Linear Transformations

Another way to view $A\mathbf{x} = \mathbf{b}$:

Matrix A is an object acting on $\mathbf{x}$ by multiplication to produce a new vector $A\mathbf{x}$ or $\mathbf{b}$.

EXAMPLE:

$$\underset{A}{\begin{bmatrix} 2 & -4 \\ 3 & -6 \\ 1 & -2 \end{bmatrix}} \underset{x}{\begin{bmatrix} 2 \\ 3 \end{bmatrix}} = \underset{b}{\begin{bmatrix} -8 \\ -12 \\ -4 \end{bmatrix}}$$

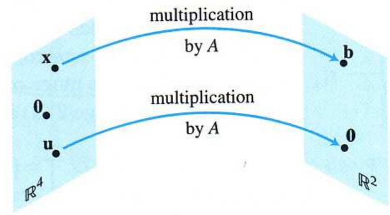


FIGURE 1 Transforming vectors via matrix multiplication.

$$\underset{A}{\begin{bmatrix} 2 & -4 \\ 3 & -6 \\ 1 & -2 \end{bmatrix}} \underset{u}{\begin{bmatrix} 2 \\ 1 \end{bmatrix}} = \underset{0}{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}$$

The correspondence from x to Ax is a function from one set of vectors to another.

Suppose A is $m \times n$. Solving $A\mathbf{x} = \mathbf{b}$ amounts to finding all ____ in $\mathbf{R}^n$ which are transformed into vector $\mathbf{b}$ in $\mathbf{R}^m$ through multiplication by A.

Matrix transformations

Examples

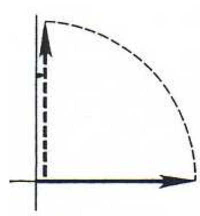
1. A multiple of the identity matrix, $A = cI$,

$$A = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}$$

stretching

2. **A rotation matrix** turns the whole space around the origin.
e.g. A turns all vectors through 90°

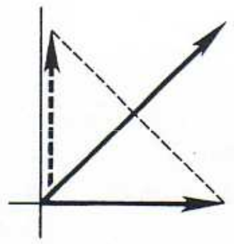
$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$



90° rotation

3. **A reflection matrix** transforms every vector into its image on the opposite side of a mirror.

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$


reflection

4. **A projection matrix** takes the whole space onto a lower dimensional subspace.
e.g.

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$


projection

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A **transformation** T from $\mathbf{R}^n$ to $\mathbf{R}^m$ is a rule that assigns to each vector $\mathbf{x}$ in $\mathbf{R}^n$ a vector $T(\mathbf{x})$ in $\mathbf{R}^m$.

Terminology:

$\mathbf{R}^n$: **domain** of T $\mathbf{R}^m$: **codomain** of T

$T(\mathbf{x})$ in $\mathbf{R}^m$ is the **image** of $\mathbf{x}$ under the transformation T

Set of all images $T(\mathbf{x})$ is the **range** of T

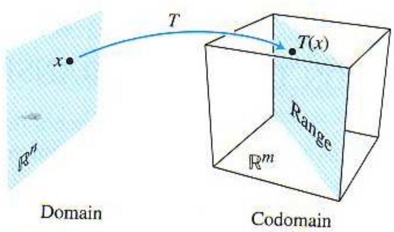


FIGURE 2 Domain, codomain, and range of $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$.

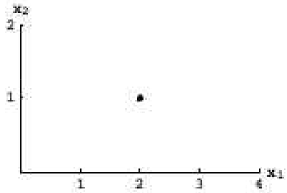
T(x) is computed as *Ax* (multiply x by *A* to transform *x*→*A*

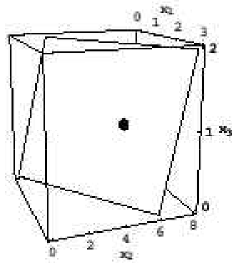
The range of *T* is the set of all linear combinations of the columns of *A*, because each image of *T(x)* is of the form *Ax*.

EXAMPLE: Let $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \end{bmatrix}$. Define a transformation $T : \mathbf{R}^2 \to \mathbf{R}^3$ by $T(\mathbf{x}) = A\mathbf{x}$.

Then if $\mathbf{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$,

$$T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$$





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EXAMPLE: Let $A = \begin{bmatrix} 1 & -2 & 3 \\ -5 & 10 & -15 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 2 \\ -10 \end{bmatrix}$ and $\mathbf{c} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$. Then define a transformation $T : \mathbf{R}^3 \to \mathbf{R}^2$ by $T(\mathbf{x}) = A\mathbf{x}$.

a. Find an **x** in $\mathbf{R}^3$ whose image under T is **b**.
b. Is there more than one **x** under T whose image is **b**. *(uniqueness problem)*
c. Determine if **c** is in the range of the transformation T. *(existence problem)*

Solution: (a) Solve ________=______ for **x**.

I.e., solve ________=______ or

$$\begin{bmatrix} 1 & -2 & 3 \\ -5 & 10 & -15 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -10 \end{bmatrix}$$

Augmented matrix:

$$\begin{bmatrix} 1 & -2 & 3 & 2 \\ -5 & 10 & -15 & -10 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 = 2x_2 - 3x_3 + 2$

x_2 is free

x_3 is free

Let $x_2 =$ _____ and $x_3 =$ _____. Then $x_1 =$ _____.

So $\mathbf{x} = \begin{bmatrix} \quad \end{bmatrix}$

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(b) Is there more than one **x** whose image under T is **b** ?
~ Is there an **x** for which $T(\mathbf{x})=\mathbf{b}$?

Free variables exist

⇓

There is more than one **x** for which $T(\mathbf{x}) = \mathbf{b}$

A uniqueness problem for a system of linear equations, translated here into the language of matrix transformations
: Is **b** the image of a unique **x** in R^n

(c) Is there an **x** for which $T(\mathbf{x}) = \mathbf{c}$? This is another way of asking if $A\mathbf{x} = \mathbf{c}$ is _______________.

Augmented matrix:

$$\begin{bmatrix} 1 & -2 & 3 & 3 \\ -5 & 10 & -15 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

c is not in the _______________ of T.

An existence problem:
Does there exist an **x** whose image is **c**?

Linear transformations

If A is $m \times n$, then the transformation $T(\mathbf{x}) = A\mathbf{x}$ has the following properties:

$$T(\mathbf{u}+\mathbf{v}) = A(\mathbf{u}+\mathbf{v}) = \_\_\_\_\_\_ + \_\_\_\_\_\_$$

$$= \_\_\_\_\_\_ + \_\_\_\_\_\_$$

and

$$T(c\mathbf{u}) = A(c\mathbf{u}) = \_\_\_\_\_ A\mathbf{u} = \_\_\_\_\_ T(\mathbf{u})$$

for all $\mathbf{u},\mathbf{v}$ in $\mathbf{R}^n$ and all scalars c.

DEFINITION

A transformation T is **linear** if:

i. $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u},\mathbf{v}$ in the domain of T.

ii. $T(c\mathbf{u}) = cT(\mathbf{u})$ for all $\mathbf{u}$ in the domain of T and all scalars c.

Linear transformations preserve the operations of vector addition and scalar multiplication

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Every matrix transformation is a **linear** transformation.

Useful fact If T is a linear transformation, then

$$T(\mathbf{0}) = \mathbf{0} \quad \text{and}$$

$$T(c\mathbf{u}+d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v}).$$

Proof:

$$T(\mathbf{0}) = T(0\mathbf{u}) = \_\_\_\_ T(\mathbf{u}) = \_\_\_\_\_.$$

$$T(c\mathbf{u}+d\mathbf{v}) = T(\quad) + T(\quad)$$

$$= \_\_\_\_ T(\quad) + \_\_\_\_ T(\quad)$$

If a transformation satisfies $T(c\mathbf{u}+d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$ for all u, v and c, d it must be linear.

$$\boxed{T(c_1\mathbf{V}_1 + c_2\mathbf{V}_2 + \ldots\ldots c_p\mathbf{V}_p) = c_1T(\mathbf{V}_1) + \ldots\ldots + c_pT(\mathbf{V}_p)}$$

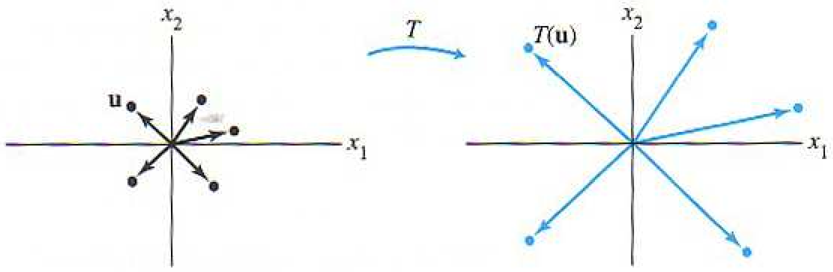
Example

Given a scalar r, define T: $R^2 \rightarrow R^2$ by $T(x)=rx$. T is called a contraction when $0 \le r \le 1$ and a dilation when $r>1$. Let $r=3$, and show that T is a linear transformation.

Solution Let **u**, **v** be in $\mathbb{R}^2$ and let c, d be scalars. Then

$$\begin{aligned} T(c\mathbf{u} + d\mathbf{v}) &= 3(c\mathbf{u} + d\mathbf{v}) && \text{Definition of } T \\ &= 3c\mathbf{u} + 3d\mathbf{v} && \\ &= c(3\mathbf{u}) + d(3\mathbf{v}) && \text{Vector arithmetic} \\ &= cT(\mathbf{u}) + dT(\mathbf{v}) \end{aligned}$$

Thus T is a linear transformation because it satisfies (4). See Fig. 5.



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EXAMPLE 5 Define a linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by

$$T(\mathbf{x}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}$$

Find the images under T of $\mathbf{u} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, and $\mathbf{u} + \mathbf{v} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$.

Solution

$$T(\mathbf{u}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}, \qquad T(\mathbf{v}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix},$$

$$T(\mathbf{u} + \mathbf{v}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 6 \\ 4 \end{bmatrix} = \begin{bmatrix} -4 \\ 6 \end{bmatrix}$$

Note that $T(\mathbf{u} + \mathbf{v})$ is obviously equal to $T(\mathbf{u}) + T(\mathbf{v})$. It appears from Fig. 6 that T rotates **u**, **v**, and **u** + **v** counterclockwise about the origin through 90°. In fact, T transforms the entire parallelogram determined by **u** and **v** into the one determined by $T(\mathbf{u})$ and $T(\mathbf{v})$. (See Exercise 28.)

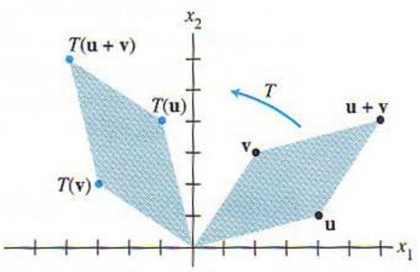


FIGURE 6 A rotation transformation.

- T rotates u, v, u+v anticlockwise about the origin through 90 degree.
- T transforms the entire parallelogram determined by u and v into the one determined by T(u) and T(v)

EXAMPLE: Let $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\mathbf{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ and

$\mathbf{y}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. Suppose $T : \mathbf{R}^2 \to \mathbf{R}^3$ is a linear transformation

which maps $\mathbf{e}_1$ into $\mathbf{y}_1$ and $\mathbf{e}_2$ into $\mathbf{y}_2$. Find the images of

$\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Solution: First, note that

$T(\mathbf{e}_1) =$ ______ and $T(\mathbf{e}_2) =$ ______.

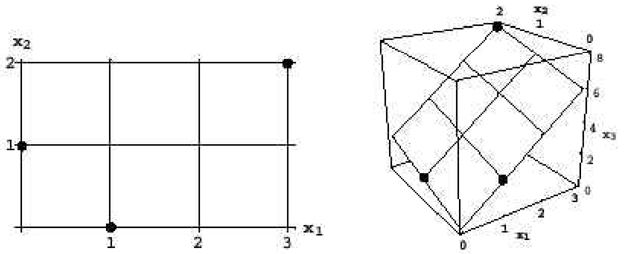
Also

$$\_\_\mathbf{e}_1 + \_\_\mathbf{e}_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Then

$$T\left(\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right) = T(\_\_\mathbf{e}_1 + \_\_\mathbf{e}_2) =$$

$$\_\_T(\mathbf{e}_1) + \_\_T(\mathbf{e}_2) =$$



$T(3\mathbf{e}_1 + 2\mathbf{e}_2) = 3T(\mathbf{e}_1) + 2T(\mathbf{e}_2)$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = T(\_\_\_\_\mathbf{e}_1 + \_\_\_\_\mathbf{e}_2) =$$

$$\_\_\_\_T(\mathbf{e}_1) + \_\_\_\_T(\mathbf{e}_2) =$$

EXAMPLE: Define $T : \mathbf{R}^3 \to \mathbf{R}^2$ such that $T(x_1, x_2, x_3) = (|x_1 + x_3|, 2 + 5x_2)$. Show that T is a not a linear transformation.

Solution: Another way to write the transformation:

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} |x_1 + x_3| \\ 2 + 5x_2 \end{bmatrix}$$

Provide a **counterexample** - example where $T(\mathbf{0}) = \mathbf{0}$, $T(c\mathbf{u}) = c\mathbf{T}(\mathbf{u})$ or $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ is violated.

A counterexample:

$$T(\mathbf{0}) = T\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} \quad \end{bmatrix} \neq \_\_\_\_$$

which means that T is not linear.

Another counterexample: Let $c = -1$ and $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Then

$$T(c\mathbf{u}) = T\left(\begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} |-1 + -1| \\ 2 + 5(-1) \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

and

$$cT(\mathbf{u}) = -1T\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = -1\begin{bmatrix} \quad \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}.$$

Therefore $T(c\mathbf{u}) \neq \_\_\_ T(\mathbf{u})$ and therefore T is not __________________.

The matrix of a linear transformation

Identity Matrix I_n is an $n \times n$ matrix with 1's on the main left to right diagonal and 0's elsewhere. The ith column of I_n is labeled $\mathbf{e}_i$.

EXAMPLE:

$$I_3 = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note that

$$I_3\mathbf{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \_\_\_\begin{bmatrix} \\ \\ \\ \end{bmatrix} + \_\_\_\begin{bmatrix} \\ \\ \\ \end{bmatrix} + \_\_\_\begin{bmatrix} \\ \\ \\ \end{bmatrix} = \_\_\_.$$

In general, for $\mathbf{x}$ in $\mathbf{R}^n$,

$$I_n\mathbf{x} = \_\_\_$$

if $T: \mathbf{R}^n \to \mathbf{R}^m$ is a linear transformation, then $T(c\mathbf{u}+d\mathbf{v})=cT(\mathbf{u})+dT(\mathbf{v})$.

Generalized Result:

$$T(c_1\mathbf{v}_1+\cdots+c_p\mathbf{v}_p)=c_1T(\mathbf{v}_1)+\cdots+c_pT(\mathbf{v}_p).$$

EXAMPLE: The columns of $I_2=\begin{bmatrix}1&0\\0&1\end{bmatrix}$ are $\mathbf{e}_1=\begin{bmatrix}1\\0\end{bmatrix}$ and $\mathbf{e}_2=\begin{bmatrix}0\\1\end{bmatrix}$. Suppose T is a linear transformation from $\mathbf{R}^2$ to $\mathbf{R}^3$ where

$$T(\mathbf{e}_1)=\begin{bmatrix}2\\-3\\4\end{bmatrix} \text{ and } T(\mathbf{e}_2)=\begin{bmatrix}5\\0\\1\end{bmatrix}.$$

Compute $T(\mathbf{x})$ for any $\mathbf{x}=\begin{bmatrix}x_1\\x_2\end{bmatrix}$.

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Solution: A vector $\mathbf{x}$ in $\mathbf{R}^2$ can be written as

$$\begin{bmatrix}x_1\\x_2\end{bmatrix}=\_\_\_\_\begin{bmatrix}1\\0\end{bmatrix}+\_\_\_\_\begin{bmatrix}0\\1\end{bmatrix}=\_\_\_\_\mathbf{e}_1+\_\_\_\_\mathbf{e}_2$$

Then

$$T(\mathbf{x})=T(x_1\mathbf{e}_1+x_2\mathbf{e}_2)=\_\_\_\_T(\mathbf{e}_1)+\_\_\_\_T(\mathbf{e}_2)$$

$$=\_\_\_\_\begin{bmatrix}2\\-3\\4\end{bmatrix}+\_\_\_\_\begin{bmatrix}5\\0\\1\end{bmatrix}=\begin{bmatrix}\ \\ \ \\ \ \end{bmatrix}$$

Note that

$$T(\mathbf{x})=\begin{bmatrix}\ & \ \\ \ & \ \\ \ & \ \end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}.$$

$$T(\mathbf{x})=\begin{bmatrix}T(\mathbf{e}_1) & T(\mathbf{e}_2)\end{bmatrix}\mathbf{x}=A\mathbf{x}$$

To get A, replace the identity matrix $\begin{bmatrix}\mathbf{e}_1 & \mathbf{e}_2\end{bmatrix}$ with
$\begin{bmatrix}T(\mathbf{e}_2) & T(\mathbf{e}_2)\end{bmatrix}$.

Theorem

Let $T : \mathbf{R}^n \to \mathbf{R}^m$ be a linear transformation. Then there exists a unique matrix A such that

$$T(\mathbf{x}) = A\mathbf{x} \text{ for all } \mathbf{x} \text{ in } \mathbf{R}^n.$$

In fact, A is the $m \times n$ matrix whose jth column is the vector $T(\mathbf{e}_j)$, where $\mathbf{e}_j$ is the jth column of the identity matrix in $\mathbf{R}^n$.

$$A = [T(\mathbf{e}_1) \quad T(\mathbf{e}_2) \quad \cdots \quad T(\mathbf{e}_n)]$$

↑

standard matrix for the linear transformation T

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EXAMPLE: $\begin{bmatrix} ? & ? \\ ? & ? \\ ? & ? \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - 2x_2 \\ 4x_1 \\ 3x_1 + 2x_2 \end{bmatrix}$

Solution:

$\begin{bmatrix} ? & ? \\ ? & ? \\ ? & ? \end{bmatrix}$ = standard matrix of the linear transformation T

$$\begin{bmatrix} ? & ? \\ ? & ? \\ ? & ? \end{bmatrix} = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) \end{bmatrix} =$$

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