Generalized Method of Moment (GMM)

Motivation

Limitation of OLS

- Nonlinearity of the function– First moment violation.
- Nonnormal distribution.

Limitation of MLE

- Sensitivity of statistical properties to the distributional assumption – Independent or jointed distribution, correctly specified of the distribution of the model.
- Computational burden.

Generalized Method of Moment (GMM)

Motivation

Limitation of MLE

ML has asymptotically optimal properties for correctly specified models.

Thus, the joint probability distribution of the data should reflect the actual data generating process.

Assuming less assumptions than ML might be better in case of incorrectly specified models in ML.

Generalized Method of Moment (GMM)

Least Squares derived by method of moments

Basic requirement for OLS is orthogonality condition. $E[x_i(y_i - x_i'\beta)] = 0, \qquad i = 1, 2, \ldots, n.$

x_i is a k x 1 vector.

The conditions on sample moments can be stated as:

$$\frac{1}{n}\sum_{i=1}^{n} x_i(y_i - x_i'\hat{\beta}) = 0$$

OLS can be derived by method of moments, using the orthogonality conditions as k-moment conditions.

Generalized Method of Moment (GMM)

Least Squares derived by method of moments

<u>Example</u>: $y_t = \beta_1 + \beta_2 x_{2t} + \beta_3 x_{3t} + \cdots + \beta_k x_{kt} + \varepsilon_t$

k-moment conditions for OLS include:

1. Zero mean condition $E[\varepsilon_t] = 0$
2. Exogeneity condition of x_{2t} $E[x_{2t}\varepsilon_t] = 0$
3. Exogeneity condition of x_{3t} $E[x_{3t}\varepsilon_t] = 0$

⋮ ⋮

k. Exogeneity condition of x_{kt} $E[x_{kt}\varepsilon_t] = 0$

Generalized Method of Moment (GMM)

ML as method of moments estimator

The first-order conditions for MLE is to maximize log-likelihood function.

$$E\left[\frac{\partial l_i}{\partial \theta}\right]_{|\theta=\theta_0} = 0, \qquad i = 1, 2, \ldots, n.$$

The conditions on sample moments can be stated as:

$$\frac{1}{n}\sum_{i=1}^{n}\frac{\partial l_i}{\partial \theta} = 0$$

ML can be derived by method of moments, using the first-order conditions for maximum of log-likelihood as k-moment conditions.

Generalized Method of Moment (GMM)

Method of moments estimator of the mean

Estimate population moments by means of sample moments. $E[y_i - \mu] = 0$

Sample mean: $$\frac{1}{n}\sum_{i=1}^{n}(y_i - \bar{y}) = 0$$

Then, $$\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$$

Mean – 1st moment condition
Variance – 2nd moment condition
Skew – 3rd moment condition
Kurtosis – 4th moment condition

Generalized Method of Moment (GMM)

Instead of using moments based on powers of x, we can use other functions.

Let $g_k()$ be any continuous function not involving the sample size n, and

$$\bar{g}_k = \frac{1}{n}\sum_{i=1}^{n} g_k(x_i), \qquad k = 1, 2, \ldots, K.$$

$$p\lim \bar{g}_k = E\left[g_k(x)\right] = \gamma_k(\theta_1, \ldots, \theta_K)$$

Generalized Method of Moment (GMM)

Assume that $g_k()$ involves some of parameters distribution.

With K parameters to be estimated, the K moment equations:

$$\begin{aligned}
&\bar{g}_1 - \gamma_1(\theta_1, \ldots, \theta_K) = 0 \\
&\bar{g}_2 - \gamma_2(\theta_1, \ldots, \theta_K) = 0 \\
&\vdots \\
&\bar{g}_K - \gamma_K(\theta_1, \ldots, \theta_K) = 0
\end{aligned}$$

Generalized Method of Moment (GMM)

With K equations and K unknown parameters, if they are independent, then, Method of Moments Estimators can be obtained by solving the system equations for

$$\hat{\theta}_k = \hat{\theta}_k \left[\bar{g}_1, \ldots, \bar{g}_K \right]$$

Example

x
20.50
31.50
47.70
26.20
44.00
8.28
30.80
17.20
19.90
9.96
55.80
25.20
29.00
85.50
15.10
28.50
21.40
17.70
6.42
84.90

Assume x is Gamma distributed with the functional form:

$$f(x) = \frac{\lambda^P}{\Gamma(P)} e^{-\lambda x} x^{P-1}, \qquad x \geq 0, P > 0, \lambda > 0$$

In estimating parameters λ and P, we can use other functions as information functions, such as: x^2, $ln\ x$, or $1/x$.

$$p\lim \frac{1}{n}\sum_{i=1}^{n} x_i = \frac{P}{\lambda} \qquad (1)$$

$$p\lim \frac{1}{n}\sum_{i=1}^{n} x_i^2 = \frac{P(P+1)}{\lambda^2} \qquad (2)$$

$$p\lim \frac{1}{n}\sum_{i=1}^{n} \ln x_i = \frac{d \ln \Gamma(P)}{dP} - \ln \lambda \qquad (3)$$

$$p\lim \frac{1}{n}\sum_{i=1}^{n} \frac{1}{x_i} = \frac{\lambda}{P-1} \qquad (4)$$

Example (cont.)

x
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8.28
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28.50
21.40
17.70
6.42
84.90

Values for the four moment functions are:

$$\frac{1}{n}\sum_{i=1}^{n}\left[x_i, x_i^2, \ln x_i, \frac{1}{x_i}\right] = [31.278,\ 1453.957,\ 3.221,\ 0.05]$$

$$= (m_1', m_2', m_*', m_{-1}')$$

Estimated parameters $\theta = (P, \lambda)$ based on any pairs of these moment functions are as follows:

$$\hat{\theta}(m_1', m_2') = (2.056,\ 0.066), \quad \hat{\theta}(m_1', m_*') = (2.41,\ 0.077)$$

$$\hat{\theta}(m_1', m_{-1}') = (2.772,\ 0.088), \quad \hat{\theta}(m_2', m_*') = (2.26,\ 0.071)$$

$$\hat{\theta}(m_{-1}', m_2') = (2.608,\ 0.08), \quad \hat{\theta}(m_{-1}', m_*') = (3.036,\ 0.102)$$

Example (cont.)

x
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6.42
84.90

If we use moment 1 & 2:

$$m_1' = p\lim \frac{1}{n}\sum_{i=1}^{n} x_i = \frac{P}{\lambda} = 31.278 \qquad (1)$$

$$m_2' = p\lim \frac{1}{n}\sum_{i=1}^{n} x_i^2 = \frac{P(P+1)}{\lambda^2} = 1453.957 \qquad (2)$$

There are two equations and two unknown (P,λ), and solve for solution, then, the estimated result will be

$$\lambda = \frac{P}{31.278} \quad \rightarrow \quad \lambda^2 = \left(\frac{P}{31.278}\right)^2 \qquad (1)$$

$$\lambda^2 = \frac{P(P+1)}{1453.957} \qquad (2)$$

$$\frac{P(P+1)}{1453.957} = \left(\frac{P}{31.278}\right)^2 \quad \rightarrow \quad \hat{P} = 2.056$$

then $\hat{\theta}\left(m_1', m_2'\right) = (2.056, \quad 0.066)$

Example (cont.)

X
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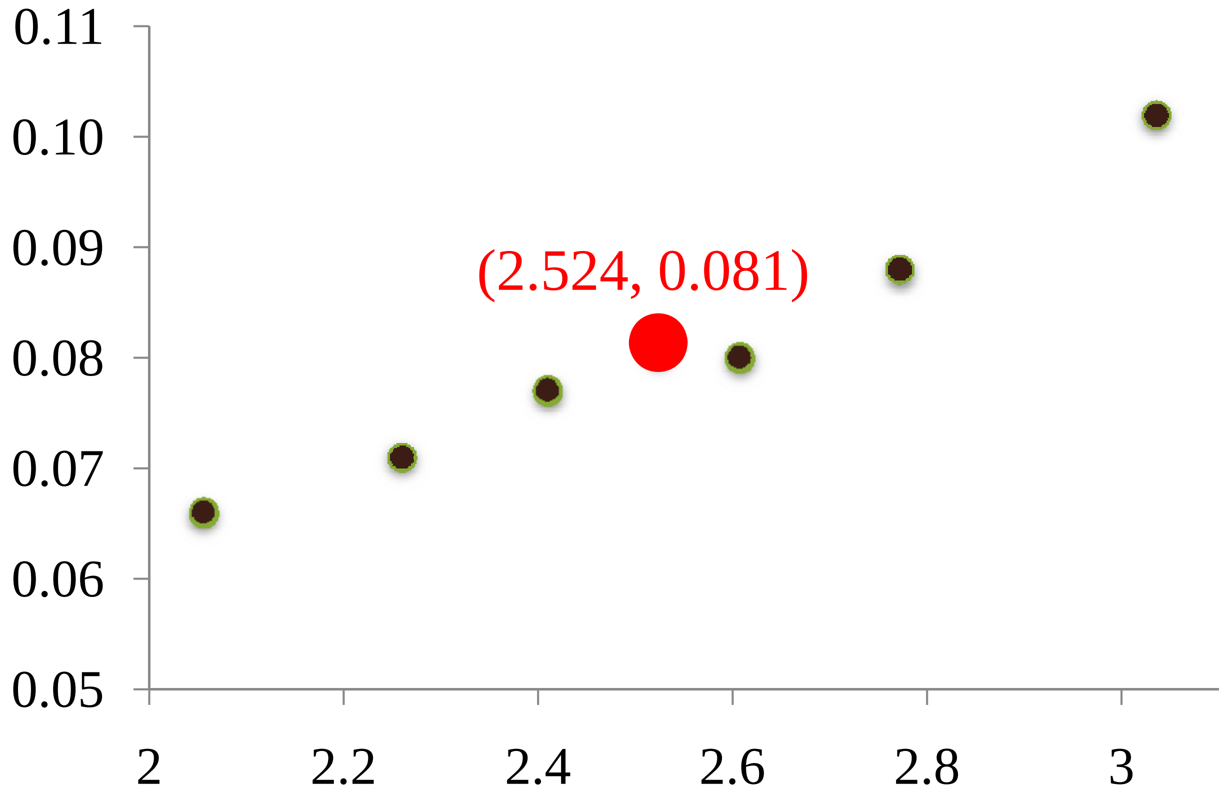
15.10
28.50
21.40
17.70
6.42
84.90

$$\hat{\theta}(m'_1, m'_2) = (2.056,\ 0.066), \qquad \hat{\theta}(m'_1, m'_*) = (2.41,\ 0.077)$$

$$\hat{\theta}(m'_1, m'_{-1}) = (2.772,\ 0.088), \qquad \hat{\theta}(m'_2, m'_*) = (2.26,\ 0.071)$$

$$\hat{\theta}(m'_{-1}, m'_2) = (2.608,\ 0.08), \qquad \hat{\theta}(m'_{-1}, m'_*) = (3.036,\ 0.102)$$



Generalized Method of Moment (GMM)

Properties of Method of Moment Estimators

MOM employs theory or a priori information to yield an assertion about a population orthogonality condition. $E\left[f(x,\theta)\right]=0$

In the simplest cases, MOM is robust to differences in the specification of data generating process.

There is no need of distribution assumptions.

However, if there are more information about data generating process, MOM may not use all information and thus turn out to be inefficient.

Generalized Method of Moment (GMM)

Exactly Identified

In case of exactly identified (m moment equations and p unknown parameters), we can solve for unique or single solution of the MOM estimator.

$$E[g_i(\theta_0)] = 0 \qquad\qquad i = 1, \cdots, n$$

MOM estimators can be computed by solving:

$$\frac{1}{n}\sum_{i=1}^{n} g_i(\hat{\theta}) = 0$$

Generalized Method of Moment (GMM)

<u>Overidentified</u>

There are cases in which there are more moment equations than parameters, thus, the system is overidentified.

Suppose the model involves p parameters and a set of $m>p$ moment equations.

Let the mx1 vector $G_n(\theta)$ be defined by

$$G_n(\theta) = \sum_{i=1}^{n} g_i(\theta_0)$$

GMM estimators can be solved by minimize:

$$\frac{1}{n} G_n' W G_n$$

Generalized Method of Moment (GMM)

Example

The model: $r_{\Delta t} = r_{t+\Delta t} - r_t = (\alpha + \beta r_t)\Delta t + \varepsilon_{t+\Delta t}$

where: $E\left[\varepsilon_{t+\Delta t}\right] = 0$ and $E\left[\varepsilon_{t+\Delta t}^2\right] = \sigma^2 r_t^{2\gamma}\Delta t$

The disturbance terms can be stated as:

$$\varepsilon_{t+1} = r_{t+1} - r_t - \alpha - \beta r_t$$

Then, the moment conditions can be stated as:

$$G(\theta) = \begin{bmatrix} E(\varepsilon_{t+1}) = 0 \\ E(\varepsilon_{t+1} r_t) = 0 \\ E(\varepsilon_{t+1}^2 - \sigma^2 r_t^{2\gamma}) = 0 \\ E\left(\left(\varepsilon_{t+1}^2 - \sigma^2 r_t^{2\gamma}\right) r_t\right) = 0 \end{bmatrix}$$

Generalized Method of Moment (GMM)

Example

The models can be divided into:

Model	α	β	σ^2	γ
1. Unrestricted model				
2. Merton		0		0
3. Vasicek				0
4. CIR SR				0.5
5. Dothan	0	0		1
6. GBM	0			1
7. Brennan & Schwartz				1
8. CIR VR	0	0		1.5
9. CEV	0			

Generalized Method of Moment (GMM)

<u>GMM Estimation</u>

<u>Step 1</u>: Specify a sufficient number of moment condition.

<u>Step 2</u>: Estimate the parameters

$\underline{m=p}$: solve for unique solution

$\underline{m>p}$: minimize distance among solutions by choosing the appropriated weighted matrix

Generalized Method of Moment (GMM)

GMM Standard Errors

Asymptotic Result

Assume that the moment conditions are valid for DGP, GMM estimator is consistent and sample average $\frac{1}{n}G_n = \frac{1}{n}\sum_{i=1}^{n} g_i$ satisfied central limit theorem.

$$\frac{1}{\sqrt{n}}G_n(\theta_0) \xrightarrow{d} N(0, J_0)$$

$$J_0 = E[g_i(\theta_0)g_i'(\theta_0)]$$

Generalized Method of Moment (GMM)

GMM Standard Errors

Asymptotic Distribution

Assume asymptotic normality:

$$G_n = G_n(\theta) \approx G_n(\theta_0) + H_{n0}(\theta - \theta_0)$$

First order condition for GMM

$$H'_{n0} W (G_{n0} + H_{n0}(\theta - \theta_0)) = 0$$

Solution for GMM $\hat{\theta} = \theta_0 - (H'_{n0} W H_{n0})^{-1} H'_{n0} W G_{n0}$

Assume: $\operatorname{plim}\left(\frac{1}{n} H_{n0}\right) = H_0$

Then, $\sqrt{n}(\hat{\theta} - \theta_0) \xrightarrow{d} N(0, V)$

where: $V = (H'_0 W H_0)^{-1} H'_0 W J_0 W H_0 (H'_0 W H_0)^{-1}$

Generalized Method of Moment (GMM)

<u>Test of moment conditions: The J-test</u>

Overidentifying Restrictions Test: To test whether the sample moment equations are appropriated. Under null hypothesis that exactly identified moment conditions hold true, the J-test is

$$G_n' J_n^{-1} G_n \approx \chi^2 (m - p)$$

Degree of freedom is *m*-*p*.

In exactly identified case, there are zero degrees of freedom.

Generalized Method of Moment (GMM)

GMM Estimation and Testing

Step 1: Specify a sufficient number of moment condition.

Step 2: Estimate the parameters

Step 3: Compute GMM standard errors

Step 4: Test of moment conditions (in overidentified models)

Quasi-Maximum Likelihood (QML)

Quasi-Maximum Likelihood

Moment conditions derived from a postulated likelihood

Instead of using moment condition based on models of economic behavior, QML method derives the moment conditions from a postulated likelihood function.

$$E\left[g_i(\theta)\right] = E\left[\partial l_i / \partial\theta\right] = 0$$

Quasi-Maximum Likelihood (QML)

<u>Quasi-Maximum Likelihood</u>

<u>Step 1</u>: Specify a probability distribution for the observed data.

<u>Step 2</u>: Derive the corresponding moment conditions.

<u>Step 3</u>: Estimate the parameters.

<u>Step 4</u>: Compute the GMM standard errors.