EE320 (2/2013)

INTRODUCTORY MATHEMATICAL ECONOMICS

MATRIX ALGEBRA AND ITS APPLICATION

PART I: BASIC MATRIX ALGEBRA

Topics

- Representation of a system of equations by matrix notations
- Review matrices and matrix operation
- Determinant and singularity of matrix
- Matrix inversion by determinant
- Cramer’s rule

Why do economists need matrix algebra?

- Suppose you are asked to solve the following system of simultaneous equations:

$$
\begin{array}{lll}
Q_{d1} = 8 - 3P_1 + P_2 - 4P_3, & Q_{s1} = 2 + 2P_1, & Q_{d1} = Q_{s1} \\
Q_{d2} = 12 + 5P_1 - 2P_2 + 2P_3, & Q_{s2} = 2 + 3P_2, & Q_{d2} = Q_{s2} \\
Q_{d3} = 7 - 6P_1 + P_2 - 3P_3, & Q_{s3} = 5 + 4P_3, & Q_{d3} = Q_{s3}
\end{array}
$$

Question: How would you derive the equilibrium prices and quantities for the 3 goods?

- Linear algebra: solution by elimination of variables
  - ➔ This could be quite tedious and prone to mistakes.
- Matrix algebra?

Why do economists need matrix algebra? (Cont'd)

- Matrix algebra can be very useful in handling a large system of simultaneous equations, commonly used in economics.
- Why?
  - It provides a compact way of writing an equation system.
  - It leads to a way of testing the existence of a solution by evaluation of a *determinant*.
  - It gives a method of finding the solution (if exists).
- Drawback: Matrix algebra is applicable only to linear-equation system.

System of Equations in Matrix Form

- Given a system of *m* linear equations in *n* variables:

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = d_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = d_2$$
$$\ldots\ldots$$
$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = d_m$$

- We can form a matrix from the above system of equations as: ***AX* = *d***, where

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix} = [a_{ij}] \qquad X = \begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{bmatrix} \qquad d = \begin{bmatrix} d_1 \\ d_2 \\ \ldots \\ d_m \end{bmatrix}$$

Coefficients Variables Constant terms

System of Equations in Matrix Form (Cont’d)

- Example:

$$Q_d = Q_s$$

$$Q_d = a - bP$$

$$Q_s = -c + dP$$

➔Write in matrix form AX = d:

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & b \\ 0 & 1 & -d \end{bmatrix} \qquad X = \begin{bmatrix} Q_d \\ Q_s \\ P \end{bmatrix} \qquad d = \begin{bmatrix} 0 \\ a \\ -c \end{bmatrix}$$

- Later on we will show that, if a certain condition is satisfied ($|A| \neq 0$), then $X = A^{-1}d$, where A^{-1} is the inverse of matrix A.

Example

- Suppose a firm sells 3 products in 3 regions.

Good	**Sale Volume (Unit: Piece)**		
	North	Central	South
Good A	100	50	120
Good B	80	110	60
Good C	90	70	130

- Matrix of sale volumes (rows-> product & columns->region) can be written as:

$$Q = \begin{bmatrix} 100 & 50 & 120 \\ 80 & 110 & 60 \\ 90 & 70 & 130 \end{bmatrix}$$

Types of Matrices

- Let A be an $m \times n$ matrix, where m is the number of rows and n is the number of columns.

1. When $m=n$, matrix A is a square matrix.
2. When $n=1$, matrix A is a column vector.
3. When $m=1$, matrix A is a row vector.
4. An identity matrix is a square matrix containing ones along the diagonal and zeros elsewhere:

$$I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

5. A null matrix is a matrix whose elements are all zero.
6. A diagonal matrix is a square matrix $A_{nxn} = [a_{ij}]$ with $a_{ij}=0$ when $i \neq j$.

Basic Matrix Operations

- Let A and B be $m\ x\ n$ matrices.

1. $A = B$ if and only if $a_{ij} = b_{ij}$ for all values of i and j.
2. If $A = [a_{ij}]_{m\ x\ n}$ and $B = [b_{ij}]_{m\ x\ n}$, then

$$A + B = [a_{ij}]_{m\ x\ n} + [b_{ij}]_{m\ x\ n} = [a_{ij} + b_{ij}]_{m\ x\ n}$$

$$A - B = [a_{ij}]_{m\ x\ n} - [b_{ij}]_{m\ x\ n} = [a_{ij} - b_{ij}]_{m\ x\ n}$$

Example:

$$\begin{bmatrix} 3 & 1 \\ 4 & 7 \end{bmatrix} + \begin{bmatrix} -5 & 2 \\ 6 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ 10 & 7 \end{bmatrix}$$

3. If α is a real number, then $\alpha A = \alpha[a_{ij}]_{m\ x\ n} = [\alpha a_{ij}]_{m\ x\ n}$

Example:

$$5\begin{bmatrix} 3 & 1 \\ 4 & 7 \end{bmatrix} = \begin{bmatrix} 15 & 5 \\ 20 & 35 \end{bmatrix}$$

Basic Matrix Operations (cont'd)

4. Rules of matrix addition and multiplication by scalars
   a) $(A + B) + C = A + (B + C)$
   b) $A + B = B + A$
   c) $A + 0 = A$
   d) $A + (-A) = 0$
   e) $(\alpha + \beta)A = \alpha A + \beta A$
   f) $\alpha (A + B) = \alpha A + \alpha B$

Matrix Multiplication

- Suppose $A = [a_{ij}]_{m\,x\,n}$ and $B = [b_{ij}]_{n\,x\,p}$.

Then, the product $C = AB$ is the $m\ x\ p$ matrix $\mathbf{C = [c_{ij}]_{m\,x\,p}}$, whose element ij is the inner product

$$\boxed{C_{ij} = \sum_{r=1}^{n} a_{ir}b_{rj} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj}}\ .$$

- Example:

Let $A_{1x2} = \begin{bmatrix} a_{11} & a_{12} \end{bmatrix}$ $\quad B_{2x3} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$

$$\Rightarrow C = AB = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} & a_{11}b_{13} + a_{12}b_{23} \end{bmatrix}$$

Example: Matrix Multiplication

- Suppose the sale volume matrix and the price vector are given by:

$$Q = \begin{bmatrix} 100 & 50 & 120 \\ 80 & 110 & 60 \\ 90 & 70 & 130 \end{bmatrix} \qquad P = \begin{bmatrix} P_A & P_B & P_C \end{bmatrix} = \begin{bmatrix} 2 & 3 & 5 \end{bmatrix}$$

- The total revenue from the sales of all products is:

$$TR = PQ = \begin{bmatrix} 2 & 3 & 5 \end{bmatrix} \begin{bmatrix} 100 & 50 & 120 \\ 80 & 110 & 60 \\ 90 & 70 & 130 \end{bmatrix} = \begin{bmatrix} TR_{North} & TR_{Central} & TR_{South} \end{bmatrix}$$

$$TR = \begin{bmatrix} 2(100)+3(80)+5(90) & 2(50)+3(110)+5(70) & 2(120)+3(60)+5(130) \end{bmatrix}$$

$$TR = \begin{bmatrix} 890 & 780 & 1070 \end{bmatrix}$$

Rule for Matrix Multiplication:

a) $AB \neq BA$ (in general)

b) $(AB)C = A(BC)$

c) $A(B+C) = AB + AC$

d) $(B + C)A = BA + CA$

- Example:

Let $A = \begin{bmatrix} 3 & 1 \\ 4 & 7 \end{bmatrix}$ $B = \begin{bmatrix} -5 & 2 \\ 6 & 0 \end{bmatrix}$

➔ $AB = \begin{bmatrix} 3 & 1 \\ 4 & 7 \end{bmatrix}\begin{bmatrix} -5 & 2 \\ 6 & 0 \end{bmatrix} = \begin{bmatrix} -15+6 & 6 \\ -20+42 & 8 \end{bmatrix} = \begin{bmatrix} -9 & 6 \\ 22 & 8 \end{bmatrix}$

➔ $BA = \begin{bmatrix} -5 & 2 \\ 6 & 0 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 4 & 7 \end{bmatrix} = \begin{bmatrix} -15+8 & -5+14 \\ 18 & 6 \end{bmatrix} = \begin{bmatrix} -7 & 9 \\ 18 & 6 \end{bmatrix}$

Matrix Transposition

- The transpose of any *m x n* matrix *A*, denoted by A' or A^T, is defined as the *n x m* matrix whose first column is the first row of *A*, and the second column is the second row of *A*, and so on.

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \longrightarrow A' = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \dots & \dots & \dots & \dots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{bmatrix}$$

- Example:

$$Q = \begin{bmatrix} 100 & 80 & 90 \\ 50 & 110 & 70 \\ 120 & 60 & 130 \end{bmatrix} \longrightarrow Q' = \begin{bmatrix} 100 & 50 & 120 \\ 80 & 110 & 60 \\ 90 & 70 & 130 \end{bmatrix}$$

Properties of transposes

a) $(A')' = A$

b) $(A + B)' = A' + B'$

c) $(AB)' = B'A'$

d) $(\alpha A)' = \alpha A'$

Note: A square matrix A is symmetric (i.e. $A = A'$) iff $a_{ij} = a_{ji}$ for all i,j

- Example:

$$A = \begin{bmatrix} 3 & 1 \\ 4 & 7 \end{bmatrix} \qquad B = \begin{bmatrix} -5 & 2 \\ 6 & 0 \end{bmatrix}$$

➔ $(AB)' = \begin{bmatrix} -9 & 6 \\ 22 & 8 \end{bmatrix}' = \begin{bmatrix} -9 & 22 \\ 6 & 8 \end{bmatrix}$

➔ $B'A' = \begin{bmatrix} -5 & 6 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 1 & 7 \end{bmatrix} = \begin{bmatrix} -15+6 & -20+42 \\ 6 & 8 \end{bmatrix} = \begin{bmatrix} -9 & 22 \\ 6 & 8 \end{bmatrix}$

Matrix Inversion

- The inverse of matrix A, denoted by A^{-1}, is defined only if A is a square matrix and must satisfy the condition

$$AA^{-1} = A^{-1}A = I.$$

- Properties of the inverse:
  a) If A^{-1} exists, then $(A^{-1})^{-1} = A$.
  b) If AB is invertible, then $(AB)^{-1} = B^{-1}A^{-1}$
  c) $(A')^{-1} = (A^{-1})'$
  d) $(cA)^{-1} = c^{-1}A^{-1}$, where $c \neq 0$
- If a square matrix A has an inverse, A is said to be nonsingular. If A is not invertible, then A is called a singular matrix.
  - The nonsingularity condition is required for a solution of a linear-equation system to exist.

Inverse Matrix and Solution of Linear-Equation System

- Given the equation system in matrix notation:

$$AX = d$$

$$A^{-1}AX = A^{-1}d$$

$$\Rightarrow \quad X = A^{-1}d$$

where X is the column vector of variables, and $A^{-1}d$ is the column vector of solution values.

  - Since A^{-1}, if it exists, is unique, the solution $A^{-1}d$ must be unique values.

- Example:

$$\begin{bmatrix} Q_d \\ Q_s \\ P \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & b \\ 0 & 1 & -d \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ a \\ -c \end{bmatrix} = \frac{-1}{b+d} \begin{bmatrix} -b & -d & -b \\ d & -d & -b \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ a \\ -c \end{bmatrix} = \frac{1}{b+d} \begin{bmatrix} ad-bc \\ ad-bc \\ a+c \end{bmatrix}$$

- Next, we will study how to determine A^{-1} by determinant.

Determinant

- One way to test whether an inverse of a matrix exists (i.e. it is nonsingular) is to use determinant.
- The determinant of a *square* matrix A, denoted by |A|, is a uniquely defined associated with that matrix.
  - Determinant of order 1: |A| = |a| = a
  - Determinant of order 2: $|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$
  - Determinant of order 3:

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

➢ $|A| = a_{11}\left(a_{22}a_{33} - a_{32}a_{23}\right) - a_{12}\left(a_{21}a_{33} - a_{31}a_{23}\right) + a_{13}\left(a_{21}a_{32} - a_{31}a_{22}\right)$

Example: Determinants of Order 3

- Evaluate the following determinant:

$$|A| = \begin{vmatrix} 5 & 7 & 9 \\ 2 & 5 & 6 \\ 9 & 0 & 12 \end{vmatrix}$$

$$= 5\begin{vmatrix} 5 & 6 \\ 0 & 12 \end{vmatrix} - 7\begin{vmatrix} 2 & 6 \\ 9 & 12 \end{vmatrix} + 9\begin{vmatrix} 2 & 5 \\ 9 & 0 \end{vmatrix}$$

$$= 5(60-0) - 7(24-54) + 9(0-45)$$

$$= 300 + 210 - 405$$

$$= 105$$

Evaluating an *n*th-Order Determinants by Laplace Expansion

<u>Definitions</u>: Let A be an n x n matrix.

- The minor of the element a_{ij} is: $$|M_{ij}| = \begin{vmatrix} a_{11} & \cdots & a_{1j-1} & a_{1j} & a_{1j+1} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \iddots & \vdots \\ a_{i1} & \cdots & a_{ij-1} & a_{ij} & a_{ij+1} & \cdots & a_{in} \\ \vdots & \iddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nj-1} & a_{nj} & a_{nj+1} & \cdots & a_{nn} \end{vmatrix}$$
- The cofactor of the element a_{ij} is: $|C_{ij}| = (-1)^{i+j}\,|M_{ij}|$
- The determinant of the matrix A of order n can be found by the Laplace expansion of *any row* or *any column* as follows:

$$|A| = \sum_{j=1}^{n} a_{ij}|C_{ij}| \qquad \text{[expansion by the } i\text{th row]}$$

$$|A| = \sum_{i=1}^{n} a_{ij}|C_{ij}| \qquad \text{[expansion by the } j\text{th column]}$$

Basic Properties of Determinants (1)

- Property I

The interchange of rows and columns does not affect the value of a determinant. That is, |A’| = |A|.

Example:

$$|A| = \begin{vmatrix} 3 & 1 \\ 4 & 7 \end{vmatrix} = 17 \qquad |A'| = \begin{vmatrix} 3 & 4 \\ 1 & 7 \end{vmatrix} = 17$$

- Property II

The interchange of any two rows (or columns) will alter the sign, but not the numerical value of the determinant.

Example:

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \qquad |B| = \begin{vmatrix} c & d \\ a & b \end{vmatrix} = bc - ad$$

Basic Properties of Determinants (2)

- Property III

If all element in any one row (or column) is multiplied by a scalar k, the determinant is multiplied by k.

Example:
$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \qquad |B| = \begin{vmatrix} a & b \\ kc & kd \end{vmatrix} = kad - kbc = k|A|$$

- Property IV

The addition (or subtraction) of a multiple of any row to (from) another row will leave the value of the determinant unaltered.

Example:
$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \qquad |B| = \begin{vmatrix} a & b \\ c+ka & d+kb \end{vmatrix} = ad+kab - bc - kab$$
$$= |A|$$

Basic Properties of Determinants (3)

- Property V

If one row (or column) is a multiple of another row (or column), the value of the determinant will be zero.

Example:

$$[A]=\begin{bmatrix} 3 & 4 & 2 \\ 15 & 20 & 10 \\ 4 & 0 & 1 \end{bmatrix}$$

➔ |A| = 0. Thus, this matrix A is linearly dependent.

Note:

The condition of *linear independence* is a sufficient condition for the *nonsingularity* of a matrix. For the rows (or columns) to be linearly independent, none must be a linear combination of the rest.

Determinantal Criterion for Nonsingularity

- Summary:

$|A| \neq 0 \Leftrightarrow$ there is row (column) independence in matrix A

A is nonsingular

A^{-1} exists

a unique solution $X^* = A^{-1}d$ exists

Matrix Inversion by Determinant

- Assume that A is an n x n nonsingular matrix, and |A|≠ 0.
- The inverse of matrix A is:

$$\boxed{A^{-1} = \frac{1}{|A|} adj(A)}$$

where

$$adj(A) \equiv C'_{nxn} \equiv \begin{bmatrix} |C_{11}| & |C_{21}| & \cdots & |C_{n1}| \\ |C_{12}| & |C_{22}| & \cdots & |C_{n2}| \\ \cdots & \cdots & \cdots & \cdots \\ |C_{1n}| & |C_{2n}| & \cdots & |C_{nn}| \end{bmatrix} \text{ and } \quad |C_{ij}| = (-1)^{i+j} \underbrace{\begin{vmatrix} a_{11} & \cdots & a_{1j-1} & a_{1j} & a_{1j+1} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \iddots & \vdots \\ a_{i1} & \cdots & a_{ij-1} & a_{ij} & a_{ij+1} & \cdots & a_{in} \\ \vdots & \iddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nj-1} & a_{nj} & a_{nj+1} & \cdots & a_{nn} \end{vmatrix}}_{|M_{ij}|}$$

Steps to Find the Inverse of a Matrix A

1. Find the determinant |A|. If |A| = 0, then the inverse does not exist.
2. Find the cofactors of all the elements of A, and arrange them as a matrix: $C = [|C_{ij}|]$
3. Take the transpose of the cofactor matrix, *C*, to get the *adj(A)*.
4. Divide the adj(A) by the determinant.

Example:

Find the inverse of $A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & b \\ 0 & 1 & -d \end{bmatrix}$

Answer:

$$A^{-1} = \frac{1}{-(b+d)} \begin{bmatrix} -b & -d & -b \\ d & -d & -b \\ 1 & -1 & 1 \end{bmatrix}$$

Finding Solutions Using Inverse Matrix

- For a problem AX = d, if |A|≠ 0, then X = A^{-1}d.
- Example:

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & b \\ 0 & 1 & -d \end{bmatrix} \qquad X = \begin{bmatrix} Q_d \\ Q_s \\ P \end{bmatrix} \qquad d = \begin{bmatrix} 0 \\ a \\ -c \end{bmatrix}$$

- Solution:

$$X^* = \begin{bmatrix} Q_d^* \\ Q_s^* \\ P^* \end{bmatrix} = A^{-1}d = \frac{1}{-(b+d)} \begin{bmatrix} -b & -d & -b \\ d & -d & -b \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ a \\ -c \end{bmatrix} = \frac{1}{b+d} \begin{bmatrix} ad-bc \\ ad-bc \\ a+c \end{bmatrix}$$

Cramer's Rule

- Given an equation system *Ax = d*, where *A* is an *n x n* nonsingular matrix, the solution value of the *j*th variable can be obtained from:

$$x_j^* = \frac{|A_j|}{|A|} = \frac{1}{|A|}\begin{vmatrix} a_{11} & a_{12} & \cdots & d_1 & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & d_2 & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & d_n & \cdots & a_{nn} \end{vmatrix}$$

where $|A_j|$ is the determinant of the matrix *A* when the *j*th column is replaced by the constant terms $d_1 \ldots d_n$.

Example: Cramer’s Rule

- Given the system of two equations:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}$$

- By Cramer’s rule, the solutions are given by:

$$x^* = \frac{\begin{vmatrix} \boxed{\begin{matrix} u \\ v \end{matrix}} & \begin{matrix} b \\ d \end{matrix} \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{ud - bv}{ad - bc} \quad \text{and} \quad y^* = \frac{\begin{vmatrix} \begin{matrix} a \\ c \end{matrix} & \boxed{\begin{matrix} u \\ v \end{matrix}} \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{av - uc}{ad - bc}$$

Another Example: Market Equilibrium

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & b \\ 0 & 1 & -d \end{bmatrix} \qquad X = \begin{bmatrix} Q_d \\ Q_s \\ P \end{bmatrix} \qquad d = \begin{bmatrix} 0 \\ a \\ -c \end{bmatrix}$$

- Using Cramer's rule, we get:

$$Q^* = \frac{\begin{vmatrix} 0 & -1 & 0 \\ a & 0 & b \\ -c & 1 & -d \end{vmatrix}}{\begin{vmatrix} 1 & -1 & 0 \\ 1 & 0 & b \\ 0 & 1 & -d \end{vmatrix}} = \frac{ad - bc}{b + d} \quad \text{and} \quad P^* = \frac{\begin{vmatrix} 1 & -1 & 0 \\ 1 & 0 & a \\ 0 & 1 & -c \end{vmatrix}}{\begin{vmatrix} 1 & -1 & 0 \\ 1 & 0 & b \\ 0 & 1 & -d \end{vmatrix}} = \frac{a + c}{b + d}$$

MATRIX ALGEBRA: APPLICATIONS

Topics

- Partial and general market equilibrium
- Excise tax and market equilibrium
- Simple macroeconomic model
- IS-LM model
- Input-output model

Partial Market Equilibrium (1)

- Given the system of equations

$$Q_d = 24 - 4P \qquad \text{-- (1)}$$

$$Q_s = -27 + 13P \qquad \text{-- (2)}$$

$$Q_d = Q_s \qquad \text{-- (3)}$$

- Write the system in matrix form *AX* = *d*:

$$A = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -13 \\ 1 & -1 & 0 \end{bmatrix} \qquad X = \begin{bmatrix} Q_d \\ Q_s \\ P \end{bmatrix} \qquad d = \begin{bmatrix} 24 \\ -27 \\ 0 \end{bmatrix}$$

Partial Market Equilibrium (2)

- Using Cramer’s rule to find Q* and P*

➢ Answer: P* = 3, Q* = 12

General Market Equilibrium (1)

- Market for good 1

$$Q_{d1} - Q_{s1} = 0 \qquad (1)$$

$$Q_{d1} = a_0 + a_1 P_1 + a_2 P_2 \qquad (2)$$

$$Q_{s1} = b_0 + b_1 P_1 + b_2 P_2 \qquad (3)$$

- Market for good 2:

$$Q_{d2} - Q_{s2} = 0 \qquad (4)$$

$$Q_{d2} = \alpha_0 + \alpha_1 P_1 + \alpha_2 P_2 \qquad (5)$$

$$Q_{d2} = \beta_0 + \beta_1 P_1 + \beta_2 P_2 \qquad (6)$$

•Write the above system of equations in the matrix form:

Define $c_i \equiv a_i - b_i$ and $\gamma_i \equiv \alpha_i - \beta_i$

$$\begin{bmatrix} c_1 & c_2 \\ \gamma_1 & \gamma_2 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} -c_0 \\ -\gamma_0 \end{bmatrix}$$

General Market Equilibrium (2)

- Find the equilibrium prices and quantities for good 1 and good 2.

Define: $c_i \equiv a_i - b_i$

$$\gamma_i \equiv \alpha_i - \beta_i$$

- Answer:

$$\boxed{P_1^* = \frac{c_2\gamma_0 - c_0\gamma_2}{c_1\gamma_2 - c_2\gamma_1}}$$

$$\boxed{P_2^* = \frac{c_0\gamma_1 - c_1\gamma_0}{c_1\gamma_2 - c_2\gamma_1}}$$

Excise Tax and Market Equilibrium: Specific Tax (1)

- Suppose a specific tax t baht per unit is imposed on consumer.

$$Q_D = Q_S \qquad (1)$$

$$Q_D = a - b(P + T) \qquad (2)$$

$$Q_S = -c + dP \qquad (3)$$

- Write the above system of the equations in the matrix form:

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & b \\ 0 & 1 & -d \end{bmatrix} \qquad X = \begin{bmatrix} Q_d \\ Q_s \\ P \end{bmatrix} \qquad d = \begin{bmatrix} 0 \\ a - bT \\ -c \end{bmatrix}$$

Excise Tax and Market Equilibrium: Specific Tax (2)

- Find the new equilibrium prices and quantity.
- Answer:

$$\boxed{P^* = \frac{a+c+dT}{b+d} = P_D^{'}}$$

$$\boxed{P^T = P_D^{'} - T = \frac{a+c-bT}{b+d} = P_S^{'}}$$

$$\boxed{Q^* = \frac{ad-bc-bdT}{b+d}}$$

Excise Tax and Market Equilibrium: Ad Varolem Tax (1)

- Suppose an *t%* tax is imposed on consumer.

$$Q_D = Q_S \qquad (1)$$

$$Q_D = a - bP^T \qquad (2)$$

$$Q_S = -c + dP \qquad (3)$$

$$P^T = (1+t)P \qquad (4)$$

- Write the above system of the equations in the matrix form:

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & b \\ 0 & 1 & -d & 0 \\ 0 & 0 & -(1+t) & 1 \end{bmatrix} \qquad X = \begin{bmatrix} Q_d \\ Q_s \\ P \\ P^T \end{bmatrix} \qquad d = \begin{bmatrix} 0 \\ a \\ -c \\ 0 \end{bmatrix}$$

Excise Tax and Market Equilibrium: Ad Varolem Tax (2)

- Find the new equilibrium prices and quantity.
- Answer:

$$\boxed{P* = \frac{a+c}{b(1+t)+d} = P_S^{'}}$$

$$\boxed{P^t = (1+t)P_s^{'} = \frac{(1+t)(a+c)}{b(1+t)+d} = P_D^{'}}$$

$$\boxed{Q* = \frac{ad-(1+t)bc}{b(1+t)+d}}$$

Simple Macroeconomic Model (1)

- Keynesian model:

$$Y = C + I_0 + G_0 + X_0 - M$$

$$C = a + bY_d$$

$$M = mY_d$$

$$Y_d = Y - T$$

$$T = tY$$

- Reduce the above system into:

Simple Macroeconomic Model (2)

- Write the above system in the matrix form, and find the equilibrium national income.

$$\begin{bmatrix} 1+m(1-t) & -1 \\ -b(1-t) & 1 \end{bmatrix} \begin{bmatrix} Y \\ C \end{bmatrix} = \begin{bmatrix} I_0 + G_0 + X_0 \\ a \end{bmatrix}$$

- Solutions:

$$\boxed{Y^* = \frac{a + I_0 + G_0 + X_0}{1 - b(1-t) + m(1-t)}}$$

$$\boxed{C^* = \frac{a[1+m(1-t)] + b(1-t)(I_0 + G_0 + X_0)}{1 - b(1-t) + m(1-t)}}$$

IS-LM Model (1)

- Commodity Market:

$Y = C + I + G_0$

$C = a + bY_d \qquad (a > 0,\ 0 < b < 1)$

$I = I_0 - ir \qquad (I_0,\ i > 0)$

$Y_d = Y - T$ where $T = tY \qquad (0 < t < 1)$

➔ $Y = a + b(1-t)Y + I_0 - ir + G_0$: IS

- Money Market:

$M^S = M_0$

$M^D = kY - hr \qquad (k, h > 0)$

➔ $M_0 = kY - hr$: LM

- Write a matrix form of the above IS-LM equations:

IS-LM Model (2)

- Use Cramer's rule to find the equilibrium national income and interest rate.

$$\begin{bmatrix} 1-b(1-t) & i \\ k & -h \end{bmatrix}\begin{bmatrix} Y \\ r \end{bmatrix} = \begin{bmatrix} a+I_0+G_0 \\ M_0 \end{bmatrix}$$

- Solutions:

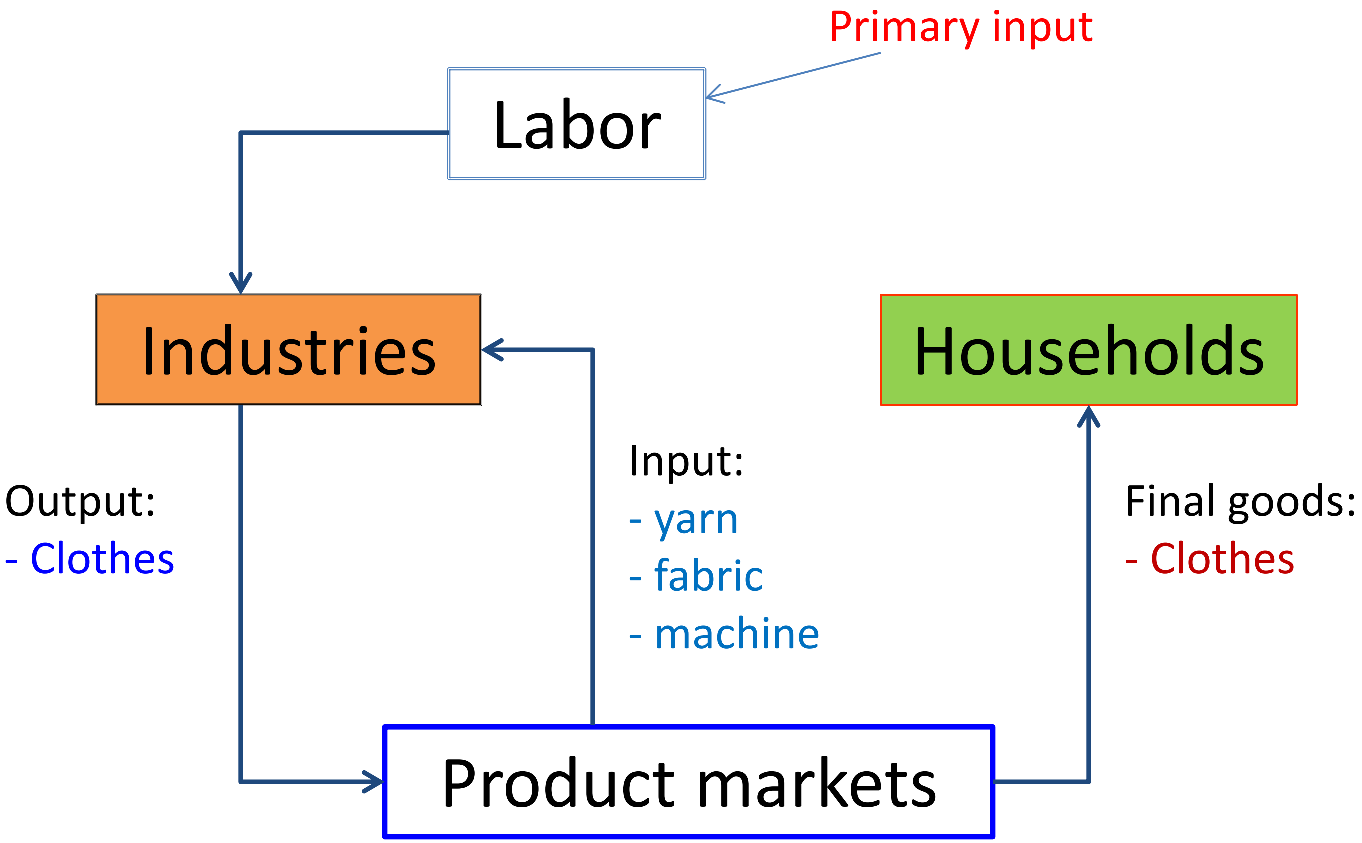
$$\boxed{Y^* = \frac{(a+I_0+G_0)h+iM_0}{ik+h[1-b(1-t)]}}$$

$$\boxed{r^* = \frac{(a+I_0+G_0)k-[1-b(1-t)]M_0}{ik+h[1-b(1-t)]}}$$

Leontif Input-Output Model (1)

- This model characterizes input-output relationships among *n* interlinked industries in an economy.
- It seeks to determine the output level that each of the *n* industries should produce to meet the total demand for that product.
- Applications: Use in production planning, e.g. planning for economic development of a country
- Assumptions:
  - Each industry produces only one homogenous commodity.
  - Each industry uses a fixed input ratio for the production of its output.
  - Production in every industry is subject to constant returns to scale.

Leontif Input-Output *Open* Model (2)



Primary input

Labor

Industries

Households

Output:
- Clothes

Input:
- yarn
- fabric
- machine

Final goods:
- Clothes

Product markets

Leontif Input-Output Model (3)

- Consider an economy with *n* industries.
  - All the outputs are produced need to meet (i) the input demand of the same *n* industries and (ii) the final demand (e.g. consumer demand)
  - 2 types of inputs: (i) intermediate inputs and (ii) primary inputs (e.g. labor)
- Notations:
  - x_i = total number of units of good i that industry i is going to produce
  - a_{ij} = the number of units of good i needed to produce <u>one</u> unit of good j (a_{ij} = input coefficient)
  - $a_{ij}x_j$ = the number of units of good i needed to produce x_j unit of good j
  - d_i = final demand for good i

Leontif Input-Output Model (4)

- Equilibrium between supply and demand for each good *i*:

$$x_i = a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n + d_i$$

- The system of equations for *n* industries:

$$\begin{aligned} x_1 &= a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n + d_1 \\ x_2 &= a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n + d_2 \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ x_n &= a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n + d_n \end{aligned}$$

- Input-coefficient matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Leontif Input-Output Model (4)

- Rearranging the above system of equations:

$$\begin{array}{llll}
(1 - a_{11})x_1 - & a_{12}x_2 - \ldots - & a_{1n}x_n & = d_1 \\
- a_{21}x_1 & +(1- a_{22})x_2 - \ldots - & a_{2n}x_n & = d_2 \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & & & \\
- a_{n1}x_1 - & a_{n2}x_2 - \ldots & +(1- a_{nn})x_n & = d_n
\end{array}$$

- Leontif model in matrix form:

$$\underbrace{\begin{bmatrix} (1-a_{11}) & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & (1-a_{22}) & \cdots & -a_{2n} \\ \vdots & \vdots & & \vdots \\ -a_{n1} & -a_{n2} & \cdots & (1-a_{nn}) \end{bmatrix}}_{\text{Leontif matrix}} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix} \quad \text{or} \quad (I - A)x = d$$

- Given that $(I - A)^{-1}$ exists, the unique solution of the system is:

$$x^* = (I - A)^{-1}d$$

Leontif Input-Output Model (5)

- A numerical example:

Industry (i)	Input Demand for Output *j*			Final Demand ($)
	Agriculture	Manufacture	Service	
Agriculture x_1	0.2	0.3	0.2	10
Manufacture x_2	0.4	0.1	0.2	5
Service x_3	0.1	0.3	0.2	6

- Questions:
  - What level of output should each industry produce?
  - Find the input coefficients of the primary input (labor) for each output, and determine the required amount of the primary input needed in this economy.

Leontif Input-Output Model (6)

- Write the input-coefficient matrix:

$$A = \begin{bmatrix} 0.2 & 0.3 & 0.2 \\ 0.4 & 0.1 & 0.2 \\ 0.1 & 0.3 & 0.2 \end{bmatrix}$$

- The input-output system in the matrix form *(I-A)x = d*:

$$\begin{bmatrix} 0.8 & -0.3 & -0.2 \\ -0.4 & 0.9 & -0.2 \\ -0.1 & -0.3 & 0.8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \\ 6 \end{bmatrix}$$

Leontif Input-Output Model (7)

- Given that $\begin{bmatrix} 0.8 & -0.3 & -0.2 \\ -0.4 & 0.9 & -0.2 \\ -0.1 & -0.3 & 0.8 \end{bmatrix}^{-1} = \dfrac{1}{0.38}\begin{bmatrix} 0.66 & 0.3 & 0.24 \\ 0.34 & 0.62 & 0.24 \\ 0.21 & 0.27 & 0.6 \end{bmatrix}$

- The solutions for the output levels to be produced are:

$$\begin{bmatrix} x_1^* \\ x_2^* \\ x_2^* \end{bmatrix} = \frac{1}{0.384}\begin{bmatrix} 0.66 & 0.3 & 0.24 \\ 0.34 & 0.62 & 0.24 \\ 0.21 & 0.27 & 0.6 \end{bmatrix}\begin{bmatrix} 10 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 24.84 \\ 20.68 \\ 18.36 \end{bmatrix}$$

- The level of primary input (labor) required for this economy:

$a_{01} = 0.3$; $a_{02} = 0.3$; $a_{03} = 0.4$

➔ $a_0 = 0.3(24.84) + 0.3(20.68) + 0.4(18.36) = 21$