha}$$

$$U = (c, l) \quad \text{where} \quad w(24 - l) = pc$$

- **Partial differentiation** is used to find the impact of a change in one independent variable on the dependent variable, holding everything else constant, we need to use.
- **Total differential** is used to find the total change in the dependent variable as one or more independent variables change at the same time.
- **Total derivative** is used when independent variable are related to one another.

# PARTIAL DIFFERENTIATION

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# Partial Differentiation

- Consider  $y = f(x_1, x_2, \dots, x_n)$  where  $x_1, \dots, x_n$  are independent of one another.
- Suppose only  $x_1$  changes by  $\Delta x_1$ , the corresponding change in  $y$  is  $\Delta y$ .

- Difference quotient:

$$\frac{\Delta y}{\Delta x_1} = \frac{f(x_1 + \Delta x_1, x_2, \dots, x_n) - f(x_1, x_2, \dots, x_n)}{\Delta x_1}$$

- Partial derivative of  $y$  with respect to  $x$ :

$$f_1 \equiv \frac{\partial y}{\partial x_1} \equiv \lim_{\Delta x_1 \rightarrow 0} \frac{\Delta y}{\Delta x_1}$$

- The process of taking partial derivatives is called “partial differentiation”.

# First-Order Partial Derivatives

- Technique of partial differentiation: allow one variable to vary while holding all other independent variables constant while.
- Example 1:  $y = f(x_1, x_2) = 3x_1^2 + x_1x_2 + 4x_2^2$ . Find  $f_1(1,3)$  and  $f_2(1,3)$ .
- Example 2:  $y = f(u,v) = (u+4)(3u+2v)$ . Find  $f_u(2,1)$  and  $f_v(2,1)$ .

# Geometric Interpretation of Partial Derivatives

- Suppose  $Q = Q(K, L)$ 
  - $\partial Q / \partial L = \text{MPP}_L$
  - $\partial Q / \partial K = \text{MPP}_K$

# Gradient Vector

- The **gradient vector** of the function  $f(x_1, x_2, \dots, x_n)$  is an  $n$ -vector of all the partial derivatives:

$$\nabla f(x_1, x_2, \dots, x_n) = (f_1, f_2, \dots, f_n)$$

where  $f_i \equiv \frac{\partial y}{\partial x_i}$ .

- **Example:** Find the gradient vector of the production function  $Q = aK^bL^{1-b}$ , where  $a > 0$  and  $0 < b < 1$ .

# Application 1: Partial Market Equilibrium

- Recall the one-commodity market model:

$$\text{Demand:} \quad Q = a - bP \quad (a, b > 0)$$

$$\text{Supply:} \quad Q = -c + dP \quad (c, d > 0)$$

- Solutions:

$$P^* = \frac{a + c}{b + d}$$

$$Q^* = \frac{ad - bc}{b + d}$$

- *Comparative-static derivatives* are the partial derivatives of  $P^*$  and  $Q^*$  with respect to parameters ( $a$ ,  $b$ ,  $c$ , and  $d$ ):
  - $\partial P^*/\partial a$ ,  $\partial P^*/\partial b$ ,  $\partial P^*/\partial c$ ,  $\partial P^*/\partial d$
  - $\partial Q^*/\partial a$ ,  $\partial Q^*/\partial b$ ,  $\partial Q^*/\partial c$ ,  $\partial Q^*/\partial d$

# Application 1: Partial Market Equilibrium

- Partial derivatives of  $P^*$  with respect to parameters  $a$ ,  $b$ ,  $c$ , and  $d$  are:

# Application 1: Partial Market Equilibrium

- Partial derivatives of  $Q^*$  with respect to parameters  $a$ ,  $b$ ,  $c$ , and  $d$  are:

## Application 2: Elasticities

- **Two variables**

If  $z = f(x, y)$ , the **partial elasticity of  $z$  w.r.t.  $x$  and  $y$**  are:

$$\varepsilon_{zx} = \frac{\partial z/z}{\partial x/x} = \frac{\partial z}{\partial x} \left( \frac{x}{z} \right) \quad \text{and} \quad \varepsilon_{zy} = \frac{\partial z/z}{\partial y/y} = \frac{\partial z}{\partial y} \left( \frac{y}{z} \right)$$

When all variables are positive, elasticities can be expressed as logarithmic derivatives:

$$\varepsilon_{zx} = \frac{\partial \ln z}{\partial \ln x} \quad \text{and} \quad \varepsilon_{zy} = \frac{\partial \ln z}{\partial \ln y}$$

- **n variables**

If  $z = f(x_1, x_2, \dots, x_n) = f(\mathbf{x})$ , the **elasticity of  $z$  w.r.t.  $x_i$  when all other variables are held constant** is:

$$\varepsilon_{zi} = \frac{\partial f(\mathbf{x})/f(\mathbf{x})}{\partial x_i/x_i} = \frac{\partial \ln z}{\partial \ln x_i}$$

## Application 2: Elasticities of Demand

- Given the demand function  $Q_1 = a - bP_1 + cP_2 + mY$   
where  $Y$  = income,  $P_2$  = the price of a substitute good.
- Own price elasticity of demand:
- Income elasticity of demand:
- Cross-price elasticity of demand:

## Application 2: Elasticities of Demand

- **Example**: Given  $Q_1 = 100 - P_1 + 0.75P_2 - 0.25P_3 + 0.0075Y$ .

At  $Y = 10,000$ ,  $P_1 = 10$ ,  $P_2 = 20$ ,  $P_3 = 40$  and  $Q_1 = 100$ , find the different **cross-price elasticities of demand**.

➤  $\epsilon_{12} =$

➤  $\epsilon_{13} =$

## Application 2: Output Elasticity

- Given a *linearly homogenous* Cobb-Douglas production function

$$Q = F(K, L) = AK^\alpha L^\beta$$

- The output elasticity of capital:

$$\varepsilon_{QK} =$$

- The output elasticity of labor:

$$\varepsilon_{QL} =$$

# Application 3: Production Function

- Example: Given a production function

$$Q = 36KL - 2K^2 - 3L^2$$

- Marginal product of capital is:  $MP_K = 36L - 4K$
- Marginal product of labor is:  $MP_L = 36K - 6L$
- If the marginal revenue (MR) at  $K = 2$  and  $L = 3$  is \$5, the marginal revenue product (MRP) for the *third unit of L* is:

$$\begin{aligned} \rightarrow MRP_{L=3} &= MR \times MP_L \\ &= \end{aligned}$$

## Application 4:

# Multipliers in Macroeconomic Models

- Consider a national-income model

$$Y = C + I_0 + G_0$$

$$C = \alpha + \beta(Y - T) \quad (\alpha > 0; 0 < \beta < 1)$$

$$T = \gamma + \delta Y \quad (\gamma > 0; 0 < \delta < 1)$$

- Equilibrium national income (in reduced form):

$$Y^* = \frac{\alpha - \beta\gamma + I_0 + G_0}{1 - \beta + \beta\delta}$$

- Comparative-static derivatives:

- $\frac{\partial Y^*}{\partial \alpha}; \frac{\partial Y^*}{\partial \beta}; \frac{\partial Y^*}{\partial I_0}; \frac{\partial Y^*}{\partial G_0}; \frac{\partial Y^*}{\partial \gamma}; \frac{\partial Y^*}{\partial \delta}$

## Application 4:

# Multipliers in Macroeconomic Models

- Comparative-static derivatives with special *policy significance*:
  - Government-expenditure multiplier:

$$\frac{\partial Y^*}{\partial G_0}$$

- Nonincome-tax multiplier:

$$\frac{\partial Y^*}{\partial \gamma}$$

- Partial derivatives of  $Y^*$  w.r.t. the income tax rate ( $\delta$ ):

$$\frac{\partial Y^*}{\partial \delta}$$

# Second-Order Partial Derivatives (1)

- Consider the function  $z = f(x, y)$ , which give rise to:

$$f_x \equiv \frac{\partial z}{\partial x} \quad \text{and} \quad f_y \equiv \frac{\partial z}{\partial y}$$

- Since  $f_x$  is a function of  $x$  (and  $y$ ), we can determine the rate of change of  $f_x$  with respect to  $x$ , while  $y$  is fixed, by a *second-order partial derivative with respect to  $x$* :

$$f_{xx} \equiv \frac{\partial}{\partial x}(f_x) \quad \text{or} \quad \frac{\partial^2 z}{\partial x^2} \equiv \frac{\partial}{\partial x} \left( \frac{\partial z}{\partial x} \right)$$

- Similarly, the *second-order partial derivative with respect to  $y$*  is:

$$f_{yy} \equiv \frac{\partial}{\partial y}(f_y) \quad \text{or} \quad \frac{\partial^2 z}{\partial y^2} \equiv \frac{\partial}{\partial y} \left( \frac{\partial z}{\partial y} \right)$$

## Second-Order Partial Derivatives (2)

- Also, since  $f_x$  is a function of  $y$  and  $f_y$  is a function of  $x$ , *cross (or mixed) partial derivatives* can be written as:

$$f_{xy} \equiv \frac{\partial^2 z}{\partial x \partial y} \equiv \frac{\partial}{\partial x} \left( \frac{\partial z}{\partial y} \right) \quad \text{and} \quad f_{yx} \equiv \frac{\partial^2 z}{\partial y \partial x} \equiv \frac{\partial}{\partial y} \left( \frac{\partial z}{\partial x} \right)$$

### Young's Theorem:

Let  $y = f(x_1, x_2, \dots, x_n)$  is twice continuously differentiable ( $C^2$ ).

Then,

$$\frac{\partial^2 y}{\partial x_i \partial x_j} = \frac{\partial^2 y}{\partial x_j \partial x_i}; i \neq j$$

## Second-Order Partial Derivatives (3)

- Let  $y = f(x_1, x_2)$ . The second-order partial derivatives can be written in a matrix form called “**Hessian matrix**”:

$$H = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}$$

- Example: Let  $z = x^2 e^{-y}$ . Find  $f_x$ ,  $f_y$ , and the Hessian matrix.

# Application: Production Function

- Consider an agricultural production function

$$Q = F(K, L, T) = AK^\alpha L^\beta T^\gamma \quad (A > 0; \alpha > 0, \beta > 0, \gamma > 0)$$

Where K = capital, L = labor, and T = land.

- Marginal product of capital is:  $F_K =$
- Marginal product of labor is:  $F_L =$
- Marginal product of land is:  $F_T =$
- Second-order partial derivatives:
- Cross partial derivatives:

# DIFFERENTIALS & TOTAL DIFFERENTIALS

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# Differentials

- Recall the definition of derivatives:

$$\frac{dy}{dx} = f'(x) = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$$

$$\rightarrow dy = f'(x)dx$$

➤  $dy$  = differential of  $y$

➤  $dx$  = differential of  $x$

- Example: Let  $y = 3x^2 + 7x - 5$ . Find  $dy$ .

- The process of finding the differential  $dy$  from a function  $y = f(x)$  is called *differentiation*. (Note: for finding  $dy/dx$ , this is called differentiation with respect to  $x$ ).

# Total Differentials

- **Total differential** of a function  $Y$  is the **sum of the approximated changes from all parameters**.
- The process of finding total differentials is called “**total differentiation**.”

- Let  $y = f(x_1, x_2)$

$$dy = f_1 dx_1 + f_2 dx_2, \quad \text{where } f_1 = \partial y / \partial x_1, f_2 = \partial y / \partial x_2$$

- Example: Let  $z = 5x^2 + 6y + 7$ . Find  $dz = ?$

# Example 1: Saving Function

- Consider a **saving function**  $S = S(Y, r)$  where  $Y$  = income,  $r$  = interest rate.

- The **total change in S** is approximated by the **total differential**:

$$dS = \left( \frac{\partial S}{\partial Y} \right) \cdot dY + \left( \frac{\partial S}{\partial r} \right) \cdot dr \quad \text{or} \quad dS =$$

- For a **given change in Y**, the resulting change in S can be approximated by:  $dS = \left( \frac{\partial S}{\partial Y} \right) \cdot dY$

where  $\frac{\partial S}{\partial Y}$  = marginal propensity to save.

- For a **given change in r (dr)**, the resulting change in S can be approximated by:  $dS = \left( \frac{\partial S}{\partial r} \right) \cdot dr$

## Example 2: Utility Function

- General case of  $n$  independent variables:

Example:  $U = U(x_1, x_2, \dots, x_n)$

- Total differential of  $U$  is:

or

- Example: Let  $U = 100x^{0.5}y^{0.5}$ . Find  $dU = ?$

→  $dU =$

# Rules of Differentials

**Rule I**       $dk = 0$       (k is a constant.)

**Rule II**       $d(cu^n) = cnu^{n-1}du$

**Rule III**       $d(u \pm v) = du \pm dv$

**Rule IV**       $d(u \cdot v) = v \cdot du + u \cdot dv$

**Rule V**       $d(u/v) = [v \cdot du - u \cdot dv]/v^2$

**Rule VI**       $d(u \pm v \pm w) = du \pm dv \pm dw$

**Rule VII**       $d(uvw) = vw \cdot du + uw \cdot dv + uv \cdot dw$

# Examples: Rules of Differentials

- Example 1:  $y = 3x_1^2 + x_1x_2^2$
- Example 2:  $y = \frac{x_1 + x_2}{2x_1^2}$
- Example 3:  $y = 3x_1(2x_2 - 1)(x_3 + 5)$

# Application: Utility Function

- Consider a utility function

$$U = Ax^a y^b,$$

- Marginal utility of  $x$ :  $MU_x = aAx^{a-1}y^b$
- Marginal utility of  $y$ :  $MU_y = bAx^a y^{b-1}$
- **Marginal rate of substitution (MRS)** as the slope of the indifference curve:

# Application: Production Function

- Consider an agricultural production function

$$Q = F(K, L) = 60K^{0.25}L^{0.75}$$

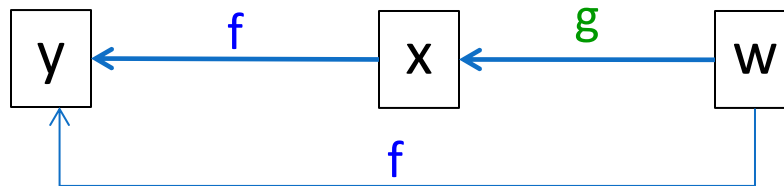
- Marginal product of capital is:
- Marginal product of labor is:
- **Marginal rate of technical substitution (MRTS)** as the slope of the isoquant:

# TOTAL DERIVATIVES

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# Total Derivatives (1)

- Now, consider the case where **independent variables are related to one another**.
  - Example: what is the rate of change of  $C(Y^*, T_0)$  w.r.t.  $T_0$ , where  $Y$  and  $T_0$  are related?
- Consider  $y = f(x, w)$  where  $x = g(w)$



➔  $y = f[g(w), w]$

➔  $dy/dw = ?$

# Total Derivatives (2)

- From  $y = f[g(w), w]$

1. Use chain rule:

$$\frac{dy}{dw} = f_x \frac{dx}{dw} + f_w = \frac{\partial y}{\partial x} \frac{dx}{dw} + \frac{\partial y}{\partial w}$$

2. Totally differentiate:

$$dy = f_x \cdot dx + f_w \cdot dw$$

➔  $\frac{dy}{dw} = f_x \frac{dx}{dw} + f_w$

- $dy/dw$  is called the “total derivative of  $y$  w.r.t.  $w$ ”.
- $\partial y/\partial w$  (partial derivative) is a component of the total derivative.

# Examples: Total Derivatives

- Example 1:  $y = f(x, w) = 3x - w^2$ , where  $x = g(w) = 2w^2 + w + 4$

$$\rightarrow \frac{dy}{dw}$$

- Example 2:  $U = U[c, g(c)]$

$$\rightarrow \frac{dU}{dc}$$

# A Variation on Total Derivatives

- Let  $y = f(x_1, x_2, w)$  where  $\begin{cases} x_1 = g(w) \\ x_2 = h(w) \end{cases}$

$$\rightarrow \frac{dy}{dw}$$

- Example:  $Q = Q(K, L, t)$ , where  $K = K(t)$  and  $L = L(t)$

$$\rightarrow \frac{dQ}{dt}$$

# Another Variation on Total Derivatives

- Let  $y = f(x_1, x_2, u, v)$  where 
$$\begin{cases} x_1 = g(u, v) \\ x_2 = h(u, v) \end{cases}$$

# Application: Utility Function

- Suppose  $U = U(C, n)$ ,

where  $C$  = consumption,  $n$  = leisure =  $24-L$ , and  $C = f(L)$ .

# IMPLICIT FUNCTION

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# Implicit Function

- Consider  $y = f(x) = 3x^4$  : Explicit function  
 $\rightarrow y - 3x^4 = 0$  : Implicit function

- **General form:**  $F(y, x_1, x_2, \dots, x_n) = 0$ .

- This function may define an **implicit function:**

$$y = f(x_1, x_2, \dots, x_n)$$

# Derivatives of Implicit Functions

- Given  $F(y, x_1, x_2, \dots, x_n) = 0$ ,

- We can write  $dF = d(0) = 0$ ,

$$\text{or } F_y dy + F_1 dx_1 + F_2 dx_2 + \dots + F_n dx_n = 0. \quad \text{-- (*)}$$

- Implicit function has the total differential:

$$dy = f_1 dx_1 + f_2 dx_2 + \dots + f_n dx_n$$

- Substituting  $dy$  in (\*) gives:

$$(F_y f_1 + F_1) dx_1 + (F_y f_2 + F_2) dx_2 + \dots + (F_y f_n + F_n) dx_n = 0$$

$$F_y f_i + F_i = 0 \quad \text{for } i = 1, 2, \dots, n.$$

$$\rightarrow \boxed{f_i \equiv \frac{\partial y}{\partial x_i} = -\frac{F_i}{F_y}} : \text{Implicit Function Rule}$$

# Example: Derivatives of Implicit Functions

- Example 1: Find  $dy/dx$  for the implicit function  $y - 3x^4 = 0$ .
- Example 2: Find  $\partial y/\partial x$  for any implicit function(s) that may be defined by  $F(y, x, w) = y^3x^2 + w^3 + yxw - 3 = 0$ .
- Example 3: Assume  $F(Q, K, L) = 0$  implicitly defines the production function  $Q = f(K, L)$ . Express  $MPP_K$  and  $MPP_L$  in relation to the function  $F$ .

# Application 1: Partial Market Equilibrium

- Suppose now  $Q_d$  is a function of  $P$  as well as  $Y_0$ .

$$Q_d = Q_s$$

$$Q_d = D(P, Y_0) \quad (\partial D / \partial P < 0; \partial D / \partial Y_0 > 0)$$

$$Q_s = S(P) \quad (\partial S / \partial P > 0)$$

- Equilibrium condition:

$$D(P, Y_0) - S(P) = 0.$$

- Equilibrium price:

$$P^* = P^*(Y_0).$$

- Equilibrium identity:

$$D(P^*, Y_0) - S(P^*) \equiv 0, \quad \text{where } D(P^*, Y_0) - S(P^*) = F(P^*, Y_0)$$

# Application 1: Partial Market Equilibrium

- Comparative-static derivatives:

$$\rightarrow \frac{dP^*}{dY_0}$$

$$\rightarrow \frac{dQ^*}{dY_0}$$

## Application 2: Production Function

- Example: Given the equation for a production isoquant

$$F(K,L) = 16K^{0.25}L^{0.75} = 2144,$$

Use the implicit function rule to find the slope of the isoquant ( $|dK/dL|$ : marginal rate of technical substitution).