Practice problem set 6 EE320 Semester 1/2019

Chapter 8:

Multivariate calculus: Unconstrained optimization

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Question 1:

Define f(x,y) for all (x,y) by

$$f\ (x,y)\ =\ e^{x+y} +\ e^{x-y} - \frac{3}{2}x\ -\ \frac{1}{2}y$$

a. Derive the Hessian matrix of $f\ (x,y)$.
b. Show that f(x,y) is monotonically convex function. That's, the function is convex for the entire domain sets defining f(x,y).
c. Find the *global* extrema of f(x,y)

Question 2:

Consider a function $f\ (x,y)\ =\ x^2\ -\ y^2\ -xy\ -x^3$

a. Find and classify the stationary points of $f\ (x,y)$
b. Find the domain set of $f\ (x,y)$ where $f\ (x,y)$ is concave, and find the largest value of $f\ (x,y)$ in that domain set.

Question 3: Suppose that there are two firms in the industry, and they are competing in quantities. The amount of the commodity sold by firm *i* is $q_i, i = 1, 2$. The market demand function is given by $P = 50 - 3q$, where $q = q_1 + q_2$. The cost functions for each firm is given by $TC_i = 25 + 5q_i, i = 1,2$.

a. Find the profit-maximizing quantity for each firm,and determine each firm's profit level.

b. Suppose that both firms merge. Compute the new profit-maximizing quantity and the new profit of the merged firm. Do firms have incentive to merge, and why?

Question 4: Given the production function

$$\boldsymbol{Q = f(K, L) = 8K^{1/2}L^{1/4}}$$

Suppose that the price per unit of Q is \$$P$, and the per unit input prices for K and L are \$$r$ and \$$w$, respectively. ($P$, w, and r are positive constants.)

a. Solve for the values and that maximizes the profit. Verify that the second-order sufficient condition is met.
b. Find the comparative statics derivatives $\frac{\partial K^*}{\partial r}$ and $\frac{\partial L^*}{\partial P}$, evaluate the signs, and interpret their economic meanings.

Question 5: Each of two firms A and B produces its own brand of a commodity, such as mineral water, in amounts denoted by x and y, and these are sold at prices p and q unit, respectively. Each firm determines its own price and produces exactly as much as is demanded. The demands for the two brands are given by

$$x = 29 - 5p + 4q$$

$$y = 16 + 4p - 6q$$

Firm A has total costs $5 + x$, whereas firm B has total costs $3 + 2y$. (Assume that the functions to be maximized have maxima, and at positive prices.)

a. Initially, the two firms collude in order to maximize their combined profits, as one monopolist would. Find the prices (p, q), the production levels (x, y), and the profits of firms A and B.
b. Then, an antitrust authority prohibits collusion, so each producer maximizes its own profit, taking the other's price as given.
    - If q is fixed, how will A choose p? (Find p as a function $p = p_A(q)$.)

- If p is fixed, how will B choose q? (Find q as a function $q = q_B(p)$.)

c. Under the assumptions in part b), what constant equilibrium prices are possible? What are the production levels and profits in this case?

Question 6: A firm produces two different kinds A and B of a commodity. The daily cost of producing Q_1 units of A and Q_2 units of B is:

$$C(Q_1, Q_2) = 0.1(Q_1^2 + Q_1Q_2 + Q_2^2).$$

Suppose that the firm sells all its output at a price per unit $P_1 = 120$ for A and $P_2 = 90$ for B.

a. Find the daily production levels that maximize profit.
b. What prices (P_1) per unit of A would imply that the optimal daily production level for A is 400 units?

Question 7:

Let the total cost function depend on goods x, y and z ;

Determine the level of x , y and z which minimize total cost and determine the minimum total cost.

Question 8:

A monopolist produces two products, A, and B. The joint-cost function is $\boldsymbol{C = 5000 + 5\,q_A + 3\,q_B}$, where c is the total cost of producing q_A units of A and q_B units of B. The demand functions for these products are given by $p_A = 205 - 2q_A - q_B$ and $p_B = 153 - q_A - q_B$, where p_A and p_B are the prices of A and B, respectively. Consider the following problem.

a. Determine the profit-maximizing level output for both products.
b. How much should the monopolist set the price of the two products?

Question 9:

A manufacturer produces products A and B for which the average costs of production are constant at 3 and 5 (dollars per unit), respectively. The quantities q_A, q_B of A and B that can be sold each week are given by the joint-demand functions $q_A = 10 - p_A + p_B$ and $q_B = 12 + p_A - 3p_B$, p_A and p_B are the prices (in dollars per unit) of A and B, respectively. Determine the prices of A and B at which the manufacturer can maximize profit.

Question 10:

Consider a market with 2 firms. Each firm sells an identical product, facing the same market demand equation given by $p = 10 - Q$. For the first firm, denoted by firm 1, the cost function is given by $C = c_1 Q_1$. For the second firm, the cost function is given by $C = c_1 {Q_2}^2$. Consider the following problem.

a. Determine the level of output that each firm will choose to produce under the Cournot equilibrium.
b. State the requirement for c_1 in order to ensure that both firms stay active in the equilibrium.
c. Do they share the same market size under the equilibrium? Explain your result with some economic intuitions.
d. Determine the level of output that each firm will produce under the collusion.

Question 11:

The profit function of a firm is $\pi(x, y) = px + qy - \alpha x^2 - \beta y^2$, where p and q are the prices per unit and $\alpha x^2 + \beta y^2$ are the costs of producing x units of the first good and y units of the other. The constants are all positive.

a. Find the values of x and y that maximize profits. Denote them by x^* and y^*. Verify that the second-order conditions are satisfied.

b. Define π(p, q) = π(x^*,y^*) as the optimal profit function. The function generates the level of maximum profit that firm attain under different combination of profit-maximizing output bundles. Verify that $\partial\pi(p,q)/\partial p = x^*$ and $\partial\pi(p,q)/\partial q = y^*$. Give these results economic interpretations.

c. Show that π(p,q) is convex in p and q. That is, you show that Hessian matrix of the optimal profit function is positive definiteness.

Practice problem set 6

EE320 Semester 1/2019

Chapter 8:

Multivariate calculus: Unconstrained optimization

(Solution)

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Question 1:

Define f(x,y) for all (x,y) by $f\ (x, y)\ =\ e^{x+y} +\ e^{x-y} - \frac{3}{2}\ x\ -\ \frac{1}{2}y$

a. Derive the Hessian matrix of $f\ (x, y)$.

$$H = \begin{bmatrix} e^{x+y} + e^{x-y} & e^{x+y} - e^{x-y} \\ e^{x+y} - e^{x-y} & e^{x+y} + e^{x-y} \end{bmatrix}$$

b. Show that f(x,y) is monotonically convex function. That's, the function is convex for the entire domain sets defining f(x,y).

|H1| $= e^{x+y} +\ e^{x-y}\ >\ 0\ \ for\ all\ x\ and\ all\ y.$

|H2| $= (e^{x+y} +\ e^{x-y})^2 - (e^{x+y} -\ e^{x-y})^2 = 4e^{x+y} *\ e^{x-y} = 4e^{2x} > 0\ \ for\ all\ x, y$

Thus, the function is monotonically convex.

c. Find the *global* extrema of f(x,y)

FOC: $[x]$: $\quad e^{x+y} +\ e^{x-y} - \frac{3}{2} = 0$ --- (1)

$[y]: \quad e^{x+y} - e^{x-y} - \frac{1}{2} = 0$ --- (2)

(1)+(2) $\Rightarrow e^{x+y} = 1 \Rightarrow x + y = ln(1) = 0$

(1)-(2) $\Rightarrow e^{x-y} = 0.5 \Rightarrow x - y = ln(0.5)$

$$x^* = 0.5 * ln(0.5), \qquad y = -0.5 * ln(0.5)$$

This point is global minimum because, as proved in b, "f" is monotonically convex.

Question 2:

Consider a function $f\,(x, y) = x^2 - y^2 - xy - x^3$

a. Find and classify the stationary points of $f\,(x, y)$

$[x]: 2x - y - 3x^2 = 0$ --(1)

$[y]: -2y - x = 0$ --(2)

From (2), y = -x/2. Plug this back into (1), this yields us,

$$2x + x/2 - 3x^2 = 0$$

$$5x - 6x^2 = 0$$

$x = 0 \; and \; 5/6 \Rightarrow y = 0 \; (x = 0) \; and \; y = -5/12 \; (x = 5/6).$

We have two stationary points: (0,0) and (⅚ ,-5/12).

b. Find the domain set of $f\,(x, y)$ where $f\,(x, y)$ is concave, and find the largest value of $f\,(x, y)$ in that domain set.

$$H = \begin{bmatrix} 2 - 6x & -1 \\ -1 & -2 \end{bmatrix}$$

|H1| = 2 - 6x

|H2| = (-2)(2-6x) - 1 = 12x - 3

To ensure the concavity of function, 2 - 6x < 0 and 12x-3 > 0. That is, we need x > ⅓ and x > ¼ . So, x > ⅓ is needed to ensure the concavity of the function. Note form (1) that, one of the stationary points is (⅚, -5/12). This point falls into the territory of domain set that generates the function with concavity property. As a result, this point is global maximum.

Question 3: Suppose that there are two firms in the industry, and they are competing in quantities. The amount of the commodity sold by firm *i* is q_i, $i = 1, 2$. The market demand function is given by $P = 50 - 3q$, where $q = q_1 + q_2$. The cost functions for each firm is given by $TC_i = 25 + 5q_i, i = 1,2$.

a. Find the profit-maximizing quantity for each firm, and determine each firm's profit level.

Ans.

Firm 1: $Max_{q_1}\ \pi_1 = [50 - 3(q_1 + q_2)]q_1 - 25 - 5q_1$

FOC: $\frac{\partial \pi_1}{\partial q_1} = 0 \Rightarrow 45 - 6q_1 - 3q_2 = 0$ -- (1)

Firm 2: $Max_{q_2}\ \pi_2 = [50 - 3(q_1 + q_2)]q_2 - 25 - 5q_2$

FOC: $\frac{\partial \pi_1}{\partial q_2} = 0 \Rightarrow 45 - 3q_1 - 6q_2 = 0$ -- (2)

(1) & (2) $\Rightarrow q_i^* = 5$ and $\pi_i^* = 50$ for i = 1, 2.

b. Suppose that both firms merge. Compute the new profit-maximizing quantity and the new profit of the merged firm. Do firms have incentive to merge, and why?

Ans.

$$Max_{q_m}\ \pi_m = [50 - 3q_m]q_m - 25 - 5q_m$$

FOC: $\frac{\partial \pi_m}{\partial q_m} = 0 \Rightarrow 45 - 6q_m = 0$

$$\Rightarrow q_m^* = 7.5 \text{ and } \pi_m^* = 143.75.$$

If the two firms split the profit equally, each firm should get the new profit of 71.875 (>50). Thus, both firms have incentives to merge. Note that the incentives to merge hold as long as the two firms commit or sign a contract not to deviate from the agreed production.

Question 4: Given the production function

$$\boldsymbol{Q = f(K, L) = 8K^{1/2}L^{1/4}}$$

Suppose that the price per unit of Q is \$$P$, and the per unit input prices for K and L are \$$r$ and \$$w$, respectively. ($P$, w, and r are positive constants.)

a. Solve for the values and that maximizes the profit. Verify that the second-order sufficient condition is met.

Ans. $\text{Max}_{K,L}\ \pi = P(\boldsymbol{8K^{1/2}L^{1/4}}) - \boldsymbol{rK - wL}$

FOC: $\pi_K = 0 \Rightarrow 4PK^{-1/2}L^{1/4} - r = 0$ -- (1)

$\pi_L = 0 \Rightarrow 2PK^{1/2}L^{-3/4} - w = 0$ -- (2)

$$K^* = \frac{2*4^3P^4}{r^3w} \text{ and } L^* = \frac{4^3P^4}{r^2w^2}$$

b. Find the comparative statics derivatives $\frac{\partial K^*}{\partial r}$ and $\frac{\partial L^*}{\partial P}$, evaluate the signs, and interpret their economic meanings.

Ans. $\frac{\partial K^*}{\partial r} = \frac{-384P^4}{r^4w} < 0$

$\frac{\partial L^*}{\partial P} = \frac{256P^3}{r^2w^2} > 0$

Question 5: Each of two firms A and B produces its own brand of a commodity, such as mineral water, in amounts denoted by x and y, and these are sold at prices p and q unit, respectively. Each firm determines its own price and produces exactly as much as is demanded. The demands for the two brands are given by

$$x = 29 - 5p + 4q$$

$$y = 16 + 4p - 6q$$

Firm A has total costs $5 + x$, whereas firm B has total costs $3 + 2y$. (Assume that the functions to be maximized have maxima, and at positive prices.)

a. Initially, the two firms collude in order to maximize their combined profits, as one monopolist would. Find the prices (p, q), the production levels (x, y), and the profits of firms A and B.

Ans. $\pi_T(p,q) = px + qy - (5 + x) - (3 + 2y) = 26p + 24q - 5p^2 - 6q^2 + 8pq - 69$. $(p^*, q^*) = (9, 8)$ and $(x^*, y^*) = (16, 4)$.

b. Then, an antitrust authority prohibits collusion, so each producer maximizes its own profit, taking the other's price as given.
   - If q is fixed, how will A choose p? (Find p as a function $p = p_A(q)$.)
   - If p is fixed, how will B choose q? (Find q as a function $q = q_B(p)$.)

Ans. $\pi_A(p) = px - (5 + x) = 34p - 5p^2 + 4pq - 4q - 34$, with q is fixed.

➔ $p = p_A(q) = \frac{1}{5}(2q + 17)$.)

$\pi_B(q) = qy - (3 + 2y) = 28q - 6q^2 + 4pq - 8p - 35$, with p is fixed.

➔ $q = q_B(p) = \frac{1}{3}(p + 7)$.

c. Under the assumptions in part b), what constant equilibrium prices are possible? What are the production levels and profits in this case?

Ans. $(p^*, q^*) = (5, 4)$ and $(x^*, y^*) = (20, 12)$. $\pi_A = 75$ and $\pi_B = 21$.

Question 6: A firm produces two different kinds A and B of a commodity. The daily cost of producing Q_1 units of A and Q_2 units of B is:

$$C(Q_1, Q_2) = 0.1(Q_1^2 + Q_1Q_2 + Q_2^2).$$

Suppose that the firm sells all its output at a price per unit $P_1 = 120$ for A and $P_2 = 90$ for B.

a. Find the daily production levels that maximize profit.

Ans. $\pi(Q_1, Q_2) = 120Q_1 + 90Q_2 - 0.1(Q_1^2 + Q_1Q_2 + Q_2^2)$.

FONC: $\pi_1(Q_1, Q_2) = 120 - 0.2Q_1 - 0.1Q_2 = 0$; $\pi_2(Q_1, Q_2) = 90 - 0.1Q_1 - 0.2Q_2 = 0$

Thus, $(Q_1, Q_2) = (500, 200)$ maximizes profit.

b. What prices (P_1) per unit of A would imply that the optimal daily production level for A is 400 units?

Ans. $\tilde{\pi}(Q_1, Q_2) = P_1Q_1 + 90Q_2 - 0.1(Q_1^2 + Q_1Q_2 + Q_2^2)$

FONC: $\tilde{\pi}_1(Q_1, Q_2) = P_1 - 0.2(400) - 0.1Q_2 = 0$;

$\tilde{\pi}_2(Q_1, Q_2) = 90 - 0.1(400) - 0.2Q_2 = 0$.

It follows that $P_1 = 105$.

Question 7:

Let the total cost function depend on goods x, y and z ;

$$TC = 1,000 + 3x^2 + 2y^2 + 2z^2 - 2xy - 40z - 20x$$

Determine the level of x , y and z which minimize total cost and determine the minimum total cost.

Answer:

From the FOC, you can solve for x =4, y=2, z=10 and minimum TC=760]

$$H = \begin{bmatrix} 6 & -2 & 0 \\ -2 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

|H1| =6 >0 ; |H2| =20 > 0 ; |H3| =80 >0. TC is a monotonically convex function. This guarantees the solution in the FOC as the minimizer of the total cost function. In fact, as the function is monotonically convex, the local solution is warranted to be a unique global solution in this example.

Question 8:

A monopolist produces two products, A, and B. The joint-cost function is $C = 5000 + 5\,q_A + 3\,q_B$,where c is the total cost of producing q_A units of A and $q_B$units of B. The demand functions for these products are given by $p_A = 205 - 2q_A - q_B$ and $p_B = 153 - q_A - q_B$, where $p_A$and $p_B$are the prices of A and B, respectively. Consider the following problem.

a. Determine the profit-maximizing level output for both products.

FOCs. [q_A]: 205 - 4q_A- $2q_B$ = 5

[q_B]: 153 - 2q_B- $2q_A$ = 3

$q_A{}^* = 25\ units$ and $q_B{}^* = 50\ units$

b. How much should the monopolist set the price of the two products?

$$p_A = 205 - 2(25) - 50 = 105$$
$$p_B = 153 - 25 - 50 = 78$$

Question 9:

A manufacturer produces products A and B for which the average costs of production are constant at 3 and 5 (dollars per unit), respectively. The quantities q_A, q_B of A and B that can be sold each week are given by the joint-demand functions $q_A = 10 - p_A + p_B$ and $q_B = 12 + p_A - 3p_B$, p_A and p_B are the prices (in dollars per unit) of A and B, respectively. Determine the prices of A and B at which the manufacturer can maximize profit.

Profit function will be written in terms of p_A and p_B.

$$\pi(*) = p_A * (10 - p_A + p_B) + p_B * (12 + p_A - 3p_B) - 3(10 - p_A + p_B)$$
$$-5(12 + p_A - 3p_B).$$

FOC: $[p_A]$: $10 - 2p_A + 2p_B + 3 - 5 = 0$

$[p_B]$: $12 + 2p_A - 6p_B - 3 + 15 = 0$

Solving for p_A and p_B, the answer is (12, 8).

You can check that the second order condition holds.

Question 10:

Consider a market with 2 firms. Each firm sells an identical product, facing the same market demand equation given by $p = 10 - Q$. For the first firm, denoted by firm 1, the cost function is given by $C = c_1 Q_1$. For the second firm, the cost function is given by $C = c_1 {Q_2}^2$. Consider the following problem.

a. Determine the level of output that each firm will choose to produce under the Cournot equilibrium.

$[Q_1]$: $10 - 2Q_1 - Q_2 - c_1 = 0 \Rightarrow 2Q_1 + Q_2 = 10 - c_1$ --(1)

$[Q_2]$: $10 - Q_1 - 2\,Q_2 - 2c_1Q_2 = 0 \Rightarrow Q_1 + 2(1 + c_1)\,Q_2 = 10$ --(2)

Solving for Q_1 and Q_2, the answer is

$$\left(2\frac{(5+9c_1-c_1{}^2)}{4c_1+3}\ ,\ \frac{c_1+10}{4c_1+3}\right)$$

b. State the requirement for c_1 in order to ensure that both firms stay active in the equilibrium.

Note first that it's always for the case $Q_2 > 0$ as $c_1 \geq 0$. (When $c_1 = 0$, each firm doesn't have to pay for the cost of production.)

To guarantee that $Q_1 > 0$, we need to ensure that $5 + 9c_1 - c_1{}^2 > 0$ and $c_1 \geq 0$.

c. Do they share the same market size under the equilibrium? Explain your result with some economic intuitions.

No, they don't. One firm might be acquiring bigger size of market share than the other. The firm with lower marginal cost would typically acquire bigger size of market share in the equilibrium.

d. Determine the level of output that each firm will produce under the collusion.

Consider the joint profit-maximization problem. The problem is turned into a multi-plant problem.

$$[Q_1]:\ 10 - 2(Q_1 + Q_2) - c_1 = 0$$
$$[Q_2]:\ 10 - 2(Q_1 + Q_2) - 2c_1Q_2 = 0$$

We yield that $Q_1 = \frac{9-c_1}{2}$ and $Q_2 = \frac{1}{2}$

Question 11:

The profit function of a firm is $\pi(x, y) = px + qy - \alpha x^2 - \beta y^2$, where p and q are the prices per unit and $\alpha x^2 + \beta y^2$ are the costs of producing x units of the first good and y units of the other. The constants are all positive.

a. Find the values of x and y that maximize profits. Denote them by $x^*$and y^*. Verify that the second-order conditions are satisfied.

$x^* = p/2\alpha$ and $y^* = q/2\beta$

$$H = \begin{bmatrix} -2\alpha & 0 \\ 0 & -2\beta \end{bmatrix}$$

$|H1| = -2\alpha < 0$ and

$|H2| = 4\alpha\beta > 0$ ⇒ the function is monotonically concave.

b. Define $\pi(p, q) = \pi(x^*, y^*)$ as the optimal profit function. The function generates the level of maximum profit that firm attain under different combination of profit-maximizing output bundles. Verify that $\partial\pi(p,q)/\partial p = x^*$ and $\partial\pi(p,q)/\partial q = y^*$. Give these results economic interpretations.

$\pi(p, q) = p^2/2\alpha + q^2/2\beta - \alpha(p/2\alpha)^2 - \beta(q/2\beta)^2$

$\partial\pi(p,q)/\partial p = p/\alpha - p/2\alpha = p/2\alpha = x^*$

$\partial\pi(p,q)/\partial q = q/\beta - q/2\beta = q/2\beta = y^*$

c. Show that π(p,q) is convex in p and q. That is, you show that Hessian matrix of the optimal profit function is positive definiteness.

Using the information in b, we know that

$$H = \begin{bmatrix} 1/2\alpha & 0 \\ 0 & 1/2\beta \end{bmatrix}$$

$|H1| = 1/2\alpha > 0$ and $|H2| = 1/(4\alpha\beta) > 0$. H is positive definite, and thus π(p,q) is convex in p and q.