Assignment 3: Due date: March 31, 2022 before 2.00 pm

On page 134-138, Chapter 4: Consumption-savings and state pricing, work on the following questions:

$$2, 3, 4, 6$$

2. Assume there is an economy with k states of nature and where the following asset pricing formula holds:

$$P_a = \sum_{s=1}^{k} \pi_s m_s X_{sa}$$
$$P_a = E[mX_a]$$

Let an individual in this economy have the utility function $\ln(C_0) + E[\delta \ln(C_1)]$, and let C_0^* be her equilibrium consumption at date 0 and C_s^* be her equilibrium consumption at date 1 in state s, $s = 1, ..., k$. Denote the date 0 price of elementary security s as p_s, and derive an expression for it in terms of the individual's equilibrium consumption.

utility function = $\ln(C_0) + E[\delta \ln(C_1)]$, C_0^* date 0 , C_s^* date 1

From $P_s = \pi_s m_s$

$$\frac{\partial U(c)}{C_0^*} = \ln(C_0^*) = \frac{1}{C_0^*}$$

$$\frac{\partial U(c)}{C_s^*} = \ln(C_s^*) = \frac{1}{C_s^*}$$

$$m_s = \frac{\delta U'(C_s^*)}{U'(C_0^*)} = \frac{\delta\left(\frac{1}{C_s^*}\right)}{\left(\frac{1}{C_0^*}\right)} = \delta \frac{C_0^*}{C_s^*}$$

$$P_s = \delta \pi_s \frac{C_0^*}{C_s^*}$$

3. Consider the one-period consumption-portfolio choice problem. The individual's first-order conditions lead to the general relationship

$$\frac{1}{E[m_{01} R_s]} = R_f$$

$$1 = E\left[m_{01} R_s\right]$$

where m_{01} is the stochastic discount factor between dates 0 and 1, and R_s is the one-period stochastic return on any security in which the individual can invest. Let there be a finite number of date 1 states where π_s is the probability of state s. Also assume markets are complete and consider the above relationship for primitive security s; that is, let R_s be the rate of return on primitive (or elementary) security s. The individual's elasticity of intertemporal substitution is defined as

$$\varepsilon = \frac{R_s}{\frac{C_s}{C_0}} \frac{\partial\left(\frac{C_s}{C_0}\right)}{\partial R_s}$$

$$\varepsilon^I \equiv \frac{R_s}{C_s/C_0} \frac{d\left(C_s/C_0\right)}{dR_s}$$

where C_0 is the individual's consumption at date 0 and C_s is the individual's consumption at date 1 in state s. If the individual's expected utility is given by

$$U\left(C_0\right) + \delta E\left[U\left(\widetilde{C}_1\right)\right]$$

where utility displays constant relative risk aversion, $U\left(C\right) = C^\gamma/\gamma$, solve for the elasticity of intertemporal substitution, ε^I.

From : $m_{01} = \delta \frac{U'(C_1)}{U'(C_0)}$

$= \delta \left(\frac{C_1}{C_0}\right)^{\gamma-1}$

$U(C) = \frac{\gamma C^{\gamma-1}}{\gamma}$

$U'(C) = C^{\gamma-1}$

$1 = E[m_{01} R_s]$

$1 = \pi_s m_s R_s$

$0 = \pi_s \delta (\gamma-1) \left(\frac{C_s}{C_0}\right)^{\gamma-2} \cdot R_s \partial\left(\frac{C_s}{C_0}\right) + \pi_s \delta \left(\frac{C_s}{C_0}\right)^{\gamma-1} \partial R_s$

$\frac{R_s}{\frac{C_s}{C_0}} \frac{\partial\left(\frac{C_s}{C_0}\right)}{\partial R_s} = \frac{1}{1-\gamma}$

$\frac{1}{R_f} = \frac{1}{1.05}$

4. Consider an economy with $k = 2$ states of nature, a "good" state and a "bad" state.[16] There are two assets, a risk-free asset with $R_f = 1.05$ and a second risky asset that pays cashflows

$p = \begin{bmatrix} \frac{1}{1.05} \\ 6 \end{bmatrix}$

$$X_2 = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$$

The current price of the risky asset is 6.

a. Solve for the prices of the elementary securities p_1 and p_2 and the risk-neutral probabilities of the two states.

$p' = \begin{bmatrix} \frac{1}{1.05} & 6 \end{bmatrix}$

$X = \begin{bmatrix} 1 & 10 \\ 1 & 5 \end{bmatrix} \rightarrow X' = \begin{bmatrix} -1 & 2 \\ 0.2 & -0.2 \end{bmatrix}$

$p_1 = p' X^{-1} e = \begin{bmatrix} \frac{1}{1.05} & 6 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 0.2 & -0.2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{1.05} + 1.2 \\ \frac{40}{21} - 1.2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0.2476$

$p_2 = p' X^{-1} e = \begin{bmatrix} \frac{1}{1.05} & 6 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 0.2 & -0.2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{1.05} + 1.2 \\ \frac{40}{21} - 1.2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0.7048$

b. Suppose that the physical probabilities of the two states are $\pi_1 = \pi_2 = 0.5$. What is the stochastic discount factor for the two states?

$m_s = \frac{p_s}{\pi_s}$

$m_1 = \frac{p_1}{\pi_1} = \frac{0.2476}{0.5} = 0.4952$

$m_2 = \frac{p_2}{\pi_2} = \frac{0.7048}{0.5} = 1.4096$

6. This question asks you to relate the stochastic discount factor pricing relationship to the CAPM. The CAPM can be expressed as

$$E[R_i] = R_f + \beta_i \gamma \qquad E[R_i] - \beta_i\gamma$$

where $E[\cdot]$ is the expectation operator, R_i is the realized return on asset i, R_f is the risk-free return, β_i is asset i's beta, and γ is a positive market risk premium. Now, consider a stochastic discount factor of the form

$$m = a + bR_m$$

where a and b are constants and R_m is the realized return on the market portfolio. Also, denote the variance of the return on the market portfolio as σ_m^2.

a. Derive an expression for γ as a function of a, b, $E[R_m]$, and σ_m^2. (Hint: you may want to start from the equilibrium expression $0 = E[m(R_i - R_f)]$.)

b. Note that the equation $1 = E[mR_i]$ holds for all assets. Consider the case of the risk-free asset and the case of the market portfolio, and solve for a and b as a function of R_f, $E[R_m]$, and σ_m^2.

c. Using the formula for a and b in part (b), show that $\gamma = E[R_m] - R_f$.

a. $0 = E[m(R_i - R_f)]$

$0 = E[(a + bR_m)(R_i - R_f)]$

$0 = aE[R_i] - aR_f + bE[R_m R_i] - bR_f E[R_m]$

$0 = a(E[R_i] - R_f) + b[E[R_m]E[R_i] + cov[R_m, R_i] - R_f E[R_m]]$

$0 = (E(R_i) - R_f)(a + bE(R_m)) +$

$$(E(R_i) - R_f) = \frac{b\,cov(R_m, R_i)}{(a + bE(R_m))}$$

$$R_f + \beta_i\gamma - R_f = \frac{-cov(R_m, R_i)}{\sigma_m^2}\,\frac{b\sigma_m^2}{a + bE[R_m]}$$

$$\beta_i\gamma = -\beta_i \frac{b\sigma_m^2}{a + bE[R_m]}$$

$$\gamma = \frac{b\sigma_m^2}{a + bE[R_m]}$$

B. $$\frac{1}{R_f} = E[m_m]$$

&

$$\frac{1}{R_f} = E[a + bR_m]$$

solve for market portfolio

$$a = \frac{1}{R_f} - bE[R_m]$$

$$1 = E[mR_m]$$

$$1 = E[(a+bR_m)R_m]$$

$$1 = aE[R_m] + b(\sigma_m^2 + E[R_m]^2)$$

$$1 = \frac{1}{R_f} - bE[R_m] \cdot E(R_m) + b(\sigma_m^2 + E[R_m]^2)$$

$$1 = \frac{E[R_m]}{R_f} + b\sigma_m^2$$

and

$$b = -\frac{E[R_m] - R_f}{R_f \sigma_m^2}$$

combine all

$$a = \frac{\sigma_m^2 + E(R_m)(E(R_m) - R_f)}{R_f \sigma_m^2}$$

d. Prove

$$a + bE[R_m] = \frac{\sigma_m^2 + E(R_m)(E(R_m) - R_f)}{R_f \sigma_m^2} - \frac{E[R_m] - R_f}{R_f \sigma_m^2}$$

$$= \frac{1}{R_f}$$

Then,

$$\gamma = E[R_m] - R_f$$

$$\gamma = \frac{b\sigma_m^2}{a + bE[R_m]}$$

$$\gamma = \frac{E[R_m] - R_f}{R_f \sigma_m^2} \sigma_m^2 R_f$$

*

we can say that $\gamma = E[R_m] - R_f$